## Engineering

## Mathematics-II

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## Lesson 1

## Linear Equations and Matrices

### 1.1 Introduction

The problem of solving system of linear equations arises in almost all areas of science and engineering. This is an important part of linear algebra and lies at the heart of it.

A linear equation on $n$ variables $x_{1}, x_{2}, \ldots, x_{n}$ is an equation of the form $a_{1} x_{1}+a_{2} x_{2}$ $+\ldots+a_{n} x_{n}=b$,
where $a_{1}, a_{2}, \ldots, a_{n}$ and $b$ are real or complex numbers, usually known in advance. A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables. The following is an example of a system of linear equations:

$$
\begin{align*}
& x_{1}-2 x_{2}+4 x_{3}=10 \\
& 2 x_{1}-3 x_{3}=-9 \tag{1.1}
\end{align*}
$$

It is convenient to represent large systems of linear equations in terms of rectangular arrays called matrices. An $m \times n$ matrix is a rectangular array of elements with $m$ number of rows and $n$ number of columns. It is denoted by $\left(a_{i j}\right)_{m \times}$ n, where $\mathrm{i}=1,2,3, \ldots, m$, and $\mathrm{j}=1,2, \ldots, \mathrm{n}$, and $\mathrm{a}_{\mathrm{ij}}$ are real or complex numbers (or elements of a field) called entries of the matrix. Almost all the concepts in linear algebra are expressed in terms of matrices.

A system of $m$ linear equations on $n$ variables $x_{1}, x_{2}, \ldots, x_{n}$ can be written as

$$
\begin{align*}
& a_{11} x_{1}+a_{12} x_{2}+\ldots+a_{1 n} x_{n}=b_{1} \\
& a_{21} x_{1}+a_{22} x_{2}+\ldots+a_{2 n} x_{n}=b_{2} \\
& a_{m 1} x_{1}+a_{m 2} x_{2}+\ldots+a_{m n} x_{n}=b_{n} \tag{1.2}
\end{align*}
$$

The $m \times n$ matrix

$$
\left(\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 n} \\
a_{21} & a_{22} & \ldots & a_{2 n} \\
\vdots & & & \\
a_{m 1} & a_{m 2} & \ldots & a_{m n}
\end{array}\right)
$$

associated with the system (1.2) is called the co-efficient matrix of the system. The $\mathrm{m} \times(\mathrm{n}+1)$ matrix

is called the augmented matrix of the system (1.2).

The augmented matrix of a system consists of the co-efficient matrix with an additional column whose entries are the constants from the right sides of the equations. If in (1.2) $b_{i}=0$ for all $i=1,2, \ldots, n$ then the system is called homogeneous otherwise non-homogeneous. We perform some operations on
matrices not only for solving system of linear equations but also for studying other topics in linear algebra.

### 1.2. Matrix Operations

As matrix notation simplifies the calculations in solving systems of linear equations, we shall discuss different kind of matrices and operations on them.

Recall that a matrix A of size $m \times n$ over a field $F$ (here we take F as the real or complex field) is denoted by $A=\left(a_{i j}\right)_{m \times n}, i=1,2,3, \ldots, m$, and $j=1,2, \ldots, n$, and $\mathrm{a}_{\mathrm{ij}}$ are from F . If $\mathrm{m}=\mathrm{n}$ then A is called a square matrix. In this case the entries $\mathrm{a}_{11}, \ldots, \mathrm{a}_{\mathrm{nn}}$ are called the main diagonal or principal diagonal and other entries are called off-diagonal entries. If $\mathrm{a}_{\mathrm{ij}}=0$ for all i and j , then A is called the null matrix or the zero matrix, and is denoted by 0 . An identity matrix, denoted by I , is a square matrix whose all diagonal entries are equal to 1 and off diagonal entries are equal to zero.

A square matrix A is called a diagonal matrix if all the off-diagonal entries are zero. A square matrix $\mathrm{A}=\left(\mathrm{a}_{\mathrm{i}}\right)_{\mathrm{n} \times \mathrm{n}}$ is called lower (respectively upper) triangular matrix if $\mathrm{a}_{\mathrm{ij}}=0$ whenever $\mathrm{i}>\mathrm{j}$ (respectively $\mathrm{i}<\mathrm{j}$ ), that is, all entries above (respectively below) the main diagonal are zero.

Two matrices of the same size $A=\left(a_{i j}\right)_{m \times n}$ and $B=\left(b_{i j}\right)_{m \times n}$ are said to be equal if $\mathrm{a}_{\mathrm{ij}}=\mathrm{b}_{\mathrm{ij}}$ for all $\mathrm{i}, \mathrm{j}$.

### 1.2.1 Addition and Scalar Multiplication

If $\mathrm{A}=\left(\mathrm{a}_{\mathrm{ij}}\right)_{\mathrm{m} \times \mathrm{n}}$ is a matrix over F and $\alpha \square \mathrm{F}$ then the scalar multiplication of A by $\alpha$ is the matrix $\quad \alpha \mathrm{A}=\left(\alpha \mathrm{a}_{\mathrm{ij}}\right)_{\mathrm{m} \times \mathrm{n}}$ i.e. each entry of A is multiplied by $\alpha$.

If $\mathrm{A}=\left(\mathrm{a}_{\mathrm{ij}}\right)_{\mathrm{m} \times \mathrm{n}}$ and $\mathrm{b}=\left(\mathrm{b}_{\mathrm{ij}}\right)_{\mathrm{m} \times \mathrm{n}}$ are matrices of the same size over F then addition of $A$ and $B$ denoted by, $A+B$, is the matrix $C=\left(c_{i j}\right)_{m \times n}$, where $c_{i j}=a_{i j}+b_{i j}$.

Scalar multiplication and addition of matrices satisfy some properties as given below.

For matrices A, B and C of the same size over $F$ and $\alpha, \beta \square F$ :
(1) $\mathrm{A}+\mathrm{B}=\mathrm{B}+\mathrm{A}$ (commutative)
(2) $(\mathrm{A}+\mathrm{B})+\mathrm{C}=\mathrm{A}+(\mathrm{B}+\mathrm{C})$ (associative)
(3) $A+0=0+A=A$, where 0 is the zero matrix of the same size as $A$.
(4) $\mathrm{A}+(-\mathrm{A})=(-\mathrm{A})+\mathrm{A}=0$, where $-\mathrm{A}=(-1)$ A i.e. if $\mathrm{A}=\left(\mathrm{a}_{\mathrm{ij}}\right)_{\mathrm{m} \times \mathrm{n}}$ then $-\mathrm{A}=(-$ $\left.\mathrm{a}_{\mathrm{ij}}\right)_{\mathrm{m} \times \mathrm{n}}$.
(5) $(\alpha+\beta) \mathrm{A}=\alpha \mathrm{A}+\beta \mathrm{A}$.
(6) $\alpha(\mathrm{A}+\mathrm{B})=\alpha \mathrm{A}+\alpha \mathrm{B}$.
(7) $\alpha(\beta \mathrm{A})=\alpha \beta \mathrm{A}$.

### 1.2.2 Matrix Multiplication

If $\mathrm{A}=\left(\mathrm{a}_{\mathrm{ij}}\right)_{\mathrm{m} \times \mathrm{n}}$ and $\mathrm{B}=\left(\mathrm{b}_{\mathrm{ij}}\right)_{\mathrm{n} \times \mathrm{p}}$ are matrices over F then multiplication or product of $A$ and $B$, denoted by $A B$, is the matrix $C=\left(c_{i j}\right)_{m \times p}$, where

$$
c_{\mathrm{ij}}=\sum_{k=1}^{\mathrm{n}} \mathrm{a}_{\mathrm{ik}} \mathrm{~b}_{\mathrm{kj}} .
$$

Matrix multiplication satisfies some properties as given below.
(1) Matrix multiplication need not be commutative, that is, one can find matrices A and $B$ such that $A B$ is not equal to $B A$.
(2) For matrices A and B if $\mathrm{AB}=0$ then it may not imply either $\mathrm{A}=0$ or $\mathrm{B}=0$.
(3) If for matrices $A, B, C$ if $A B=A C$, it may not imply $B=C$, that is matrix multiplication does not obey cancellation law.
(4) If $A$, B and C are matrices of sizes $m \times n, n \times p$, and $p \times q$ respectively then (A B) $C=A(B C)$ (associative).
(5) If $A$ is a matrix of size $m \times n$ and both $B$ and $C$ are matrices of size $n \times p$ then $A(B+C)=A B+A C$ (left distributive).
(6) If $A, B$ are matrices of size $m \times n$ each and $C$ is a matrix of size $n \times p$ then $(A+B) C=A C+B C$ (right distributive).
(7) For any square matrix $\mathrm{A}, \mathrm{AI}=\mathrm{IA}=\mathrm{A}$, where I is the identity matrix of the same size as A.

For matrix $A=\left(a_{i j}\right)_{m \times n}$, the transpose of $A$, denoted by $A^{T}$, is the matrix $A^{T}=\left(a_{j i}\right)_{n}$ $\times \mathrm{m}$. In other words $A^{T}$ is obtained from $A$ by writing the rows of $A$ as the columns of $\mathrm{A}^{\mathrm{T}}$ in order. Some properties of transpose operation are as given below.
(1) For any matrix $A,\left(A^{T}\right)^{T}=A$.
(2) For matrices A and B of the same size

$$
(A+B)^{T}=A^{T}+B^{T} .
$$

(3) For matrices $A$ and $B$ over $F$ of sizes $m \times n$ and $n \times p$ respectively, $(A B)^{T}=B^{T} A^{T}$.

### 1.2.3 Some Special Matrices

Here we shall discuss about some of the special type of matrices which will be used in the subsequent lectures.

We consider a square matrix $A=\left(a_{i j}\right)_{n \times n}$. If $A$ is a real matrix and satisfies $A=A^{T}$ then $A$ is called symmetric. In this case $a_{i j}=a_{j i}$ for all $i, j$. If $A$ satisfies $A^{T}=-A$ then A is called skew-symmetric. In this case $\mathrm{a}_{\mathrm{ij}}=-\mathrm{a}_{\mathrm{ji}}$ for all $\mathrm{i}, \mathrm{j}$, and therefore all diagonal entries are equal to zero.

Here we take a complex square matrix $\mathrm{A}=\left(\mathrm{a}_{\mathrm{ij}}\right)_{\mathrm{n} \times \mathrm{n}}$. The conjugate of A is the matrix $\overline{\mathrm{A}}=\left(\overline{\mathrm{a}}_{\mathrm{ij}}\right)_{\mathrm{n} \times \mathrm{n}}$, where $\square \overline{\mathrm{a}}_{\mathrm{ij}}$ is the complex conjugate of $\mathrm{a}_{\mathrm{ij}}$. Matrix A is said to be Hermitian if $(\square \mathrm{A})^{\mathrm{T}}=\mathrm{A}$. In this case $\mathrm{a}_{\mathrm{ij}}=\overline{\mathrm{a}}_{\mathrm{ji}}$ and in particular $\mathrm{a}_{\mathrm{ii}}=\overline{\mathrm{a}}_{\mathrm{i} .}$. Thus for Hermitian matrices diagonal entries are real numbers. Matrix A is said to be skewHermitian if $(\square \mathrm{A})^{\mathrm{T}}=-\mathrm{A}$. By the similar argument $\mathrm{a}_{\mathrm{ij}}=-\overline{\mathrm{a}}_{\mathrm{ji}}$ and so diagonal entries are either 0 or pure imaginary for skew-Hermitian matrices. One sees that symmetric and Hermitian matrices agree for real matrices. Similarly, skewsymmetric and skew-Hermitian matrices also agree for real matrices.

A complex square matrix $A=\left(\mathrm{a}_{\mathrm{ij}}\right)_{\mathrm{n} \times \mathrm{n}}$ is called unitary if $\mathrm{A}(\square \mathrm{A})^{\mathrm{T}}=(\square \mathrm{A})^{\mathrm{T}} \mathrm{A}=\mathrm{I}$, where $I$ is the identity matrix of the same size as A. In case of real matrices unitary
matrices are called orthogonal, that is, a real matrix $A$ is orthogonal if $A A^{T}=A^{T} A$ $=\mathrm{I}$.

### 1.2.4 Elementary Row/Column Operations

For any matrix $A$, each of the following is called an elementary row (resp. columns) operation on A :
(1) Interchange of two rows (resp. columns).
(2) Addition of scalar multiple of one row (resp. column) to another row (resp. column).
(3) Multiplication of a row (resp. column) by a non-zero scalar.

### 1.3 Determinant of Matrices

Let $\mathrm{A}=\left(\mathrm{a}_{\mathrm{ij}}\right)_{\mathrm{n} \times \mathrm{n}}$ be a square matrix with $\mathrm{a}_{\mathrm{ij}} \square \mathbb{R}$ or

We define determinant of A, denoted by det A or | A |, recursively as below. For $\mathrm{n}=2$,

$$
|\mathrm{A}|=\left|\begin{array}{ll}
\mathrm{a}_{11} & \mathrm{a}_{12} \\
\mathrm{a}_{21} & \mathrm{a}_{22}
\end{array}\right|=\mathrm{a}_{11} \mathrm{a}_{22}-\mathrm{a}_{12} \mathrm{a}_{21} .
$$

For $\mathrm{n} \geq 3$,

$$
\operatorname{det} \mathrm{A}=|\mathrm{A}|=\sum_{\mathrm{j}=1}^{\mathrm{m}}(-1)^{\mathrm{i}+\mathrm{j}} \mathrm{a}_{\mathrm{ij}} \mathrm{~m}_{\mathrm{ij}} .
$$

Where i is a fixed integer with $1 \leq \mathrm{i} \leq \mathrm{n}$, and $\mathrm{m}_{\mathrm{ij}}$ is the determinant of the matrix obtained from A by deleting ith row and jth column.

One may also find determinant of A by using following properties of determinant:
(1) For identity matrix I of any size, det $\mathrm{I}=1$.
(2) $\operatorname{det} \mathrm{A}=\operatorname{det} \mathrm{A}^{\mathrm{T}}$
(3) If any two rows (or columns) are interchanged, then the value of the determinant is multiplied by $(-1)$.
(4) If each element of a row is multiplied by a scalar $\alpha$ then the value of the determinant is multiplied by $\alpha$. Therefore $|\alpha \mathrm{A}|=\alpha^{\mathrm{n}}|\mathrm{A}|$.
(5) If a non-zero scalar multiple of the elements of some row (or column) is added to the corresponding elements of some other row (or column), then the value of the determinant remains unchanged.
(6) Determinant of diagonal or triangular matrices is the product of its diagonal entries.
(7) If $A$ and $B$ are the matrices of the same order then $\operatorname{det}(A B)=\operatorname{det}(A) \operatorname{det}(B)$.

### 1.4 Conclusions

Matrices and operations on them will be used in almost all the subsequent lectures. In the next lecture we shall solve systems of linear equations. A solution of a system of linear equations on $n$ variables $x_{1}, x_{2}, \ldots, x_{n}$ is a list ( $s_{1}, s_{2}, \ldots, s_{n}$ ) of numbers such that each equation is a true statement when the values $s_{1}, s_{2}, \ldots, s_{n}$ are substituted for $\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{\mathrm{n}}$ respectively. The set of all possible solutions is called the solution set of the given system. Two systems are called equivalent if they have the same solution set. That is, every solution of the first system is a solution of the second system and vice versa. Getting solution set of a system of two linear equations in two variables is easy because it is just finding the intersection of two lines. However, solving a large system is not so straight-
forward. For this we represent a system in matrix notation and then we perform some operations on the associated matrices. From the resultant matrices either we draw conclusion that the system has no solution or find solutions of the system.

Keywords: Algebra of matrices, special matrices, elementary row operations, determinant of matrices, linear systems.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

Linear Algebra, A. R. Rao and P. Bhimasankaram, Hindustan Book Agency, New Delhi, 2000.

Linear Algebra and Its Applications, Fourth Edition, Gilbert Strang, Thomson Books/Cole, 2006.

Matrix Methods: An Introduction, Second Edition, Richard Bronson, Academic press, 1991.

## Lesson 2

## Rank of a Matrix and Solution of a Linear System

### 2.1 Introduction

In this lecture we shall discuss about the rank of matrices, the consistency of systems of linear equations, and finally present the Gauss elimination method for solving the linear systems. For this we need an important form of matrices called echelon form which is obtained by applying elementary row (or column) operations.

### 2.2 Echelon Form of a Matrix

Echelon form of a matrix is useful in solving system of linear equations, finding rank of a matrix and checking many more results in linear algebra.

An $\mathrm{m} \times \mathrm{n}$ matrix A is said to be in (row) echelon form if
(i) All the zero rows of A are at the bottom.
(ii) For the non-zero rows of A , as the row number increases, the number of zero entries at the beginning of the row also increases.

In the echelon form of a matrix some people consider one more condition that the $1^{\text {st }}$ non-zero entry in a non-zero row is equal to 1 . However this condition is not required for us and therefore not included in the definition of echelon form of a matrix. One finds row echelon form of a matrix by applying elementary row operations. By applying elementary column operations, one gets column echelon form of the matrix.

Example 2.2.1: Find the row echelon form of

$$
A=\left(\begin{array}{lll}
1 & 3 & 5 \\
1 & 4 & 3 \\
1 & 1 & 9
\end{array}\right)
$$

We keep $1^{\text {st }}$ row as it. Then we make $1^{\text {st }}$ entry of the second row zero by applying elementary row operations. So replacing $2^{\text {nd }}$ row $R_{2}$ by $R_{2}-R_{1}$ one gets

$$
\left(\begin{array}{ccc}
1 & 3 & 5 \\
0 & 1 & -2 \\
1 & 1 & 9
\end{array}\right) .
$$

Then we make at least $1^{\text {st }}$ two entries of the $3^{\text {rd }}$ row of the above matrix zero. For this we replace $R_{3}$ by $R_{3}-R_{1}$ in the above matrix and get

$$
\left(\begin{array}{ccc}
1 & 3 & 5 \\
0 & 1 & -2 \\
0 & -2 & 4
\end{array}\right) .
$$

Finally by replacing $R_{3}$ by $R_{3}+2 R_{2}$ one gets the echelon form of $A$ and is given by

$$
\left(\begin{array}{ccc}
1 & 3 & 5 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{array}\right) .
$$

### 2.3 Rank of a Matrix

The rank of a matrix has several equivalent definitions. Here we take the rank of a matrix A as the number of non-zero rows in the row echelon form of A . It is also defined as the number of nonzero columns in the column echelon form of the matrix. Whatever way the definition may be given the rank of a matrix will
be the same, is a fixed number. Therefore the rank of a matrix A has the following properties.
(1) Matrix $A$ and its transpose have the same rank, that is, $\operatorname{rank}(A)=\operatorname{rank}\left(A^{T}\right)$.
(2) If $A$ is a matrix of size $m \times n$ then $\operatorname{rank}(A)$ is at the $\operatorname{most} \min \{m, n\}$.
(3) If $B$ is a sub-matrix of $A$ then rank (B) is less than or equal to rank(A).

Example 2. 3.1: Here we find rank of the matrix

$$
A=\left(\begin{array}{ccccc}
2 & -2 & 3 & 4 & -1 \\
-1 & 1 & 2 & 5 & 2 \\
0 & 0 & -1 & -2 & 3 \\
1 & -1 & 2 & 3 & 0
\end{array}\right)
$$

Here we find echelon form of the matrix A. First row will be kept as it is. Replacing $\mathrm{R}_{4}$ by $\mathrm{R}_{4}+\mathrm{R}_{2}$ and then $\mathrm{R}_{2}$ by $2 \mathrm{R}_{2}+\mathrm{R}_{1}$ the matrix will be

$$
A\left(\begin{array}{ccccc}
2 & -2 & 3 & 4 & -1 \\
0 & 0 & 7 & 14 & 3 \\
0 & 0 & -1 & -2 & 3 \\
0 & 0 & 4 & 8 & 2
\end{array}\right) .
$$

Replacing $R_{4}$ by $R_{4}+4 R_{3}$ and then replacing $R_{3}$ by $7 R_{3}+R_{2}$ the matrix will be

$$
\left(\begin{array}{ccccc}
2 & -2 & 3 & 4 & -1 \\
0 & 0 & 7 & 14 & 3 \\
0 & 0 & 0 & 0 & 24 \\
0 & 0 & 0 & 0 & 14
\end{array}\right)
$$

Finally replacing $R_{4}$ by $R_{4}-\frac{14}{24} R_{3}$ the resultant matrix will be in echelon form as given below:

$$
\left(\begin{array}{ccccc}
2 & -2 & 3 & 4 & -1 \\
0 & 0 & 7 & 14 & 3 \\
0 & 0 & 0 & 0 & 24 \\
0 & 0 & 0 & 0 & 0
\end{array}\right) .
$$

Now there are three non-zero rows in the echelon form of the given matrix A. Therefore rank of A is equal to 3 .

### 2.4 Solution of a Linear System

Recall that a system of $m$ linear equations in $n$ variables $x_{1}, x_{2}, \ldots, x_{n}$ will be of the form

$$
\begin{aligned}
& a_{11} x_{1}+a_{12} x_{2}+\ldots+a_{1 n} x_{n}=b_{1} \\
& a_{21} x_{1}+a_{22} x_{2}+\ldots+a_{2 n} x_{n}=b_{2} \\
& \cdot \\
& \cdot \\
& a_{m 1} x_{1}+a_{m 2} x_{2}+\ldots+a_{m n} x_{n}=b_{m}
\end{aligned}
$$

where $a_{i j}$ 's and $b_{i}$ 's are real or complex numbers.

By using matrix notation this system can be expressed as $\mathrm{Ax}=\mathrm{b}$, where A is the $\mathrm{m} \times \mathrm{n}$ matrix

$$
A=\left(\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 n} \\
a_{21} & a_{22} & \ldots & a_{2 n} \\
\vdots & & & \\
a_{m 1} & a_{m 2} & \ldots & a_{m n}
\end{array}\right)
$$

$x$ is the $n \times 1$ matrix $x=\left(\begin{array}{c}x_{1} \\ \vdots \\ x_{n}\end{array}\right)$, $b$ is the $m \times 1$ matrix $\left(\begin{array}{c}b_{1} \\ b_{2} \\ \vdots \\ b_{m}\end{array}\right)$.


#### Abstract

A system of linear equations has either (i) no solution or (ii) exactly one solution or (iii) infinitely many solutions. The system is said to be consistent if it has at least one solution, that is (ii) or (iii) of the above hold, and is inconsistent if it has no solution.


The following theorem gives conditions for existence of solution of the system $\mathrm{Ax}=\mathrm{b}$.

Theorem 2.4.1: Let $\mathrm{Ax}=\mathrm{b}$ be a system of m linear equations on n variables. Let the augmented matrix $\square \mathrm{A}$ of A be (A b). Then
(i) The system is consistent if rank $\mathrm{A}=\operatorname{rank} \square \mathrm{A}$.
(ii) The system has a unique solution if $\operatorname{rank} \mathrm{A}=\operatorname{rank} \square \mathrm{A}=\mathrm{n}$.
(iii) The system has infinitely many solutions if rank $\mathrm{A}=\operatorname{rank} \square \mathrm{A}=\mathrm{k}<\mathrm{n}$.

Remark 2.4.1: Recall that if $b=0$ then the system $A x=0$ is called homogeneous. In this case $A=\square A$ and so from the above theorem a homogeneous system is always consistent.In fact $\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in(0,0, \ldots$, 0 ) is always a solution of $\mathrm{Ax}=0$.

### 2.5 Gauss Elimination Method for Solving a System

Gauss-Elimination method is a matrix method constantly used to solve large systems of linear equations. The main steps in this method are as follow:

1. Consider the augmented matrix of the system.
2. Convert the augmented matrix in to row echelon form. Decide whether the system is consistent or not. If yes then go to the next step, stop otherwise.
3. Write the system of equations corresponding to the matrix in echelon form obtained in step 2. Now this system is either solvable by back-substitution or having some free variables (variables which do not occur at the beginning of any equation of the system in this step) for which we assign arbitrary real/complex value and then solve the system.

We explain the above method through some examples.

Example 2.5.1: Consider the system of linear equations

$$
\begin{aligned}
& 2 x-2 y+3 z+4 u=-1 \\
& -x+y+2 z+5 u=3 \\
& -z-2 u=3 \\
& x-y+2 z+3 u=0
\end{aligned}
$$

The augmented matrix of this system is

$$
\left(\begin{array}{ccccc}
2 & -2 & 3 & 4 & -1 \\
-1 & 1 & 2 & 5 & 2 \\
0 & 0 & -1 & -2 & 3 \\
1 & -1 & 2 & 3 & 0
\end{array}\right)
$$

Notice that this is the same matrix A appears in Example 3.1. So its row echelon form will be

$$
\left(\begin{array}{ccccc}
2 & -2 & 3 & 4 & -1 \\
0 & 0 & 7 & 14 & 3 \\
0 & 0 & 0 & 0 & 24 \\
0 & 0 & 0 & 0 & 0
\end{array}\right) .
$$

Observe that the rank of the co-efficient matrix is 2 and that of the augmented matrix is 3. Therefore according to Theorem 4.1(i) the given system is inconsistent.

Example 2.5.2: Here we shall solve the system

$$
\begin{aligned}
& 2 x+y-2 z=10 \\
& 3 x+2 y+2 z=1 \\
& 5 x+4 y+3 z=4
\end{aligned}
$$

The augmented matrix is $\left(\begin{array}{cccc}2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4\end{array}\right)$.

Row echelon form of this matrix is $\left(\begin{array}{cccc}2 & 1 & -2 & 10 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & -14 & 42\end{array}\right)$.

Notice that $1^{\text {st }}$ three columns is the row echelon form of the co-efficient matrix and its rank is equal to three which is same as the rank of the augmented matrix. Therefore the system is consistent and since the number of variables is also equal to three from Theorem 4.1(ii) the system has a unique solution.

The system corresponding to the echelon form of the augmented matrix is:

$$
\begin{aligned}
& 2 x+y-2 z=10 \\
& y+10 z=-28 \\
& -14 z=42
\end{aligned}
$$

From the last equation we get $\mathrm{z}=-3$. Then by back substitution we get $\mathrm{y}=2$ and $x=1$ from the $2^{\text {nd }}$ and $1^{\text {st }}$ equations respectively. Hence $(1,2,-3)$ is the unique solution of the given system.

Example 2.5.3: Here we shall solve the system

$$
\begin{aligned}
& x+2 y-3 z=6 \\
& 2 x-y+4 z=2 \\
& 4 x+3 y-2 z=14
\end{aligned}
$$

Augmented matrix of the system is $\left(\begin{array}{cccc}1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14\end{array}\right)$
and its row echelon form is $\left(\begin{array}{cccc}1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & 0 & 0 & 0\end{array}\right)$.

The rank of the co-efficient matrix and the rank of the augmented matrix are same and is equal to 2 which is less than the number of variables. Therefore the system has infinite number of solutions. From the row echelon form of the augmented matrix the system will be

$$
\begin{aligned}
& x+2 y-3 z=6 \\
& -5 y+10 z=-10
\end{aligned}
$$

Here z is the free variable. So it can take any real value. Let $\mathrm{z}=\alpha, \alpha$ is a real number. Then from the second equation of the above system $\mathrm{y}=2+2 \alpha$ and
then from the first equation $x=2-\alpha$. Hence the set of all solutions of the system is

$$
\{(2-\alpha, 2+2 \alpha, \alpha): \alpha \square \mathbb{R}\}
$$

### 2.6 Conclusions

In this lecture we have observed that homogeneous systems are always consistent. We shall see in an other lecture that the set of all solutions of a homogeneous system has linearity property and therefore these systems are of special interest. In a subsequent lecture we shall learn about the linearity property of sets, which is the basic thing of the subject Linear Algebra.

Keywords: Echelon form of matrices, rank of a matrix, solution of linear system, Gauss elimination method.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

Linear Algebra, A. R. Rao and P. Bhimasankaram, Hindustan Book Agency, New Delhi, 2000.

Linear Algebra and Its Applications, Fourth Edition, Gilbert Strang, Thomson Books/Cole, 2006.

Matrix Methods: An Introduction, Second Edition, Richard Bronson, Academic press, 1991.

## Lesson 3

## Inverse of Matrices by Determinants and Gauss-Jordan Method

### 3.1 Introduction

In lecture 1 we have seen addition and multiplication of matrices. Here we shall discuss about the reciprocal or inverse of matrices. Matrix inverse is one of the basic concepts that is useful in several topics of linear algebra. Not every matrix has an inverse. In this lecture we shall find conditions for existence of inverse of a matrix and discuss two different methods for getting it.

### 3.2 Inverse of a Matrix

Let A be a square matrix of size $n$. A square matrix $B$ of size $n$ is said to be inverse of $A$ if and only if $A B=B A=I$, where $I$ is the identity matrix of size $n$.

Example 3.2.1: Let $\mathrm{A}=\left[\begin{array}{ll}1 & 2 \\ 3 & 4\end{array}\right]$ and $\mathrm{B}=\left[\begin{array}{cc}-2 & 1 \\ \frac{1}{2} & -\frac{1}{2}\end{array}\right]$.

Notice that $\mathrm{AB}=\mathrm{BA}=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$. So A and B are inverse of each other.

Inverse of a matrix A is denoted by $\mathrm{A}^{-1}$. If a square matrix has an inverse then it is called invertible or non-singular, otherwise it is non-invertible or singular. Not all square matrices are invertible.

Theorem 3.2.1: A square matrix has an inverse if and only if its determinant is non-zero.

Some of the properties of inverse of a matrix are as listed below:
(1) Inverse of a matrix if exists is unique.
(2) Inverse of inverse of a matrix is the matrix itself, that is, $\left(A^{-1}\right)^{-1}=A$.
(3) Inverse and transpose operations are interchangeable, that is, $\left(A^{T}\right)^{-1}=\left(A^{-1}\right)^{T}$.
(4) If A and B are invertible matrices then $(\mathrm{AB})^{-1}=\mathrm{B}^{-1} \mathrm{~A}^{-1}$.

### 3.3 Inverse by Determinants

Recall that for a square matrix $\mathrm{A}=\left(\mathrm{a}_{\mathrm{ij}}\right)$, the minor of any entry $\mathrm{a}_{\mathrm{ij}}$ is the determinant of the square matrix obtained from $A$ by removing $\mathrm{i}^{\text {th }}$ row and $\mathrm{j}^{\text {th }}$ column. Moreover the cofactor of $\mathrm{a}_{\mathrm{ij}}$ is equal to the minor of $\mathrm{a}_{\mathrm{ij}}$ multiplied by $(-1)^{\mathrm{i}}$ ${ }^{+\mathrm{j}}$. The cofactor matrix associated with an $\mathrm{n} \times \mathrm{n}$ matrix A is an $\mathrm{n} \times \mathrm{n}$ matrix $\mathrm{A}^{\mathrm{c}}$ obtained from A by replacing each entry of A by its cofactor. The adjugate $\mathrm{A}^{*}$ of A is the transpose of the cofactor matrix of $A$.

The following theorem gives an idea to find inverse of a matrix.

Theorem 3.3.1: For any square matrix A,

$$
\mathrm{AA}^{*}=\mathrm{A}^{*} \mathrm{~A}=(\operatorname{det} \mathrm{A}) \mathrm{I}
$$

where $I$ is the identity matrix of the same size as $A$.

Corollary 3.3.1 If det $\mathrm{A} \neq 0$ then

$$
A\left(\frac{A^{*}}{\operatorname{det} A}\right)=\left(\frac{A^{*}}{\operatorname{det} A}\right) A=I .
$$

Thus we have the following formula for inverse of a matrix given in the theorem below.

Theorem 3.3.2: For any square matrix A with $\operatorname{det} \mathrm{A} \neq 0$,

$$
\mathrm{A}^{-1}=\frac{1}{\operatorname{det} \mathrm{~A}} \mathrm{~A}^{*} .
$$

Example 3.3.1: Here we find inverse of the matrix

$$
A=\left(\begin{array}{ccc}
2 & 0 & -1 \\
0 & 1 & 2 \\
3 & 1 & 1
\end{array}\right)
$$

We first check the value of determinant of $A$. Since $\operatorname{det} A=1 \neq 0$, the inverse of $A$ exists.

One can check that the cofactor matrix $\mathrm{A}^{\mathrm{c}}$ of A is given by

$$
\mathrm{A}^{\mathrm{c}}=\left(\begin{array}{ccc}
-1 & 6 & -3 \\
-1 & 5 & -2 \\
1 & -4 & 2
\end{array}\right) .
$$

Then the adjugate $A^{*}$ of $A$ is

$$
A^{*}=\left(\begin{array}{ccc}
-1 & -1 & 1 \\
6 & 5 & -4 \\
-3 & -2 & 2
\end{array}\right) .
$$

Since $\operatorname{det} A=1, A^{-1}=\frac{1}{\operatorname{det} A} A^{*}=A^{*}$.

Hence,

$$
\mathrm{A}^{-1}=\left(\begin{array}{ccc}
-1 & -1 & 1 \\
6 & 5 & -4 \\
-3 & -2 & 2
\end{array}\right) .
$$

Existence of inverse of a matrix can be linked with rank of A through the result below given in the theorem.

Theorem 3.3.3: For a matrix $A$ of size $n, \operatorname{det} A \neq 0$ if and only if rank $A=n$. In other words inverse of A exists if and only if rank $\mathrm{A}=\mathrm{n}$.

### 3.4 Inverse by Gauss-Jordan Elimination

Next we shall find inverse of a square matrix A of size n by Gauss-Jordan elimination method. The following steps are followed in this method:

Step 1: If either $\operatorname{det} A \neq 0$ or rank $A=n$ then proceed to next step, otherwise inverse of A does not exist.

Step 2: Form the augmented matrix (AI) where I is the $\mathrm{n} \times \mathrm{n}$ identity matrix.
Step 3: Apply elementary row operations to (A I) so that first $n$ column of it will form an upper triangular matrix, say U. So now the resultant matrix is ( U B).

Step 4: Again apply elementary row operations to (UB) till first $n$ columns form the identity matrix. If the resultant matrix is ( I K ) then K is the inverse of matrix A .

We shall consider an example below to explain this method.

Example 3.4.1: Here we shall find inverse of the matrix
$\mathrm{A}=\left(\begin{array}{ccc}2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3\end{array}\right)$. One checks that rank of A is equal to 3 or $\operatorname{det} \mathrm{A} \neq 0$, and so
inverse of A exists. The augmented matrix is $\left(\begin{array}{ccc|ccc}2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1\end{array}\right)$.

Replacing $R_{1}$ by ${ }_{2}^{1} R_{2}$ one gets

$$
\longrightarrow\left(\begin{array}{ccc|ccc}
1 & 0 & 0 & 3 & -1 & 1 \\
0 & 1 & 0 & -15 & 6 & -5 \\
0 & 0 & 1 & 5 & -2 & 2
\end{array}\right)
$$

The last matrix is of the form (I K). Therefore the inverse of A is given by

$$
A^{-1}=\left(\begin{array}{ccc}
3 & -1 & 1 \\
-15 & 6 & -5 \\
5 & -2 & 2
\end{array}\right) .
$$

### 3.5 Conclusions

Several other methods are also there to find inverse of a matrix and for particular type of matrices like upper or lower triangular matrices one can derive an easier formula for the inverse. Applying inverse of a matrix one can find solution of the system $A x=b$ if $A$ is a square matrix of size $n$ and rank of $A$ is $n$. In this case $x=A^{-}$ ${ }^{1} \mathrm{~b}$ is the solution.

Keywords: Invertible matrices, Adjugate of a matrix, Gauss-Jordan elimination method, Augmented matrix.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

Linear Algebra, A. R. Rao and P. Bhimasankaram, Hindustan Book Agency, New Delhi, 2000.

Linear Algebra and Its Applications, Fourth Edition, Gilbert Strang, Thomson Books/Cole, 2006.

Matrix Methods: An Introduction, Second Edition, Richard Bronson, Academic press, 1991.

## Lesson 4

## Vector Spaces, Linear Dependence and Independence

### 4.1 Introduction

In this lecture we discuss about the basic algebraic structure involved in linear algebra. This structure is known as the vector space. A vector space is a non-empty set that satisfies some conditions with respect to addition and scalar multiplication. Recall that by a scalar we mean a real or a complex number. The set of all real numbers $\mathbb{R}$ is called the real field and the set of all complex numbers $\square$ is called the complex field. Here onwards by a field F we mean the set of real or the set of complex numbers. The elements of vector spaces are usually known as vectors and that in field F are called scalars. In this lecture we also discuss about linearly dependency or independency of vectors.

### 4.2 Vector Spaces

A non-empty set V together with two operations called addition (denoted by + ) and scalar multiplication (denoted by.), in short ( $\mathrm{V},+$, . ), is a vector space over a field F if the following hold:
(1) V is closed under scalar multiplication, i.e. for every element $\alpha \square \mathrm{F}$ and $\mathrm{u} \square \mathrm{V}$, $\alpha . u \square$ V. (In place of $\alpha$.u usually we write simply $\alpha u$ ).
(2) ( $\mathrm{V},+$ ) is a commutative group, that is, (i) forevery pair of elements $\mathrm{u}, \mathrm{v} \square \mathrm{V}$, $\mathrm{u}+\mathrm{v} \square \mathrm{V}$ (ii) elements of V are associative and commutative with respect to + (iii) V has the zero element, denoted by 0 , with respect to + , i.e, $\mathrm{u}+0=0+\mathrm{u}=0$, for every element $u$ of V and finally (iv) every element u of V has additive inverse, i.e, there exists $\mathrm{v} \square \mathrm{V}$ such that $\mathrm{u}+\mathrm{v}=\mathrm{v}+\mathrm{u}=0$.
(3) For $\alpha, \beta \square \mathrm{F}$ and $\mathrm{u} \square \mathrm{V},(\alpha+\beta) . \mathrm{u}=\alpha . \mathrm{u}+\beta . \mathrm{u}$.
(4) For $\alpha \square \mathrm{F}$ and $\mathrm{u}, \mathrm{w} \square \mathrm{V}, \alpha .(\mathrm{u}+\mathrm{w})=\alpha . \mathrm{u}+\alpha . \mathrm{w}$.
(5) For $\alpha, \beta \square \mathrm{F}$ and $\mathrm{u} \square \mathrm{V}$, $\alpha .(\beta . \mathrm{u})=(\alpha \beta) . \mathrm{u}$
(6) $1 . \mathrm{u}=\mathrm{u}$, for all $\mathrm{u} \square \mathrm{V}$, where 1 is the multiplicative identity of F .

If V is vector space over F then elements of V are called vectors and elements of F are called scalars.

For vectors $\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{n}}$ in V and scalars $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{\mathrm{n}}$ in F the expression $\alpha_{1} \mathrm{v}_{1}, \alpha_{2} \mathrm{v}_{2}, \ldots, \alpha_{\mathrm{n}} \mathrm{v}_{\mathrm{n}}$ is called a linear combination of $\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{n}}$. Notice that V contains all finite linear combinations of its elements hence it is also called a linear space.

Examples 4.2.1: Here we give example of some vector spaces.
(1) $\square$ is a vector space over $\mathbb{R}$. But $\mathbb{R}$ is not a vector space over $\square$ as it is not closed under scalar multiplication.
(2) If $F=\mathbb{R}$ or $\quad F^{n}$ Hedt $\left.\left(x_{1}, x_{2}, \ldots, x_{n}\right): x_{i} \square F, 1 \leq i \leq n\right\}$ is a vector space over F where addition and scalar multiplication are as defined below:

For $\mathrm{x}=\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{\mathrm{n}}\right), \mathrm{y}=\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \ldots, \mathrm{y}_{\mathrm{n}}\right) \square \mathrm{F}^{\mathrm{n}}$ and $\alpha \square \mathrm{F}$, $x+y=\left(x_{1}+y_{1}, x_{2}+y_{2}, \ldots, x_{n}+y_{n}\right)$.
$\alpha \mathrm{x}=\left(\alpha \mathrm{x}_{1}, \alpha \mathrm{x}_{2}, \ldots, \alpha \mathrm{x}_{\mathrm{n}}\right)$.
$\mathrm{F}^{\mathrm{n}}$ is also called the n -tuple space.
(3) The Space of $\mathbf{m} \times \mathbf{n}$ Matrices: Here $\mathrm{F}^{\mathrm{m} \times \mathrm{n}}$ is the set of all $\mathrm{m} \times \mathrm{n}$ matrices over F. $F^{m \times n}$ is a vector space over $F$ with respect to matrix addition and matrix scalar multiplication.
(4) The space of polynomials over F: Let $\mathrm{P}(\mathrm{F})$ be the set of all polynomials over F, i.e.,

$$
\mathrm{P}(\mathrm{~F})=\left\{\mathrm{a}_{0}+\mathrm{a}_{1} \mathrm{x}+\ldots+\mathrm{a}_{\mathrm{n}} \mathrm{x}^{\mathrm{n}}: \mathrm{a}_{\mathrm{i}} \square \mathrm{~F}, 1 \leq \mathrm{i} \leq \mathrm{n}, \mathrm{n} \geq 0 \text { is an integer }\right\} .
$$

$P(F)$ is a vector space over $F$ with respect to addition and scalar multiplication of polynomials, that is,

$$
\begin{aligned}
& \left(a_{0}+a_{1} x+\ldots+a_{n} x^{n}\right)+\left(b_{0}+b_{1} x+\ldots+b_{m} x^{m}\right) \\
& =c_{0}+c_{1} x+c_{2} x^{2}+\ldots+c_{k} x^{k}
\end{aligned}
$$

where

$$
c_{i}=a_{i}+b_{i}, k=\max \{m, n\}, a_{i}=b_{j}=0
$$

for

$$
\begin{aligned}
& i>n \text { and } j>m \text {. And } \\
& \alpha\left(a_{0}+a_{1} x+\ldots .+a_{n} x^{n}\right)=\alpha a_{0}+\alpha a_{1} x+\ldots i+\alpha a_{n} x^{n} .
\end{aligned}
$$

The following results can be verified easily (proof of which can be taken as exercise).

Theorem 4.2.1: If $V$ is a vector space over $F$ then
(a) $\alpha .0=0$, for $\alpha \square \mathrm{F}$, here 0 is the additive identity of V or the zero vector.
(b) $0 . \mathrm{u}=0$, for $\mathrm{u} \square \mathrm{V}$, here 0 in the left hand side is the scalar zero i.e. additive identity of F and 0 in right hand side is the zero vector in V .
(c) $(-\alpha) . \mathrm{u}=-(\alpha . \mathrm{u})$, for all $\alpha \square \mathrm{F}, \mathrm{u} \square \mathrm{V}$.
(d) If $u \neq 0$ in $V$ then $\alpha \cdot u=0$ implies $\alpha=0$.

### 4.3 Subspaces

For every algebraic structure we have the concept of sub-structures. Here we discuss about subspaces of vector spaces.

Let V be a vector space over F. A subset W of V is called a subspace of V if W is closed under ' + ' and ' . ' (which are the addition and scalar multiplication of V). In other words (i) for $\mathrm{u}, \mathrm{v} \square \mathrm{W}, \mathrm{u}+\mathrm{v} \square \mathrm{W}$ and (ii) for $\mathrm{u} \square \mathrm{W}$ and $\alpha \square \mathrm{F}, \alpha \mathrm{u} \square \mathrm{W}$.

The above two conditions of a subspace can be combined and expressed in a single statement that: W is a subspace of V if and only if for $\mathrm{u}, \mathrm{v} \square \mathrm{W}$ and scalars $\alpha, \beta$ $F, \alpha u+\beta v \square W$.

Example 4.3.1: Here we give some example of subspaces.
(1) The zero vector of the vector space $V$ alone i.e. $\{0\}$ and the vector space $V$ itself are subspaces of V . These subspaces are called trivial subspaces of V .
(2) Let $V=\mathbb{R}^{2}$, the Euclidean plane, and $W$ be the straight line in $\mathbb{R}^{2}$ passing through $(0,0)$ and (a, b), i.e. $W=\left\{(x, y) \square \mathbb{R}^{2}: a x+b y=0\right\}$. Then $W$ is a subspace of $\mathbb{R}^{2}$. Whereas the straight lines which do not pass through the origin are not subspaces of $\mathbb{R}^{2}$.
(3) The set of all $n \times n$ symmetric matrices over $F$ forms a subspace of $F^{n \times n}$ ( $F$ is a field).
(4) The set of all $n \times n$ Hermitian matrices is not a subspace of $\square^{n \times n}$ (the collection of all $\mathrm{n} \times \mathrm{n}$ complex matrices), because if A is a Hermitian matrix then diagonal entries of A are real and so iA is not a Hermitian matrix (However the set of all $\mathrm{n} \times \mathrm{n}$ Hermitian matrices forms a vector space over $\mathbb{R}$ ).

### 4.4 Linear Span

Let V be a vector space over F and S be a subset of V . The liner span of S , denoted by $\mathcal{L}(\mathrm{S})$, is the collection of all possible finite linear combinations of elements in S . Then $\mathcal{L}(\mathrm{S})$ satisfies the following properties given in the theorem below.

Theorem 4.4.1: For any subset $S$ of a vector space $V$
(1) $\mathcal{L}(S)$ is a subspace of $V$.
(2) $\mathcal{L}(\mathrm{S})$ is the smallest subspace of V containing S , i.e. if W is any subspace of V containing S then $\mathcal{L}(\mathrm{S})$ contained in W .

Example 4.4.1: In $\mathbb{R}^{2}$ if $S=\{(2,3)\}$ then $\mathcal{L}(S)$ is the straight line passing through $(0,0)$ and $(2,3)$ i.e. $\mathcal{L}(S)=2 x+3 y=0$. If $S=\{(1,0),(0,1)\}$ then $\mathcal{L}(S)=\mathbb{R}^{2}$.

### 4.5 Linearly Dependency/Independency

A vector space can be expressed in terms of very few elements of it, provided that, these elements spans the space and satisfy a condition called linearly independency. Short-cut representation of a vector space is essential in many subjects like Information and Coding Theory.

Consider a vector space V over a field F and a set $\mathrm{S}=\left\{\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{k}}\right\}$ of vectors in V . S is said to be linearly dependent if there exist scalars $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{\mathrm{k}}$ (in F), not all zero such that

$$
\alpha_{1} v_{1}+\alpha_{2} v_{2}+\ldots+\alpha_{k} v_{k}=0 .
$$

If S is not linearly dependent then it is called linearly independent. In other words S is linearly independent, if whenever $\alpha_{1} \mathrm{v}_{1}+\alpha_{2} \mathrm{v}_{2}+\ldots+\alpha_{\mathrm{n}} \mathrm{v}_{\mathrm{n}}=0$, all scalars $\alpha_{\mathrm{i}}$ have to be zero. This suggests a method to verify linearly dependency or independency of a given set of finite number of vectors, as given in the next subsection.

### 4.5.1 Verification of Linearly Dependency/Independency

Suppose the given set of vectors is $\mathrm{S}=\left\{\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{k}}\right\}$.
Step 1: Equate the linear combination of these vectors to the zero vector, that is, $\alpha_{1} \mathrm{v}_{1}+\alpha_{2} \mathrm{v}_{2}+\ldots+\alpha_{\mathrm{k}} \mathrm{v}_{\mathrm{k}}=0$, where $\alpha_{\mathrm{i}}$ 's are scalars that we have to find.

Step 2: Solve for scalars $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}$. If all are equal to zero then S is a linearly independent set, otherwise (i.e. at least one $\alpha_{i}$ is non-zero) the $S$ is linearly dependent.

Properties 4.5.1: Some properties of linearly dependent/independent vectors are as given below.
(1) A superset of a linearly dependent set is linearly dependent.
(2) A subset of a linearly independent set is linearly independent.
(3) Any set which contains the zero vector is linearly dependent.

Example 4.5.1: Let $\mathrm{V}=\mathbb{R}^{3}$ be the vector space (over $\mathbb{R}$ ) and $\mathrm{S}_{1}=\{(1,2,3),(1,0$, $2),(2,1,5)\}$ and $\mathrm{S}_{2}=\{(2,0,6),(1,2,-4),(3,2,2)\}$ be subsets of V. We check linearly dependency/independency of $\mathrm{S}_{1}$ and $\mathrm{S}_{2}$.

First consider the set $S_{1}$. Let $\alpha_{1}, \alpha_{2}, \alpha_{3}$ be scalars such that

$$
\alpha_{1}(1,2,3)+\alpha_{2}(1,0,2)+\alpha_{3}(2,1,5)=(0,0,0)
$$

Then we have

$$
\left(\alpha_{1}+\alpha_{2}+2 \alpha_{3}, 2 \alpha_{1}+\alpha_{3}, 3 \alpha_{1}+2 \alpha_{2}+5 \alpha_{3}\right)=(0,0,0)
$$

And is equivalent to the system

$$
\begin{aligned}
& \alpha_{1}+\alpha_{2}+2 \alpha_{3}=0 \\
& 2 \alpha_{1}+\alpha_{3}=0 \\
& 3 \alpha_{1}+2 \alpha_{2}+5 \alpha_{3}=0
\end{aligned}
$$

On solving this system we get $\alpha_{1}=\alpha_{2},=\alpha_{3}=0$, so $\mathrm{S}_{1}$ is linearly independent.
Next for $S_{2}$, we can take $\alpha_{1}=\alpha_{2}=1$ and $\alpha_{3}=-1$ and get

$$
\alpha_{1}(2,0,6)+\alpha_{2}(1,2,-4)+\alpha_{3}(3,2,2)=0 .
$$

So $S_{2}$ is a linearly dependent set.

We can also test linearly dependency/independency of vectors in $\mathrm{F}^{\mathrm{n}}$ (in particular in $\mathbb{R}^{\mathrm{n}}$ ) using echelon form of a matrix. This method has been explained in the example below.

Example 4.5.2: Let $\mathrm{V}=\mathbb{R}^{4}$ and $\mathrm{S}=\{(1,2,1,-2),(2,1,3,-1),(2,0,1,4)\}$ and $\mathrm{S}^{1}$ $=\{(0,1,2,-1),(1,2,0,3),(1,3,2,2),(0,1,1,1)\}$ be subsets of V. We will check linearly dependency/independency of $S$ and $S^{1}$.

We consider S first. We write the vectors in S as a matrix taking the vectors as rows and then apply elementary row operations and convert it to echelon form. If there is a zero row in the echelon form then the set is linearly dependent otherwise linearly independent.

$$
\begin{aligned}
& \left(\begin{array}{cccc}
1 & 2 & 1 & -2 \\
2 & 1 & 3 & -1 \\
2 & 0 & 1 & 4
\end{array}\right) \xrightarrow{R_{2} \rightarrow R_{2}-2 R_{1}}\left(\begin{array}{cccc}
1 & 2 & 1 & -2 \\
0 & -3 & 1 & 3 \\
2 & 0 & 1 & 4
\end{array}\right) \\
& \xrightarrow{R_{3} \rightarrow R_{3}-2 R_{1}}\left(\begin{array}{cccc}
1 & 2 & 1 & -2 \\
0 & -3 & 1 & 3 \\
0 & -2 & -1 & 8
\end{array}\right) \xrightarrow{R_{3} \rightarrow 3 R_{3}-2 R_{1}}\left(\begin{array}{cccc}
1 & 2 & 1 & -2 \\
0 & -3 & 1 & 3 \\
0 & 0 & -5 & 18
\end{array}\right)
\end{aligned}
$$

The last matrix is in echelon form and all the rows are non-zero. Hence $S$ is linearly independent.

Next we consider

$$
S^{1}=\{(0,1,2,-1),(1,2,0,3),(1,3,2,2),(0,1,1,1)\} .
$$

While forming the matrix we may not have to take $1^{\text {st }}$ vector in $\mathrm{S}^{1}$ as $1^{\text {st }}$ row, $2^{\text {nd }}$ vector as $2^{\text {nd }}$ row and so on. Since we have to convert the matrix into echelon form we may take $1^{\text {st }}$ row of the matrix a vector in S for which the $1^{\text {st }}$ entry is non-zero. So let the matrix be

$$
\left(\begin{array}{cccc}
1 & 2 & 0 & 3 \\
0 & 1 & 1 & 1 \\
0 & 1 & 2 & -1 \\
1 & 3 & 2 & 2
\end{array}\right)
$$

We convert this to echelon form by applying elementary row operations and is given by

$$
\left(\begin{array}{cccc}
1 & 2 & 0 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

There is a zero row in the echelon form so $S^{1}$ is linearly dependent.

### 4.6 Conclusions

Vector spaces are the main ingredients of the subject linear algebra. Here we have studied an important property of the vectors that is linearly dependency/independency. This property will be used in almost all the lectures. In the next lecture also we discuss about some basic terminologies associated with a vector space.

Keywords: Vectors, scalars, vector spaces, subspaces, linearly dependent or independent vectors.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

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## Lesson 5

## Basis and Dimension of Vector Spaces

### 5.1 Introduction

In the previous lecture we have already said that vector spaces can be represented in a short-cut form in terms of few linearly independent vectors. The set of these few vectors have a name called basis. The number of elements in a basis is fixed and this number is called the dimension of the vector space. In this lecture we shall discuss on these two important terms basis and dimension of a vector space. We shall also give an another definition of the rank of a matrix in terms of linearly independent rows/columns and finally present the rank-nullity theorem.

### 5.2 Basis and Dimension

Let V be a vector space over F. A subset S of V is called a basis for V if the following hold
(i) S is a linearly independent set
(ii) S spans V i.e., $\mathcal{L}(\mathrm{S})=\mathrm{V}$ (or in other words every element of V can be written as a finite linear combination of vectors in S).

If V contains a finite basis $\mathcal{B}$ then V is called a finite dimensional vector space and dimension of V is the number of elements in $\mathcal{B}$. If V is not finite dimensional then it is infinite dimensional vector space. Dimension of a vector space is well defined because of the theorem below.

Theorem 5.2.1: If a vector space V has a basis with k number of vectors then every basis of V contains k vectors (in other words all bases of a vector space are of the same cardinality).

Next we shall see some examples of vector spaces with their bases and dimensions.

## Example 5.2.1:

(1) $\{(2,0,6),(1,2,-4),(3,2,2)\}$ is not a basis for $\mathbb{R}^{3}$ as it is not linearly independent because $(2,0,6)+(1,2,-4)=(3,2,2)$.
(2) $S=\{(2,0,0),(3,4,0)\}$ is also not a basis for $\mathbb{R}^{3}$ as it does not span $R^{3}$ because ( $0,0, \alpha$ ), $\alpha \neq 0$, cannot be written as linear combination of vectors in $S$.
(3) The set $\{(1,0,0, \ldots, 0),(0,1,0,0, \ldots, 0), \ldots,(0,0, \ldots, 1)\}$ of vectors in $\mathbb{R}^{n}$ forms a basis for $\mathbb{R}^{\mathrm{n}}$. This basis is called standard basis of $\mathbb{R}^{\mathrm{n}}$. So dimension of $\mathbb{R}^{\mathrm{n}}$ is n.
(4) The collection of all polynomials over $\mathrm{F}, \mathrm{P}(\mathrm{F})$ is an infinite dimensional vector space over F because $\mathrm{S}=\left\{1, \mathrm{x}, \mathrm{x}^{2}, \mathrm{x}^{3}, \ldots \ldots\right\}$ is a linearly independent set and spans $P(F)$ but no finite subset of $S$ spans $P(F)$. However $P_{n}(F)$, the set of all polynomials of degree $\leq \mathrm{n}$, is a finite dimensional vector space with $\left\{1, \mathrm{x}, \mathrm{x}^{2}, \mathrm{x}^{3}, \ldots, \mathrm{x}^{\mathrm{n}}\right\}$ as a basis. Hence dimension of $\mathrm{P}(\mathrm{F})$ is equal to $\mathrm{n}+1$.
(5) The set $\mathbb{R}^{2 \times 2}$ of all $2 \times 2$ real matrices is a finite dimensional vector space over $\mathbb{R}$ with $\left\{\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right),\left(\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right),\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right)\right\}$ as a basis. So $\operatorname{dim} \mathbb{R}^{2 \times 2}=4$.

Next we shall list some of the well known properties of an n-dimensional vector space.

Theorem 5.2.2: The following results are true in an n-dimensional vector space V :
(i) Every basis of V contains n number of vectors.
(ii) A set of $\mathrm{n}+1$ or more vectors in V is a linearly dependent set.
(iii) If S is a set of n vectors in V and $\mathcal{L}(\mathrm{S})=\mathrm{V}$ then S is linearly independent.
(iv) If S is a set of n linearly independent vectors in V then $\mathcal{L}(\mathrm{S})=\mathrm{V}$. In other words S is a basis of V .
(v) If $S=\left\{\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{m}}\right\}$ is a set of m vectors in $\mathrm{V}, \mathrm{m} \leq \mathrm{n}$, then S can be extended to a basis of $V$ i.e. there exist vectors, $u_{m+1}, \ldots, u_{n}$, such that $S=\{$ $\left.\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{m}}, \mathrm{u}_{\mathrm{m}+1}, \ldots, \mathrm{u}_{\mathrm{n}}\right\}$ is a basis for V .
(vi) If $S=\left\{\mathrm{w}_{1}, \mathrm{w}_{2}, \ldots, \mathrm{w}_{\mathrm{k}}\right\}, \mathrm{k} \geq \mathrm{n}$, is a set of vectors in V such that $\mathcal{L}(\mathrm{S})=\mathrm{V}$, then S contains a basis for V .
(vii) If W is a subspace of V then $\operatorname{dim} \mathrm{W} \leq \operatorname{dim} \mathrm{V}$.

In the following example we shall use some of the results of Theorem 2.2 to check for a basis.

Example 5.2.3: Here we show that $S=\{(1,0,-1),(1,1,1),(1,2,4)\}$ is a basis for $\mathbb{R}^{3}$ in two different ways. Here we shall use the fact that dimension of $\mathbb{R}^{3}$ is 3.

Method 1: We will show that S is a linearly independent set. We get the echelon form of the matrix formed by the vectors in S . The matrix and its row reduced matrices are as follow:

$$
\begin{aligned}
& \left(\begin{array}{ccc}
1 & 0 & -1 \\
1 & 1 & 1 \\
1 & 2 & 4
\end{array}\right) \xrightarrow{\mathrm{R}_{2} \rightarrow \mathrm{R}_{2}-\mathrm{R}_{1}}\left(\begin{array}{ccc}
1 & 0 & -1 \\
0 & 1 & 2 \\
1 & 2 & 4
\end{array}\right) \\
& \xrightarrow{\mathrm{R}_{3} \rightarrow \mathrm{R}_{3}-\mathrm{R}_{1}}\left(\begin{array}{ccc}
1 & 0 & -1 \\
0 & 1 & 2 \\
0 & 2 & 5
\end{array}\right) \xrightarrow{\mathrm{R}_{3} \rightarrow \mathrm{R}_{3}-2 \mathrm{R}_{2}}\left(\begin{array}{ccc}
1 & 0 & -1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{array}\right)
\end{aligned}
$$

The last matrix is in echelon form and no zero row is there in it. So S is a linearly independent set of 3 vectors and since dimension of $\mathbb{R}^{3}$ is 3 , by Theorem 2.2(iv) S is a basis of $\mathbb{R}^{3}$.

Method 2: Next by applying Theorem 2.2(iii) we show that $S$ is a basis of $\mathbb{R}^{3}$.
Here we show that every vector in $\mathbb{R}^{3}$ can be expressed as a linear combination of vectors in $S$. Let $\left(x_{1}, x_{2}, x_{3}\right) \square \mathbb{R}^{3}$ be an arbitrary vector and $\alpha, \beta, \gamma \square \mathbb{R}$ such that

$$
\begin{aligned}
\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right) & =\alpha(1,0,-1)+\beta(1,1,1)+\gamma(1,2,4), \\
& =(\alpha+\beta+\gamma, \beta+2 \gamma,-\alpha+\beta+4 \gamma)
\end{aligned}
$$

So $\alpha+\beta+\gamma=x_{1}, \beta+2 \gamma=x_{2},-\alpha+\beta+4 \gamma=x_{3}$, and is a linear system with unknowns $\alpha, \beta, \gamma$. On solving we get

$$
\alpha=2 x_{1}-3 x_{2}+x_{3}, \beta=-2 x_{1}+5 x_{2}-2 x_{3}, \gamma=x_{1}-2 x_{2}+x_{3}
$$

Thus for every vector in $\mathbb{R}^{3}$ we have found scalars to express the vector as a linear combination of vectors in $S$. Hence $S$ forms a basis for $\mathbb{R}^{3}$.
[In particular if $\left(x_{1}, x_{2}, x_{3}\right)=(1,2,3)$ then $\alpha=-1, \beta=2, \gamma=0$ i.e.

$$
(1,2,3)=(-1)(1,0,-1)+2(1,1,1)+0(1,2,4) .]
$$

In the next example we shall find basis and dimension of a subspace generated by a set of vectors.

Example 5.2.4: We consider the subspace W of $\mathbb{R}^{5}$ generated by the vectors $\mathrm{u}=$ $(1,3,1,-2,-3), v=(1,4,3,-1,-4), w=(2,3,-4,-7,-3), x=(3,8,1,-7$, $-8)$.

Here we find a basis and the dimension of W .

The dimension of W will be the maximum number of linearly independent vectors in $\{u, v, w, x\}$. To determine this we take help of echelon form of the matrix whose rows are the vectors $\mathrm{u}, \mathrm{v}, \mathrm{w}$, and x . The matrix is

$$
\left(\begin{array}{ccccc}
1 & 3 & 1 & -2 & -3 \\
1 & 4 & 3 & -1 & -4 \\
2 & 3 & -4 & -7 & -3 \\
3 & 8 & 1 & -7 & -8
\end{array}\right)
$$

Replacing $R_{2}$ by $R_{2}-R_{1}, R_{3}$ by $R_{3}-2 R_{1}$ and $R_{4}$ by $R_{4}-3 R_{1}$ the matrix will be

$$
\left(\begin{array}{ccccc}
1 & 3 & 1 & -2 & -3 \\
0 & 1 & 2 & 1 & -1 \\
0 & -3 & -6 & -3 & 3 \\
0 & -1 & -2 & -1 & 1
\end{array}\right)
$$

Replacing $R_{3}$ by $R_{3}+3 R_{2}$ and $R_{4}$ by $R_{4}+R_{2}$ in the above matrix we get

$$
\left(\begin{array}{ccccc}
1 & 3 & 1 & -2 & -3 \\
0 & 1 & 2 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

which is in echelon form.

In the echelon form there are two non-zero rows only. Therefore dimension of W is equal to two and these non-zero rows form a basis for W. So $\{(1,3,1,-2,-3)$, $(0,1,2,1,-1)\}$ is a basis for W .

### 5.3 The Rank-Nullity Theorem

Here we give a definition of the rank of a matrix in terms of linearly independent rows or columns. The rank of a matrix A is defined as the maximum number of linearly independent rows in A. This is same as the dimension of the subspace spanned by the rows of A . This subspace is also called the row space of A . similarly one defines the column space of A . It is known that the dimension of the row space of $A$ is same as the dimension of the column space of $A$. Therefore the rank of a matrix is also equal to the dimension of its column space. From this one can also conclude that a matrix and its transpose have the same rank.

For any matrix A its nullity may be defined as below. Recall that a homogeneous system of $m$ linear equations on $n$ variables is of the form $A X=0$, where $A$ is a $m$ $\times \mathrm{n}$ matrix and X is the $\mathrm{n} \times 1$ matrix ( $\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{\mathrm{n}}$ ). Homogeneous systems are always consistent because $(0,0, \ldots, 0)$ is always a solution of it. Also this is true because of the fact that the co-efficient and augment matrices of this system have the same rank.

Let $S$ be the collection of all solutions of $A X=0$. One can easily check that $S$ is a subspace of $\mathbb{R}^{\mathrm{n}}$ and this subspace is called the solution space of the system. The dimension of the solution space of the system $\mathrm{AX}=0$ is called the nullity of A . Now we are ready to state the famous rank-nullity theorem for matrices.

Theorem 5.3.1: Let A be an $m \times n$ matrix. Then rank $A+n u l l i t y$ of $A=n$.

We illustrate the above theorem through some examples below.

Example 5.3.1: We verify the rank-nullity theorem for the matrix A $=\left(\begin{array}{ccc}1 & 2 & -1 \\ 2 & 5 & 2 \\ 1 & 4 & 7 \\ 1 & 3 & 3\end{array}\right)$.

We convert A into row echelon form and is given by $\left(\begin{array}{ccc}1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)$.

From this we get that rank of A is equal to 2 since there are two non-zero rows in the row echelon form of A.

The homogeneous system corresponding to $A$ is $A X=0$, where $X$ is the $3 \times 1$ matrix say $\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right)^{\mathrm{T}}$. So the system is

$$
\begin{aligned}
& x_{1}+2 x_{2}-x_{3}=0 \\
& 2 x_{1}+5 x_{2}+2 x_{3}=0 \\
& x_{1}+4 x_{2}+7 x_{3}=0 \\
& x_{1}+3 x_{2}+3 x_{3}=0
\end{aligned}
$$

From the echelon form of the matrix A, the above system is equivalent to

$$
\begin{aligned}
& x_{1}+2 x_{2}-x_{3}=0 \\
& x_{2}+4 x_{3}=0
\end{aligned}
$$

Here $x_{3}$ is the free variable. Let $x_{3}=\alpha, \alpha \square \mathbb{R}$. Then $x_{2}=-4 \alpha$ and $x_{1}=9 \alpha$. So the solution space of the system $\mathrm{AX}=0$ is $\mathrm{S}=\{(9 \alpha,-4 \alpha, \alpha)-: \alpha \square \mathbb{R}\}$.

A basis for $S$ is $\{(9,-4,1)\}$ because this vector generates $S$, that is, all other vectors in $S$ are scalar multiple of the vector $(9,-4,1)$. Therefore nullity of $A=\operatorname{dim} S=1$. Now rank of $A+$ nullity of $A=3$ which verifies the rank-nullity theorem.

### 5.4 Conclusions

In this lecture we have learned that if we know a basis for a vector space then the whole vector space can be generated by taking all possible finite linear combinations of the basis vectors. Because of this wonderful structure, vector spaces are widely used in coding and decoding of messages in Information and Coding theory. We shall find application of the rank-nullity theorem in some of the subsequent lectures.

Keywords: Finite dimensional vector spaces, basis, dimension, homogeneous system of equations, rank-nullity theorem.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

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## Lesson 6

## Eigenvalues and Eigenvectors of Matrices

### 6.1 Introduction

The concept of eigenvalues and eigenvectors of matrices is very basic and having wide application in science and engineering. Eigenvalues are useful in studying differential equations and continuous dynamical systems. They provide critical information in engineering design and also naturally arise in fields such as physics and chemistry.

### 6.2 Eigenvalues and Eigenvectors

Let A be square matrix of size $n$ over a real or complex field $F$. An element $\lambda$ in $F$ is called an eigenvalue of A if there exists a non-zero vector x in $\mathrm{F}^{\mathrm{n}}$ (or a $\mathrm{n} \times 1$ matrix) such that $A x=\lambda x$.

If $\lambda$ is an eigenvalue of $A$ then all the non-zero vectors $x$ satisfying $A x=\lambda x$ are called eigenvectors corresponding to $\lambda$. For a single eigenvalue there may be several eigenvectors associated with it. In fact all these eigenvectors form a subspace as we shall see below.

Theorem 6.2.1: Let $A$ be an $n \times n$ matrix, $\lambda$ be an eigenvalue of $A$, and $S$ be the set of all eigenvectors corresponding to $\lambda$. Then $\operatorname{SU}\{0\}$ is a subspace of $\mathrm{F}^{\mathrm{n}}$.

Proof: Let $x_{1}, x_{2}$ be eigenvectors corresponding to $\lambda$. Then
$A\left(x_{1}+x_{2}\right)=A x_{1}+A x_{2}=\lambda x_{1}+\lambda x_{2}=\lambda\left(x_{1}+x_{2}\right)$.
$\mathrm{A}\left(\alpha \mathrm{x}_{1}\right)=\alpha \mathrm{Ax} \mathrm{x}_{1}=\alpha \lambda \mathrm{x}_{1}=\lambda\left(\alpha \mathrm{x}_{1}\right)$. So $\mathrm{x}_{1}+\mathrm{x}_{2}, \alpha \mathrm{x}_{1} \square \mathrm{~S}$ and hence the result.

If $S$ is the set of all eigenvectors corresponding to an eigenvalue $\lambda$ then the subspace $\operatorname{SU}\{0\}$ is called the eigenspace corresponding to the eigenvalue $\lambda$.
Example 6.2.1: For the matrix $A=\left(\begin{array}{ll}5 & 4 \\ 1 & 2\end{array}\right)$ over the real field $\mathbb{R}, 6$ is an eigenvalue because for the vector $\mathrm{x}=\binom{4}{1}$ in $\mathbb{R}^{2}$
$A x=\left(\begin{array}{ll}5 & 4 \\ 1 & 2\end{array}\right)\binom{4}{1}=\binom{24}{6}=6\binom{4}{1}=6 x$.

Similarly $y=\binom{2}{\frac{1}{2}}$ is also an eigenvector of A corresponding to the eigenvalue 6.

Next we shall find all eigenvalues and associated eigenvectors of a matrix systematically.

### 6.2.1 Method to find Eigenvalues and Eigenvectors

If $\lambda$ is an eigenvalue of A and x is a corresponding eigenvector then

$$
\begin{equation*}
\mathrm{Ax}=\lambda \mathrm{x} \text { or }(\mathrm{A}-\lambda \mathrm{I}) \mathrm{x}=0 \tag{6.1}
\end{equation*}
$$

where I is the $\mathrm{n} \times \mathrm{n}$ identity matrix. Note that (6.1) is a homogeneous system of linear equations. If (6.1) has a non-zero solution then rank $(A-\lambda I)<n$. Then $A-$
$\lambda \mathrm{I}$ is not invertible and one gets that $\operatorname{det}(\mathrm{A}-\lambda \mathrm{I})=0$. Therefore if $\lambda$ is an eigenvalue of A then it satisfies the equation $\operatorname{det}(\mathrm{A}-\lambda \mathrm{I})=0$ (because it will have a non-zero eigenvector). Since $\operatorname{det}(A-\lambda I)$ is a polynomial in $\lambda$ of degree $n$, we obtain all values of $\lambda$ by solving $\operatorname{det}(A-\lambda I)=0$, and this equation will have $n$ solutions with counting multiplicities. We summarize the above discussions as follows:

1. Eigenvalues of $A$ are the solutions of $\operatorname{det}(A-\lambda I)=0$.
2. If $A$ is of size $n$ then $A$ has $n$ number of eigenvalues with counting multiplicities.
3. If $\lambda$ is an eigenvalue of $A$ then all non-zero solutions of the system $(A-\lambda I) x=$ 0 are the eigenvectors of A corresponding to $\lambda$, here $\mathrm{x}=\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{\mathrm{n}}\right)^{\mathrm{T}}$.

Eigenvalues of matrices are sometimes called characteristic values. The equation $\operatorname{det}(\mathrm{A}-\lambda \mathrm{I})=0$ is called the characteristic equation and $\operatorname{det}(\mathrm{A}-\lambda \mathrm{I})$ is called the characteristic polynomial associated with A.

We explain this method of finding eigenvalues and eigenvectors of a matrix through an example below.

Example 6.2.2: Find all eigenvalues and their corresponding eigenvectors of the matrix

$$
A=\left(\begin{array}{lll}
5 & 4 & 2 \\
4 & 5 & 2 \\
2 & 2 & 2
\end{array}\right)
$$

Solution: The characteristic polynomial of A is

$$
\begin{aligned}
\operatorname{det}(\mathrm{A}-\lambda \mathrm{I}) & =\left|\begin{array}{ccc}
5-\lambda & 4 & 2 \\
4 & 5-\lambda & 2 \\
2 & 2 & 2-\lambda
\end{array}\right| \\
& =-(\lambda-10)(\lambda-1)^{2}
\end{aligned}
$$

So the characteristic equation is $(\lambda-10)(\lambda-1)^{2}=0$ and the eigenvalues $\lambda$ are $\lambda=10,1,1$.

Eigenvectors Corresponding to $\boldsymbol{\lambda}=\mathbf{1 0}$ : Here we solve the system $(\mathrm{A}-10 \mathrm{I}) \mathrm{x}=0$.
or $\left(\begin{array}{ccc}-5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8\end{array}\right)\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)=0$
where $\mathrm{x}=\left(\begin{array}{l}\mathrm{x}_{1} \\ \mathrm{x}_{2} \\ \mathrm{x}_{3}\end{array}\right)$.

Echelon form of the co-efficient matrix is $\left(\begin{array}{ccc}-5 & U_{4} & -2 \\ 0 & -9 & 18 \\ 0 & 0 & 0\end{array}\right)$.

So the given system of equations will be

$$
\begin{array}{r}
-5 x_{1}+4 x_{2}+2 x_{3}=0 . \\
-9 x_{2}+18 x_{3}=0 .
\end{array}
$$

Here $\mathrm{x}_{3}$ is free variable. So let $\mathrm{x}_{3}=\alpha, \alpha \neq 0, \alpha \square \mathbb{R}$. Then we get $\mathrm{x}_{2}=2 \alpha$ and $\mathrm{x}_{1}=$ $\mathrm{x}_{2}=2 \alpha$. So the set of all eigenvectors corresponding to $\lambda=10$ is $\{(2 \alpha, 2 \alpha, \alpha): \alpha \square$ $\mathbb{R}, \alpha \neq 0\}$.

Eigenvectors corresponding to $\boldsymbol{\lambda}=\mathbf{1}$ : Here we have to solve the system $(\mathrm{A}-\mathrm{I}) \mathrm{x}=0$.

$$
\text { or }\left(\begin{array}{lll}
4 & 4 & 2 \\
4 & 4 & 2 \\
2 & 2 & 1
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right)=0 \text {. }
$$

Echelon form of the co-efficient matrix is $\left(\begin{array}{lll}4 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)$. So the system will be

$$
\begin{aligned}
4 x_{1}+4 x_{2}+2 x_{3} & =0 . \\
\text { or } \quad 2 x_{1}+2 x_{2}+2 x_{3} & =0 .
\end{aligned}
$$

Here $x_{2}$ and $x_{3}$ are both free variables. So let $x_{2}=\alpha, x_{3}=\beta, \alpha, \beta \square \mathbb{R}$, and $\alpha=0$, $\beta=0$ cannot hold simultaneously. Then $x_{1}=-\frac{1}{2}(2 \alpha+\beta)$. The set of all eigenvectors corresponding to $\lambda=1$ is $\left\{\left(-\frac{1}{2}(2 \alpha+\beta), \alpha, \beta\right): \alpha, \beta \square \mathbb{R}, \alpha\right.$ and $\beta$ do not take the zero value simultaneously $\}$

### 6.2.2 Properties of Eigenvalues and Eigenvectors

In the following we present some properties of eigenvalues and eigenvectors of matrices:
(1) The sum of the eigenvalues of a matrix $A$ is equal to the sum of all diagonal entries of A (called trace of A). This property provides a procedure for checking eigenvalues.
(2) A matrix is invertible if and only it has non-zero eigenvalues.

This can be verified easily as $\operatorname{det} \mathrm{A}=(\mathrm{A}-0 \mathrm{I})=0$ if and only if 0 is an eigenvalues of A . Also recall that $\operatorname{det} \mathrm{A}=0$ if and only if A is not invertible.
(3) The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.

This is true because determinant of an upper (or lower) triangular matrix is equal to the product of the (main) diagonal entries.
(4) If $\lambda$ is an eigenvalue of $A$ and if $A$ is invertible then $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$. Further if x is an eigenvector of A corresponding to $\lambda$ then it is also an eigenvector of $\mathrm{A}^{-1}$ corresponding to $\frac{1}{\lambda}$.

The above is true because if $x$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$ then $A x=\lambda x$. Multiplying both sides by $A^{-1}, x=\lambda A^{-1} \mathrm{x} \quad$ or $A^{-1} x=\frac{1}{\lambda} x$.
(5) If $\lambda$ is an eigenvalue of $A$ then $\alpha \lambda$ is an eigenvalue of $\alpha A$ where $\alpha$ is any real or complex number. Further if x is an eigenvector of A corresponding to the eigenvalue $\lambda$ then $x$ is also an eigenvector of $\alpha A$ corresponding to eigenvalue $\alpha \lambda$. This is true because $(\alpha A) x=(\alpha \lambda) x$.
(6) If $\lambda$ is an eigenvalue of $A$ then $\lambda^{k}$ is an eigenvalue of $A^{k}$ for any positive integer k . Further if x is an eigenvector of A corresponding to the eigenvalue $\lambda$
then x is also an eigenvector of $\mathrm{A}^{\mathrm{k}}$ corresponding to the eigenvalue $\lambda^{k}$. This is true because if x is an eigenvector of A corresponding the eigenvalue $\lambda$ then $A^{k} x=A^{k-1}(A x)=A^{k-1}(\lambda x)=\lambda\left(A^{k-1} x\right)=\lambda^{2}\left(A^{k-2} x\right)=\ldots=\lambda^{k} x$.
(7) If $\lambda$ is an eigenvalue of $A$, then for any real or complex number $\mathrm{c}, \lambda-\mathrm{c}$ is an eigenvalue of $\mathrm{A}-\mathrm{cI}$. Further if x is an eigenvector of A corresponding to the eigenvalue $\lambda$ then x is also an eigenvector of $\mathrm{A}-\mathrm{cI}$ corresponding to the eigenvalue $\lambda$ - $c$

This is true because $(\mathrm{A}-\mathrm{cI}) \mathrm{x}=\mathrm{Ax}-\mathrm{cx}=\lambda \mathrm{x}-\mathrm{cx}=(\lambda-\mathrm{c}) \mathrm{x}$ for an eigenvalue $\lambda$ and its corresponding eigenvector x of A .
(8) Every eigenvalue of $A$ is also an eigenvalue of $A^{T}$. One verifies this from the fact that determinant of a matrix is same as the determinant of this transpose and
$A-\lambda I\left|=\left|\left(A^{T}\right)^{T}-\lambda I^{T}\right|=\left|\left(A^{T}-\lambda I\right)^{T}\right|=\left|A^{T}-I \lambda\right|\right.$.
(9) The product of all the eigenvalues (with counting multiplicity) of a matrix equals the determinant of the matrix.
(10) Eigenvectors corresponding to distinct eigenvalues are linearly independent.

### 6.3 Conclusions

Some more properties of eigenvalues and eigenvectors will be discussed in the next lecture. In a subsequent lecture we shall show that eigenvalues and eigenvectors are used for diagonalization of matrices.

Keywords: Characteristic equation, eigenvalues, eigenvectors, properties of eigenvalues and eigenvectors.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

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## Lesson 7

## The Cayley Hamilton Theorem and Applications

### 7.1 Introduction

The Cayley Hamilton theorem is one of the most powerful results in linear algebra. This theorem basically gives a relation between a square matrix and its characteristic polynomial. One important application of this theorem is to find inverse and higher powers of matrices.

### 7.2 The Cayley Hamilton Theorem

The Cayley Hamilton theorem states that:

Theorem 7.2.1: Every square matrix satisfies its own characteristic equation.

That is if A is a matrix of size $n$ and $\chi_{A}(\lambda)=a_{0}+a_{1} \lambda+\ldots+a_{n-1} \lambda^{n-1}+\lambda^{n}=0$ is the characteristic equation of A then

$$
\chi_{A}(A)=a_{0} I+a_{1} A+\ldots+a_{n-1} A^{n-1}+A^{n}=0_{n \times n}
$$

where $0_{n \times n}$ is the zero matrix of size $n$, and for any positive integer $i, A^{i}$ is the product A $\times$ A $\ldots \times$ A of i number of A.

Example7.2.1: Let $\mathrm{A}=\left(\begin{array}{ll}1 & 2 \\ 4 & 3\end{array}\right)$. Characteristic equation is $\lambda^{2}-4 \lambda-5=0$. One can check that $\mathrm{A}^{2}=\left(\begin{array}{cc}9 & 8 \\ 16 & 17\end{array}\right), 4 \mathrm{~A}=\left(\begin{array}{cc}4 & 8 \\ 16 & 12\end{array}\right)$. So

$$
\begin{aligned}
A^{2}-4 A-5 I & =\left(\begin{array}{cc}
9 & 8 \\
16 & 17
\end{array}\right)-\left(\begin{array}{cc}
4 & 8 \\
16 & 12
\end{array}\right)-\left(\begin{array}{ll}
5 & 0 \\
0 & 5
\end{array}\right) . \\
& =\left(\begin{array}{cc}
9-4-5 & 8-8-0 \\
16-16-0 & 17-12-5
\end{array}\right)=\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right) .
\end{aligned}
$$

The Cayley-Hamilton theorem can be used to find inverse as well as higher powers of a matrix.

### 7.3 Method to Find Inverse

Here we consider a square matrix A of size n and its characteristic polynomial $\chi_{\mathrm{A}}$ $(\lambda)=\operatorname{det}(A-\lambda I)=a_{0}+a_{1} \lambda+\ldots+a_{n}-1 \lambda^{n-1}+\lambda^{n}$. The following is a well known result for matrices.

Theorem 7.3.1: If $\chi_{A}(\lambda)=\operatorname{det}(A-\lambda I)=a_{0}+a_{1} \lambda+\ldots+a_{n-1} \lambda^{n-1}+\lambda^{n}$ is the characteristic polynomial of a square matrix A then determinant of A is equal to ( 1) ${ }^{n} a_{0}$.

The following is an immediate consequence of the above theorem.

Corollary 7.3.1: $A$ is invertible if and only if $\mathrm{a}_{0} \neq 0$.

In light of the above results to find inverse of $A$ we should have $a_{0} \neq 0$. By the Cayley- Hamilton theorem we have

$$
\begin{array}{ll} 
& \mathrm{a}_{0} \mathrm{I}+\mathrm{a}_{1} \mathrm{~A}+\ldots+\mathrm{a}_{\mathrm{n}-1} \mathrm{~A}^{\mathrm{n}-1}+\mathrm{A}^{\mathrm{n}}=0 . \\
\text { or } \quad & \mathrm{A}\left(\mathrm{a}_{1} \mathrm{I}+\mathrm{a}_{2} \mathrm{~A}+\ldots+\mathrm{A}^{\mathrm{n}-1}\right)=-\mathrm{a}_{0} \mathrm{I} .
\end{array}
$$

or

$$
\mathrm{A}\left\{-\frac{1}{\mathrm{a}_{0}}\left(\mathrm{a}_{1} \mathrm{I}+\mathrm{a}_{2} \mathrm{~A}+\ldots+\mathrm{A}^{\mathrm{n}-1}\right)\right\}=\mathrm{I} .
$$

Therefore $A^{-1}=-\frac{1}{a_{0}}\left(a_{1} I+a_{2} A+\ldots+A^{n-1}\right)$ which is a formula for inverse of $A$.

We will illustrate this method in the example below.
Example 7.3.1: Here we find inverse of the matrix $A=\left(\begin{array}{ccc}2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1\end{array}\right)$ applying
Cayley- Hamilton theorem. One finds that the characteristic equation of A is $\operatorname{det}(\mathrm{A}-\lambda \mathrm{I})=-\lambda^{3}+\lambda^{2}+\lambda-1=0$.

The matrix A is invertible because $\mathrm{a}_{0}=-1 \neq 0$. By the Cayley-Hamilton theorem $-A^{3}+A^{2}+A-I=0$.
or $A\left(-A^{2}+A+I\right)=I$.
or $\quad \mathrm{A}^{-1}=-\mathrm{A}^{2}+\mathrm{A}+\mathrm{I}=-\left(\begin{array}{lll}1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1\end{array}\right)+\left(\begin{array}{ccc}2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1\end{array}\right)+\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$
$=\left(\begin{array}{ccc}2 & -1 & -1 \\ 3 & -2 & -1 \\ 0 & 0 & 1\end{array}\right)$.

### 7.4 Computation of powers of $A$

Applying Cayley-Hamilton theorem we can also find higher powers of a square matrix. For this we need a famous theorem of algebra called the division algorithm, which is stated below.

Theorem 7.4.1: (Division Algorithm) For any polynomials $f(x)$ and $g(x)$ over a field $F$ there exist polynomials $q(x)$ and $r(x)$ such that $f(x)=q(x) g(x)+r(x)$ where $r(x)=0$ or $\operatorname{deg} r(x)<\operatorname{deg} g(x)$.

The polynomial $\mathrm{r}(\mathrm{x})$ is called remainder polynomial.

Here we shall discuss about a method that finds value of higher degree polynomial on a square matrix A and in particular the value of higher power of A. The method as follows:

Step 1: Let A be a square matrix of size $n$ and $f(A)$ be a polynomial in $A$ of any finite degree $m$, usually $m>n$.

Step 2: Compute the characteristic polynomial $\chi(\mathrm{A})$ of A . From division algorithm we get $\mathrm{f}(\mathrm{A})=\mathrm{q}(\mathrm{A}) \chi(\mathrm{A})+\mathrm{r}(\mathrm{A})$, where $\mathrm{q}(\mathrm{A})$ and $\mathrm{r}(\mathrm{A})$ are polynomials in A and deg $\mathrm{r}(\mathrm{A})<\operatorname{deg} \chi(\mathrm{A})$ or $\mathrm{r}(\mathrm{A})=0$.

Step 3: From Cayley-Hamilton theorem we get $\chi(A)=0$. Therefore $f(A)=r(A)$, that is $f(A)$ is equal to a polynomial in $A$ of degree less than $n$. Then we compute $r(A)$ which involves at the most $n$ unknown constants and up to ( $n-1$ )th powers of A, that is, $\mathrm{r}(\mathrm{A})$ can be written as:

$$
r(A)=a_{0} I+a_{1} A+\ldots+a_{n-1} A^{n-1}
$$

To find $r(A)$ one has to compute the co-efficients $a_{0}, a_{1}, \ldots, a_{n-1}$ and powers of A. We use the eigenvalues of $A$ to find these co-efficients. This procedure is divided into two cases depending on the eigenvalues are distinct or not.

Step 4: In this case we assume that $A$ has distinct eigenvalues $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{\mathrm{n}}$. From Cayley-Hamilton theorem we have $f(A)=r(A)$. Therefore
$f\left(\lambda_{i}\right)=r\left(\lambda_{i}\right)$ for all $i=1,2, \ldots, n$, that is
$\mathrm{f}\left(\lambda_{1}\right)=\mathrm{r}\left(\lambda_{1}\right)=\mathrm{a}_{0}+\mathrm{a}_{1} \lambda_{1}+\mathrm{a}_{2} \lambda_{1}{ }^{2}+\ldots+\mathrm{a}_{\mathrm{n}-1} \lambda_{1}{ }^{\mathrm{n}-1}$.
$\mathrm{f}\left(\lambda_{2}\right)=\mathrm{r}\left(\lambda_{2}\right)=\mathrm{a}_{0}+\mathrm{a}_{1} \lambda_{2}+\mathrm{a}_{2} \lambda_{2}{ }^{2}+\ldots+\mathrm{a}_{\mathrm{n}-1} \lambda_{2}{ }^{\mathrm{n}-1}$
三
$f\left(\lambda_{n}\right)=r\left(\lambda_{n}\right)=a_{0}+a_{1} \lambda_{n}+a_{2} \lambda_{n}{ }^{2}+\ldots+a_{n-1} \lambda_{n}{ }^{n-1}$

Solving this system one finds the values $a_{0}, a_{1}, \ldots, a_{n-1}$, since $f\left(\lambda_{i}\right)$ and $\lambda_{i}, 1 \leq i \leq$ n, are known.

Step 5: In this step we consider the case that A has multiple eigenvalues. If $\lambda_{i}$ is an eigenvalue of A of multiplicity $k$ then we differentiate the equation $f\left(\lambda_{i}\right)=r\left(\lambda_{i}\right) k-$ 1 times, and get $k$ equations:
$f\left(\lambda_{i}\right)=r\left(\lambda_{i}\right)$.
$\left.\frac{d \mathrm{f}(\lambda)}{d \lambda}\right|_{\lambda=\lambda_{i}}=\left.\frac{d \mathrm{r}(\lambda)}{d \lambda}\right|_{\lambda=\lambda_{i}}$.
111
$\left.\frac{d^{(k-1)} \mathrm{f}(\lambda)}{d \lambda}\right|_{\lambda=\lambda_{i}}=\left.\frac{d^{(k-1)} \mathrm{r}(\lambda)}{d \lambda}\right|_{\lambda=\lambda_{i}}$.

This is how one gets a system of $n$ equations using all the eigenvalues of $A$ and from this system the values of $a_{0}, a_{1}, \ldots, a_{n}$ can be determined.

Example 7.4.1: Here we shall find the value of $f(A)=A^{78}$, for $A=\left[\begin{array}{cc}2 & -1 \\ 2 & 5\end{array}\right]$, applying Cayley-Hamilton theorem. Characteristic polynomial of $A$ is $\operatorname{det}(\mathrm{A}-\lambda \mathrm{I})=\lambda^{2}-7 \lambda+12$. Eigenvalues are 3 and 4. Since characteristic polynomial of A is of degree 2 the remainder will be of degree at the most one.

Therefore

$$
\begin{align*}
& \mathrm{A}^{78}=\mathrm{a}_{0} \mathrm{I}+\mathrm{a}_{1} \mathrm{~A}  \tag{7.1}\\
& 3^{78}=\mathrm{a}_{0} \mathrm{I}+3 \mathrm{a}_{1} \\
& 4^{78}=\mathrm{a}_{0} \mathrm{I}+4 \mathrm{a}_{1}
\end{align*}
$$

On solving we get $\mathrm{a}_{1}=-3^{78}+4^{78}$ and $\mathrm{a}_{0}=4 \times 3^{78}-3 \times 4^{78}$. Putting this value in (7.1),
$\mathbf{A}^{78}=\left(\begin{array}{cc}2 \times 3^{78}-4^{78} & 3^{78}-4^{78} \\ -2 \times 3^{78}+2 \times 4^{78} & -3^{78}+2 \times 4^{78}\end{array}\right)$.

Example 7.4.2: For the matrix $A=\left(\begin{array}{lll}1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2\end{array}\right)$, we find the value of $f(A)=A^{10}-$ $5 A^{6}+2 A^{3}$.

Eigenvalues of the matrix A are 1, 1 and 2. Since the characteristic polynomial is of degree 3 we get
$f(A)=a_{0} I+a_{1} A+a_{2} A^{2}=r(A)$.

For eigenvalue 2 we get the equation

$$
\begin{equation*}
2^{10}-5 \times 2^{6}+2 \times 2^{3}=\mathrm{a}_{0}+2 \mathrm{a}_{1}+4 \mathrm{a}_{2} \tag{7.2}
\end{equation*}
$$

Since 1 is a eigenvalue of multiplicity two we get equations

$$
\begin{align*}
\mathrm{f}(1)=\mathrm{r}(1) \text { and }\left.\frac{d \mathrm{f}(\lambda)}{d \lambda}\right|_{\lambda=1} & =\left.\frac{d \mathrm{r}(\lambda)}{d \lambda}\right|_{\lambda=1} . \text { That is, } \\
-2 & =\mathrm{a}_{0}+\mathrm{a}_{1}+\mathrm{a}_{2} \text { and }-14=\mathrm{a}_{1}+2 \mathrm{a}_{2} \tag{7.3}
\end{align*}
$$

From (7.2) and (7.3) we have the system

$$
\begin{aligned}
& a_{0}+2 a_{1}+4 a_{2}=720 \\
& a_{0}+a_{1}+a_{2}=-2 \\
& a_{1}+2 a_{2}=-14
\end{aligned}
$$

On solving this system we get $\mathrm{a}_{0}=748, \mathrm{a}_{1}=-1486$ and $\mathrm{a}_{2}=736$.

Thus $f(A)=A^{10}-5 A^{6}+2 A^{3}=748 I-1486 A+736 A^{2}$.

$$
A^{2}=\left(\begin{array}{lll}
1 & 0 & 3 \\
0 & 1 & 0 \\
0 & 0 & 4
\end{array}\right) .
$$

Now $f(A)=748\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)+(-1486)\left(\begin{array}{lll}1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2\end{array}\right)+736\left(\begin{array}{lll}1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4\end{array}\right)=$

$$
\left(\begin{array}{ccc}
-2 & 0 & 722 \\
0 & -2 & 0 \\
0 & 0 & 1720
\end{array}\right)
$$

### 7.5. Conclusions

In this lecture we have seen that how powerful the Cayley-Hamilton theorem and the concept of eigenvalues are? In the next lecture also we shall use the theory of eigenvalues for diagonalization of matrices.

Keywords: Cayley Hamilton theoem, division algorithm, inverse of matrices, power of marices.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

Linear Algebra, A. R. Rao and P. Bhimasankaram, Hindustan Book Agency, New Delhi, 2000.

Linear Algebra and Its Applications, Fourth Edition, Gilbert Strang, Thomson Books/Cole, 2006.

Matrix Methods: An Introduction, Second Edition, Richard Bronson, Academic press, 1991.


## Lesson 8

## Diagonalization of Matrices

### 8.1 Introduction

Diagonalizable matrices are of particular interest in linear algebra because of their application to computation of several matrix operations and functions easily. Not all matrices are diagonalizable. In this lecture we learn technique to identify matrices that are diagonalizable.

### 8.2 Similar Matrices

Diagonalizable matrices are defined through similar matrices. Two square matrices $A$ and $B$ are said to be similar if there exists an invertible matrix $P$ such that $A=P^{-}$ ${ }^{1} \mathrm{~B} P$ or equivalently $\mathrm{PA}=\mathrm{BP}$.

Example 8.2.1: (i) Matrices $A=\left(\begin{array}{ll}3 & 5 \\ 3 & 1\end{array}\right)$ and $B=\left(\begin{array}{ll}2 & 4 \\ 4 & 2\end{array}\right)$ are similar because PA $=\mathrm{PB}$, where $\mathrm{P}=\left(\begin{array}{ll}4 & 0 \\ 1 & 5\end{array}\right)$. Note that P is invertible as det $\mathrm{P}=20 \neq 0$. However matrices $R=\left(\begin{array}{ll}2 & 0 \\ 0 & 2\end{array}\right)$ and $S=\left(\begin{array}{ll}2 & 1 \\ 0 & 2\end{array}\right)$ are not similar because otherwise the matrix $P_{1}$ satisfying $P_{1} R=S P_{1}$ will be of the form $\left(\begin{array}{ll}a & b \\ 0 & 0\end{array}\right)$ and is a non-invertible (or singular) matrix.

In the following we shall present an important result on similar matrices.

Theorem 8.2.1: Similar matrices have the same characteristic equation (and hence the same eigenvalues).

Proof: Let $A$ and $B$ be similar matrices. We have to show that $\operatorname{det}(A-\lambda I)=\operatorname{det}(B-\lambda I) . A=P^{-1} B P$, where $P$ is an invertible matrix.

$$
\begin{gathered}
\operatorname{det}(\mathrm{A}-\lambda \mathrm{I})=\operatorname{det}\left(\mathrm{P}^{-1} \mathrm{~B} P-\mathrm{P}^{-1} \lambda \mathrm{I} \mathrm{P}\right)=\operatorname{det}\left(\mathrm{P}^{-1}(\mathrm{~B}-\lambda \mathrm{I}) \mathrm{P}\right) \\
=\operatorname{det}\left(\mathrm{P}^{-1}\right) \operatorname{det}(\mathrm{B}-\lambda \mathrm{I}) \operatorname{det}(\mathrm{P})=\operatorname{det}(\mathrm{B}-\lambda \mathrm{I}) .
\end{gathered}
$$

The above theorem also gives a criteria for checking that the given matrices are similar or not.

Example 8.2.1: Matrices $A=\left(\begin{array}{ll}1 & 2 \\ 4 & 3\end{array}\right)$ and $B=\left(\begin{array}{ll}4 & 1 \\ 3 & 2\end{array}\right)$ are not similar because their characteristic polynomials are $\lambda^{2}-4 \lambda-5$ and $\lambda^{2}-6 \lambda+5$ respectively.

### 8.3 Algebraic and Geometric Multiplicities

For diagonalization of matrices we need to understand the algebraic and geometric multiplicities of eigenvalues. Let $\lambda_{0}$ be an eigenvalue of $A$. The geometric multiplicity of $\lambda_{0}$ is the dimension of the eigenspace of $\lambda_{0}$, that is the dimension of the solution space of $\left(A-\lambda_{0} I\right) x=0$, which is also the nullity of $\left(A-\lambda_{0} I\right)$. Whereas the algebraic multiplicity of $\lambda_{0}$ is the largest positive integer k such that ( $\lambda$ $\left.-\lambda_{0}\right)^{\mathrm{k}}$ is a factor of the characteristic polynomial of A.

Example 8.3.1: Consider the matrix $A=\left(\begin{array}{ccc}-3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2\end{array}\right)$. Characteristic polynomial of A is $\operatorname{det}(\mathrm{A}-\lambda \mathrm{I})=(\lambda+2)^{2}(\lambda-4)$. So -2 is an eigenvalue of multiplicity two and therefore algebraic multiplicity of the eigenvalue -2 is equal to 2 . One can check that rank of $(\mathrm{A}+2 \mathrm{I})$ is equal to two hence its nullity is equal to one. So geometric multiplicity of the eigenvalue -2 is equal to 1 . The following theorem gives a relation between these two multiplicities.

Theorem 8.3.1: The algebraic multiplicity of an eigenvalue is not less than its geometric multiplicity.

### 8.4 Diagonalizable Matrices

A square matrix is said to be diagonalizable if it is similar to a diagonal matrix. In other words A is diagonalizable if and only if there is an invertible matrix P such that $\mathrm{P}^{-1} \mathrm{~A} P$ is a diagonal matrix.

The following theorem gives a criteria for diagonalizable matrices.

Theorem 8.4.1: Let $A_{n \times n}$ be an square matrix with eigenvalues $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$. Let $\gamma_{1}, \gamma_{2}, \ldots, \gamma_{k}$ be the geometric multiplicity of $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$ respectively. Then $A$ is diagonalizable if and only if $\gamma_{1}+\gamma_{2}+\ldots+\gamma_{\mathrm{k}}=\mathrm{n}$.

From theorems 8.3.1 and 8.4.1 we get the following result.

Corollary 8.4.1: A matrix $A_{n \times n}$ is diagonalizable if and only if for every eigenvalue $\lambda$ of A , the algebraic multiplicity of $\lambda$ is equal its geometric multiplicity.

Corollary 8.4.2: If $\mathrm{A}_{\mathrm{n} \times \mathrm{n}}$ has n distinct eigenvalues then A is diagonalizable.

### 8.5 Algorithm to Diagonalize a Matrix

Input: A square matrix $\mathrm{A}_{\mathrm{n} \times \mathrm{n}}$.
Output: A diagonal matrix similar to A.
(1) Find eigenvalues of A say $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}, k \leq n$.
(2) Find geometric multiplicity $\gamma_{i}$ of $\lambda_{i}, 1 \leq i \leq k$.
(3) If $\gamma_{1}+\gamma_{2}+\ldots+\gamma_{\mathrm{k}}=\mathrm{n}$ then continue otherwise return that A is not diagonalizable.
(4) Find basis for eigenspace of each $\lambda_{i}$. Let $\left\{\mathrm{X}_{j}{ }^{\lambda_{i}}: 1 \leq \mathrm{j} \leq \gamma_{i}\right\}$ be a basis for the eigenspace corresponding to $\lambda_{\mathrm{i}}, 1 \leq \mathrm{i} \leq \mathrm{k}$.
(5) Take $\mathrm{P}=\left(\mathrm{x}_{1} \cdots \mathrm{x}_{\gamma_{1}}{ }^{\lambda_{1}} \mathrm{x}_{1}^{\lambda_{2}} \mathrm{x}_{2}^{\lambda_{2}} \cdots \mathrm{x}_{\gamma_{2}}{ }^{\lambda_{2}} \cdots \mathrm{x}_{1}{ }^{\lambda_{k}} \mathrm{x}_{2}^{\lambda_{k}} \cdots \mathrm{x}_{\gamma_{k}}{ }^{\lambda_{k}}\right)$ be the $\mathrm{n} \times \mathrm{n}$ matrix such that each $X_{j}{ }^{\lambda_{i}}$ is a column vector i.e. a matrix of size $n \times 1$. Obviously P is invertible.
(6) $\mathrm{P}^{-1} \mathrm{~A} \mathrm{P}=\operatorname{diag}\left(\lambda_{1}, \lambda_{1, \ldots}, \lambda_{1}, \lambda_{2}, \lambda_{2}, . . \lambda_{2}, \ldots . \lambda_{\mathrm{k}}, \lambda_{\mathrm{k}}, \ldots \lambda_{\mathrm{k}}\right)$ is the diagonal matrix similar to A .

Example 8.5.1: Consider the matrix $A=\left(\begin{array}{ccc}5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2\end{array}\right)$.

A has two eigenvalues $\lambda_{1}=10$ and $\lambda_{2}=1$, where algebraic multiplicities of $\lambda_{1}$ and $\lambda_{2}$ are 1 and 2 respectively. Recall that the eigenspace of $\lambda_{1}$ is

$$
\mathrm{S}_{1}=\{(2 \alpha, 2 \alpha, \alpha): \alpha \square \mathrm{R}\} \text { (here we include the zero vector also). }
$$

$\operatorname{dim} S_{1}=1=\gamma_{1}$, the geometric multiplicity of $\lambda_{1}$. Eigen space of $\lambda_{2}$ is

$$
S_{2}=\left\{\left(-\frac{1}{2}(2 \alpha+\beta), \alpha, \beta\right): \alpha, \beta \square \mathrm{R}\right\} \text { and } \operatorname{dim} \mathrm{S}_{2}=2=\gamma_{2} .
$$

Now $\gamma_{1}+\gamma_{2}=3=$ size of the matrix A. So A is diagonalizable.

A basis for $S_{1}$ is $\{(2,2,1)\}$. A basis for $S_{2}$ is $\left\{(-1,1,0),\left(-\frac{1}{2}, 0,1\right)\right\}$ (obtained by taking $\alpha=1, \beta=0$ and then $\alpha=0, \beta=1$ ). So

$$
\begin{gathered}
\mathrm{P} \\
=\left(\begin{array}{ccc}
2 & -1 & -\frac{1}{2} \\
2 & 1 & 0 \\
1 & 0 & 1
\end{array}\right) .
\end{gathered}
$$

One checks that $\mathrm{P}^{-1} \mathrm{AP}=\left(\begin{array}{ccc}10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$ and is similar A .
Not all matrices are diagonalizable and we will see such an example below.

Example 8.5.2: As we have seen in Example 8.3.1 that for the matrix $\mathrm{A}=$ $\left(\begin{array}{lll}-3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2\end{array}\right)$ the eigenvalues are $\lambda_{1}=-2, \lambda_{2}=4, \lambda_{1}$ is of multiplicity 2 . Also the algebraic multiplicity of $\lambda_{1}$ is 2 and the geometric multiplicity of it is 1 . Therefore A is not a diagonalizable matrix.

### 8.6 Computation of Functions of Diagonalizable Matrices

In the following theorem we shall list some properties of diagonal and diagonalizable matrices.

Theorem 8.6.1: The following are true for a diagonal or a diagonalizable matrix D:
(I) If $D=\left(\begin{array}{ll}a & 0 \\ 0 & b\end{array}\right)_{n \times n}$ is a diagonal matrix the $k^{\text {th }}$ power of $D$ is equal to $\left(\begin{array}{cc}a^{k} & 0 \\ 0 & b^{k}\end{array}\right)_{n \times n}$.
(II) If A is a diagonalizable matrix with $\mathrm{A}=\mathrm{P}^{-1} \mathrm{D} \mathrm{P}$, where D is a diagonal matrix, then $A^{k}=P^{-1} D^{k} P$. (For $k=2$ one verifies that $A^{2}=A \cdot A=\left(P^{-1} D P\right)\left(P^{-1} D P\right)=P^{-1}$ $\mathrm{D}\left(\mathrm{PP}^{-1}\right) \mathrm{D} P=\mathrm{P}^{-1} \mathrm{D}^{2} \mathrm{P}$.)
(III) If $\mathrm{P}(\mathrm{x})=\mathrm{a}_{0}+\mathrm{a}_{1} \mathrm{x}+\ldots+\mathrm{a}_{\mathrm{n}} \mathrm{x}^{\mathrm{n}}$ be any polynomial and A be a diagonalizable matrix with $A=M D M^{-1}$, where $D$ is diagonal, then $P(A)=M P(D) M^{-1}$. (One can get this by taking $P(A)=M a_{0} I M^{-1}+M a_{0} D M^{-1}+\ldots+M a_{0} D^{n} M^{-1}$. )

Example 8.6.1: Here, We compute $A^{30}$ for $A=\left(\begin{array}{cc}0 & 1 \\ -2 & 3\end{array}\right)$. This matrix is diagonalizable as $A=M D M^{-1}$, where $M=\left(\begin{array}{ll}1 & 1 \\ 1 & 2\end{array}\right)$ and $D=\left(\begin{array}{ll}1 & 0 \\ 0 & 2\end{array}\right)$. Thus by Theorem 8.6.1(i) and (ii), $\mathrm{A}^{30}=\mathrm{MD}^{30} \mathrm{M}^{-1}=\left(\begin{array}{ll}1 & 1 \\ 1 & 2\end{array}\right)\left(\begin{array}{cc}1 & 0 \\ 0 & 2^{30}\end{array}\right)\left(\begin{array}{cc}2 & -1 \\ -1 & 1\end{array}\right)$.

$$
=\left(\begin{array}{ll}
2-2^{30} & 2^{30}-1 \\
2-2^{31} & 2^{31}-1
\end{array}\right)
$$

Example 8.6.2: If $P(x)=x^{17}-3 x^{5}+2 x^{2}+1$ then we find the value of $P(A)=A^{17}-$ $3 A^{5}+2 A^{2}+I$, for the same matrix $A$ in Example 8.6.2. By Theorem 8.6.1 (iii), and Example 8.61, $\mathrm{P}(\mathrm{A})=\mathrm{A}^{17}-3 \mathrm{~A}^{5}+2 \mathrm{~A}^{2}+\mathrm{I}$

$$
\begin{aligned}
& =\left(\begin{array}{cc}
2-2^{17} & 2^{17}-1 \\
2-2^{18} & 2^{18}-1
\end{array}\right)-3\left(\begin{array}{ll}
2-2^{5} & 2^{5}-1 \\
2-2^{6} & 2^{6}-1
\end{array}\right)+2\left(\begin{array}{ll}
2-2^{2} & 2^{2}-1 \\
2-2^{3} & 2^{3}-1
\end{array}\right)+\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) . \\
& =\left(\begin{array}{cc}
89-2^{17} & -88+2^{17} \\
176-2^{18} & -175+2^{18}
\end{array}\right) .
\end{aligned}
$$

### 8.7 Conclusions

Here we have seen that finding higher powers of a diagonalizable matrix or value of any polynomial on a diagonalizable matrix can be computed easily.

Keywords: Similar matrices, diagonalizable matrices, algebraic multiplicity, geometric multiplicity, functions of diagonalizable matrices.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

Linear Algebra, A. R. Rao and P. Bhimasankaram, Hindustan Book Agency, New Delhi, 2000.

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Matrix Methods: An Introduction, Second Edition, Richard Bronson, Academic press, 1991.

## Lesson 9

## Linear and Orthogonal Transformations

### 9.1 Introduction

In order to compare mathematical structures of same type we study operation presenting mappings from one structure to another. In case of vector spaces such a mapping is called a linear transformation. Matrices and linear transformations are closely related, in fact one can be obtained from the other easily. Orthogonal transformations are particular type of linear transformations.

### 9.2. Linear Transformations

Let V and W be vector space over the same field F . A mapping $\mathrm{T}: \mathrm{V} \rightarrow \mathrm{W}$ is called a linear transformation if
(i) $T(u+v)=T(u)+T(v)$, for $u, v \square$.
(ii) $T(\alpha u)=\alpha T(u)$ for all $u \square V$ and $\alpha \boxminus F$.
(Combiningly these two statements can be written as:
$\mathrm{T}(\alpha \mathrm{u}+\beta \mathrm{v})=\alpha \mathrm{T}(\mathrm{u})+\beta \mathrm{T}(\mathrm{v})$, for $\mathrm{u}, \mathrm{v} \square \mathrm{V}$ and $\alpha, \beta \square \mathrm{F})$.

Example 9.2.1: Let $T_{1}, T_{2}$ be mappings from $\mathbb{R}^{3}$ to $\mathbb{R}^{2}$ defined as:

$$
\mathrm{T}_{1}\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right)=\left(\mathrm{x}_{1}+\mathrm{x}_{2}, \mathrm{x}_{3}\right) \text { and } \mathrm{T}_{2}\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right)=\left(\mathrm{x}_{1} \mathrm{x}_{2}, \mathrm{x}_{3}\right) .
$$

$\mathrm{T}_{1}$ is a linear transformation because

$$
\begin{aligned}
& \mathrm{T}_{1}\left(\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right)+\left(\mathrm{y}_{1}, \mathrm{y}_{2}, y_{3}\right)\right)=\mathrm{T}_{1}\left(\mathrm{x}_{1}+\mathrm{y}_{1}, \mathrm{x}_{2}+\mathrm{y}_{2}, \mathrm{x}_{3}+\mathrm{y}_{3}\right) \\
& =\left(\mathrm{x}_{1}+\mathrm{y}_{1}+\mathrm{x}_{2}+\mathrm{y}_{2}, \mathrm{x}_{3}+\mathrm{y}_{3}\right) .
\end{aligned}
$$

$$
\begin{aligned}
& =\left(x_{1}+x_{2}, x_{3}\right)+\left(y_{1}+y_{2}, y_{3}\right) . \\
& =T_{1}\left(x_{1}, x_{2}, x_{3}\right)+T_{2}\left(y_{1}, y_{2}, y_{3}\right) .
\end{aligned}
$$

$T_{1}\left(\alpha\left(x_{1}, x_{2}, x_{3}\right)\right)=T_{1}\left(\alpha x_{1}, \alpha x_{2}, \alpha x_{3}\right)$.
$=\left(\alpha \mathrm{x}_{1}+\alpha \mathrm{x}_{2}, \alpha \mathrm{x}_{3}\right)=\alpha\left(\mathrm{x}_{1}+\mathrm{x}_{2}, \mathrm{x}_{3}\right)=\alpha \mathrm{T}\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right)$.
$\mathrm{T}_{2}$ is not a linear transformation because
$T_{2}\left(\left(x_{1}, x_{2}, x_{3}\right)+\left(y_{1}, y_{2}, y_{3}\right)\right)=T_{2}\left(x_{1}+y_{1}, x_{2}+y_{2}, x_{3}+y_{3}\right)$.
$=\left(\left(x_{1}+y_{1}\right)\left(x_{2}+y_{2}\right),\left(x_{3}+y_{3}\right)\right)$
$\neq\left(\mathrm{x}_{1} \mathrm{x}_{2}, \mathrm{x}_{3}\right)+\left(\mathrm{y}_{1} \mathrm{y}_{2}, \mathrm{y}_{3}\right)=\mathrm{T}\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right)+\mathrm{T}\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \mathrm{y}_{3}\right)$.

A linear transformation $\mathrm{T}: \mathrm{V} \rightarrow \mathrm{W}$ is called an isomorphism if T is a one to one mapping. Vector spaces V and W are said to be isomorphic if there is a an isomorphism from V on to W . A vector space V is trivially isomorphic to itself because the identity mapping is an isomorphism from V onto itself. If V and W are isomorphic and T is an isomorphism from V on to W then $\mathrm{T}^{-1}: \mathrm{W} \rightarrow \mathrm{V}$ is also an isomorphism.

In the theorem below we list some properties of isomorphisms.

Theorem 9.2.1: Let $\mathrm{T}: \mathrm{V} \rightarrow \mathrm{W}$ be a linear transformation. Then
(1) $T(0)=0$. Further if $T$ is an isomorphism then $T(v)=0$ implies $v=0$.
(2) If T is an isomorphism and $\mathrm{S}=\left\{\mathrm{V}_{1}, \mathrm{~V}_{2}, \ldots, \mathrm{~V}_{\mathrm{k}}\right\}$ is a linearly independent set of vectors in V then $\left\{\mathrm{T}\left(\mathrm{V}_{1}\right), \mathrm{T}\left(\mathrm{V}_{2}\right), \ldots, \mathrm{T}\left(\mathrm{V}_{\mathrm{k}}\right)\right\}$ is a linearly independent set in W.

An important result for finite dimensional vector spaces is given in the theorem below.

Theorem 9.2.2: Two finite dimensional vector spaces over the same field are isomorphic if and only if they have the same dimension.

Corollary 9.2.1: Every n-dimension vector space over $F$ is isomorphic to $F^{n}$. In particular every n-dimensional vector space over $\mathbb{R}$ is isomorphic to $\mathbb{R}^{n}$.

Next we shall define the null space and range space of a linear transformation. Let $\mathrm{T}: \mathrm{V} \rightarrow \mathrm{W}$ be a linear transformation. The kernel of T , Ker T , is the set Ker $\mathrm{T}=\{\mathrm{v}$ $\square \mathrm{V}: \mathrm{T}(\mathrm{v})=0\}$. The set $\mathrm{T}(\mathrm{V})=\{\mathrm{T}(\mathrm{v}): \mathrm{v} \square \mathrm{V}\}$ is called the range of T , denoted by rang(T). It is an well-known result that $\operatorname{Ker} \mathrm{T}=\{0\}$ if and only if T is an isomorphism One can verify easily that Ker T is a subspace of V , called the null space of T , and $\mathrm{Rang}(\mathrm{T})$ is also a subspace of W , called the range space of T . If V and We are finite dimensional vector spaces then dimension of Ker T is called the nullity of T and the dimension of rang( T ) is called the rank of T . One should not get confuse with these terminologies because very shortly we are going show that linear transformations can be represented as matrices and the vice versa.

Example 9.2.2: (1) Consider the linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2}$ defined as:
$T\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(x_{1}, x_{2}, 0\right)$. Then Ker $T$ is the $z$-axis and rang $(T)=\mathbb{R}^{2}$.
(2) Consider the linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2}$ given by $T\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{1}, 0\right.$, 0 ). Then Ker T is the yz -plane and rang ( T ) is the x -axis.

Like matrices one can also have the rank-nullity theorem for linear transformations.

Theorem 9.2.3: Let V and W be finite dimensional vector spaces and $\mathrm{T}: \mathrm{V} \rightarrow \mathrm{W}$ be a linear transformation. Then nullity of $\mathrm{T}+\operatorname{rank}$ of $\mathrm{T}=\operatorname{dim} \mathrm{V}$.

### 9.3 Linear Transformations from Matrices

Every linear transformation can be represented as a matrix and every matrix can produce a linear transformation. So people sometime treat matrices as linear transformations and vice versa. Here we shall discuss about the method to get a linear transformation from a matrix.

Let V and W be finite dimensional vector spaces over F with $\operatorname{dim} \mathrm{V}=\mathrm{n}$ and $\operatorname{dim} \mathrm{W}$ $=n$, and $A_{m \times n}=\left(a_{i j}\right)_{m \times n}$ be a matrix over F (same field). From Corollary 9.2.1 every vector in V can be expressed as an n-tuple of elements in F , in other words, we can take $V=F^{n \times 1}$, i.e. $V$ consists of $n \times 1$ matrices (or column vectors $\left(\begin{array}{c}x_{1} \\ \vdots \\ x_{n}\end{array}\right)$, $x_{i}$ F). Similarly elements of $W$ can be taken as column vectors $\left(\begin{array}{c}x_{1} \\ \vdots \\ x_{m}\end{array}\right), x_{i} \square$ F, i.e.

$$
\begin{aligned}
& W=F^{m \times 1} \text {. Then the mapping } T: V \rightarrow W \text { defined as } T\left(\begin{array}{c}
x_{1} \\
\vdots \\
x_{n}
\end{array}\right)=A_{m \times n}\left(\begin{array}{c}
x_{1} \\
\vdots \\
x_{n}
\end{array}\right)_{n \times 1} \\
& =\left(\begin{array}{c}
\sum_{j=1}^{n} a_{1 j} x_{j} \\
\vdots \\
\sum_{j=1}^{n} a_{m j} x_{j}
\end{array}\right) \text { is a linear transformation because } \\
& A\left\{\left(\begin{array}{c}
x_{1} \\
\vdots \\
x_{n}
\end{array}\right)+\left(\begin{array}{c}
y_{1} \\
\vdots \\
y_{n}
\end{array}\right)\right\}=\left\{\begin{array}{c}
\sum_{j=1}^{n} a_{1 j}\left(x_{j}+y_{j}\right) \\
\vdots \\
\sum_{j=1}^{n} a_{m j}\left(x_{j}+y_{j}\right)
\end{array}\right\}= \\
& \left\{\begin{array}{c}
\sum_{j=1}^{n} a_{1 j} x_{j} \\
\vdots \\
\sum_{j=1}^{n} a_{m j} x_{j}
\end{array}\right\}+\left\{\begin{array}{c}
\sum_{j=1}^{n} a_{1 j} y_{j} \\
\vdots \\
\sum_{j=1}^{n} a_{m j} y_{j}
\end{array}\right\}=A\left(\begin{array}{c}
x_{1} \\
\vdots \\
x_{n}
\end{array}\right)+A\left(\begin{array}{c}
y_{1} \\
\vdots \\
y_{n}
\end{array}\right) \\
& \text { and } \\
& \mathrm{A}\left(\begin{array}{c}
\alpha \mathrm{x}_{1} \\
\vdots \\
\alpha \mathrm{X}_{\mathrm{n}}
\end{array}\right)=\alpha\left\{\mathrm{A}\left(\begin{array}{c}
\mathrm{x}_{1} \\
\vdots \\
\mathrm{X}_{\mathrm{n}}
\end{array}\right)\right\}
\end{aligned}
$$

Notice that if $A$ is an $m \times n$ matrix then we get a linear transformation from an $n$ dimensional vector space to an m-dimensional vector space.

Example 9.3.1: Let $\mathrm{A}=\left(\begin{array}{ccc}1 & 3 & -2 \\ 0 & 4 & 1\end{array}\right)_{2 \times 3}$ be a matrix over $\mathbb{R}$. The mapping T: $\mathbb{R}^{3} \rightarrow \mathbb{R}^{2}$ given by:

$$
\begin{aligned}
& \mathrm{T}\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{x}_{3}\right)^{\mathrm{T}}=\left(\begin{array}{ccc}
1 & 3 & -2 \\
0 & 4 & 1
\end{array}\right)\left(\begin{array}{c}
\mathrm{x}_{1} \\
\mathrm{x}_{2} \\
\mathrm{x}_{3}
\end{array}\right) . \\
&=\binom{\mathrm{x}_{1}+3 \mathrm{x}_{2}-2 \mathrm{x}_{3}}{4 \mathrm{x}_{2}+\mathrm{x}_{3}} \text { is a linear transformation. }
\end{aligned}
$$

### 9.4 Matrix Representation of a Linear Transformation

Let V and W be vector spaces over F and $\mathrm{T}: \mathrm{V} \rightarrow \mathrm{W}$ be a linear transformation. Let $\operatorname{dim} \mathrm{V}=\mathrm{n}, \operatorname{dim} \mathrm{W}=\mathrm{m},\left\{\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{n}}\right\}$ and $\left\{\mathrm{w}_{1}, \mathrm{w}_{2}, \ldots, \mathrm{w}_{\mathrm{m}}\right\}$ be bases for V and W respectively. Note that $\mathrm{T}\left(\mathrm{v}_{1}\right), \mathrm{T}\left(\mathrm{v}_{2}\right), \ldots, \mathrm{T}\left(\mathrm{v}_{\mathrm{n}}\right)$ are vectors in W and so these vectors can be expressed as linear combinations of vectors in $\left\{\mathrm{w}_{1}, \mathrm{w}_{2}, \ldots\right.$, $\left.\mathrm{w}_{\mathrm{m}}\right\}$. So let

$$
\begin{aligned}
& T\left(v_{1}\right)=a_{11} W_{1}+a_{12} W_{2}+\ldots+a_{1 m} W_{m} . \\
& T\left(v_{2}\right)=a_{21} w_{1}+a_{22} W_{2}+\ldots+a_{2 m} W_{m} . \\
& \quad \vdots \\
& T\left(v_{n}\right)=a_{n 1} W_{1}+a_{n 2} W_{2}+\ldots+a_{n m} W_{m}, \quad \text { where } a_{i j} \square F \text {. }
\end{aligned}
$$

Then the matrix A given below is a matrix representation of T :

$$
A=\left(\begin{array}{cccc}
a_{11} & a_{21} & \cdots & a_{n 1} \\
a_{12} & a_{22} & \cdots & a_{n 2} \\
\vdots & \vdots & & \vdots \\
a_{1 m} & a_{2 m} & \cdots & a_{n m}
\end{array}\right)_{m \times n}
$$

Note that if we consider different bases in V and W then we may get different matrix representations of T (of course these matrices are all similar). In the above if we represent $T\left(v_{i}\right)=\left(a_{i 1}, a_{i 2}, \ldots, a_{i n}\right)^{T}$. then the matrix corresponding to $T$ can be written as:

$$
A=\left(T\left(v_{1}\right) T\left(v_{2}\right) \ldots T\left(v_{n}\right)\right) .
$$

Example 9.4.1: Consider the linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2}$ defined by $T\left(x_{1}\right.$, $\left.\mathrm{x}_{2}, \mathrm{x}_{3}\right)=\left(\mathrm{x}_{1}+\mathrm{x}_{2}, 2 \mathrm{x}_{3}\right)$.

Take bases $B=\{(1,1,0),(0,1,4),(1,2,3)\}$ and $B_{1}=\{(1,0),(0,2)\}$ in $\mathbb{R}^{3}$ and $\mathbb{R}^{2}$ respectively.

$$
\begin{aligned}
& \mathrm{T}(1,1,0)=(2,0)=2(1,0)+0(0,2) . \\
& \mathrm{T}(0,1,4)=(1,8)=1(1,0)+4(0,2) . \\
& \mathrm{T}(1,2,3)=(3,6)=3(1,0)+3(0,2) .
\end{aligned}
$$

So the matrix representation of T is the matrix $\left(\begin{array}{lll}2 & 1 & 3 \\ 0 & 4 & 3\end{array}\right)$.

### 9.5 Orthogonal Transformations

Before defining orthogonal transformations we recall same terminologies defined in the vector space $\mathbb{R}^{n}$. For any two vectors $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and $\mathrm{y}=\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \ldots, \mathrm{y}_{\mathrm{n}}\right)$ in $\mathbb{R}^{\mathrm{n}}$ the standard inner product of x and y , denoted by $\langle\mathrm{x}, \mathrm{y}\rangle$, is given by $\langle\mathrm{x}, \mathrm{y}\rangle=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} \mathrm{y}_{\mathrm{i}}$. Since $\langle\mathrm{x}, \mathrm{x}\rangle \geq 0$, positive square root of $\langle\mathrm{x}, \mathrm{x}\rangle$ denoted by $\|\mathrm{x}\|$, is called the norm (or length) of x . Two vectors x and y in $\mathbb{R}^{\mathrm{n}}$ are said to be orthogonal if $\langle\mathrm{x}, \mathrm{y}\rangle=0$. A basis $\left\{\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{n}}\right\}$ of $\mathbb{R}^{\mathrm{n}}$ is said to be an orthonormal basis if $\left\langle\mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{j}}\right\rangle=0$ for $1 \leq \mathrm{i} \neq \mathrm{j} \leq \mathrm{n}$, and $\left\|\mathrm{v}_{\mathrm{k}}\right\|=1$ for all $k=1,2, \ldots, n$.

Recall that a real square matrix $A$ of size $n$ is said to be orthogonal if $A A^{T}=A^{T} A$ $=I$, where $I$ is the $n \times n$ identity matrix. Orthogonal matrices satisfy the following properties: (1) $A^{T}=A^{-1}$ (2) det $A= \pm 1$ and (3) Product of two orthogonal matrices of the same size is orthogonal.

A linear transformation $\mathrm{T}: \mathbb{R}^{\mathrm{n}} \rightarrow \mathbb{R}^{\mathrm{n}}$ is called an orthogonal transformation if $\langle T(u), T(v)\rangle=\langle u, v\rangle$ for every vectors $u$ and $v$ in $\mathbb{R}^{n}$. So an orthogonal transformation not only preserves the addition and scalar multiplication, it also preserves the length of every vector.

An orthogonal transformation is also called an isometry because of the following result.

Theorem 9.5.1: A linear transformation $\mathrm{T}: \mathbb{R}^{\mathrm{n}} \rightarrow \mathbb{R}^{\mathrm{n}}$ is an orthogonal transformation if and only if $\|\mathrm{T}(\mathrm{v})\|=\|\mathrm{v}\|$ for all vectors v in $\mathbb{R}^{\mathrm{n}}$.

Example 9.5.1: The mapping $T: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ defined as $T(x, y)=\left(\frac{2 x-y}{\sqrt{5}}, \frac{x+2 y}{\sqrt{5}}\right)$ is an orthogonal transformation. One can check that T preserves addition and scalar multiplication and hence is a linear transformation. Next we show that

$$
\begin{gathered}
\|\mathrm{T}(\mathrm{x}, \mathrm{y})\|=\|(\mathrm{x}, \mathrm{y})\| \text {, for all vectors }(\mathrm{x}, \mathrm{y}) \text { in } \mathbb{R}^{2} . \\
\|\mathrm{T}(\mathrm{x}, \mathrm{y})\|=\frac{1}{\sqrt{5}}\left\{(2 \mathrm{x}-\mathrm{y})^{2}+(\mathrm{x}+2 \mathrm{y})^{2}\right\}^{\frac{1}{2}} . \\
=\frac{1}{\sqrt{5}}\left\{5 \mathrm{x}^{2}+5 \mathrm{y}^{2}\right\}^{\frac{1}{2}}=\sqrt{\mathrm{x}^{2}+\mathrm{y}^{2}}=\|(\mathrm{x}, \mathrm{y})\| .
\end{gathered}
$$

In the following theorem we show that the matrix associated with an orthogonal transformation is also orthogonal.

Theorem 9.5.2: Let $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be an orthogonal transformation and $A$ be the matrix representation of $T$ with respect to the standard basis $\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ in $\mathbb{R}^{n}$. Then A is an orthogonal matrix.

Proof: The matrix representation of T can be written as $\mathrm{A}=\mathrm{T}\left(\mathrm{e}_{1}\right), \mathrm{T}\left(\mathrm{e}_{2}\right), \ldots$, $T\left(e_{n}\right)$ ). Since $T$ is an orthogonal transformation, $\left\langle T\left(e_{i}\right), T\left(e_{j}\right)\right\rangle=\left\langle e_{i}, e_{j}\right\rangle$. The standard basis in $\mathbb{R}^{\mathrm{n}}$ is orthonormal. So $\left\langle\mathrm{T}\left(\mathrm{e}_{i}\right), \mathrm{T}\left(\mathrm{e}_{\mathrm{j}}\right)\right\rangle$ is equal to 1 if $\mathrm{i}=\mathrm{j}$ and is zero otherwise (i.e. $\mathrm{i} \neq \mathrm{j}$ ). Thus $\mathrm{AA}^{\mathrm{T}}=\mathrm{I}$ and A is orthogonal.

### 9.6 Conclusions

Linear transformations are used to recognize identical structures in linear algebra. Using these transformations one can transfer problems in a complicated space to a simpler space and then workout. Orthogonal transformations are also applied for reduction of matrices to some important foms.

Keywords: Linear transformations, isomorphic vector spaces, kernel, matrix representation, orthogonal transformations.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

Linear Algebra, A. R. Rao and P. Bhimasankaram, Hindustan Book Agency, New Delhi, 2000.

Linear Algebra and Its Applications, Fourth Edition, Gilbert Strang, Thomson Books/Cole, 2006.

Matrix Methods: An Introduction, Second Edition, Richard Bronson, Academic press, 1991.

## Lesson 10

## Quadratic Forms

### 10.1 Introduction

The study of quadratic forms began with the pioneering work of Witt. Quadratic forms are basically homogeneous polynomials of degree 2. They have wide application in science and engineering.

### 10.2 Quadratic Forms and Matrices

Let $\mathrm{A}=\left(\mathrm{a}_{\mathrm{ij}}\right)$ be a real square matrix of size n and x be a column vector $\mathrm{x}=\left(\mathrm{x}_{1}, \mathrm{x}_{2}\right.$, $\left.\ldots, x_{n}\right)^{T}$. A quadratic form on $n$ variables is an expression $Q=x^{T} A x$.

In other words,

$$
\begin{aligned}
& \mathrm{Q}=\mathrm{x}^{\mathrm{T}} \mathrm{~A} x=\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{n}\right)\left(\begin{array}{ccc}
\mathrm{a}_{11} & \cdots & \mathrm{a}_{1 \mathrm{n}} \\
\vdots & & \\
\mathrm{a}_{\mathrm{n} 1} & \cdots & \mathrm{a}_{\mathrm{n}}
\end{array}\right)\left(\begin{array}{c}
\mathrm{x}_{1} \\
\vdots \\
\mathrm{x}_{\mathrm{n}}
\end{array}\right) . \\
& =\mathrm{a}_{11} \mathrm{x}_{1}{ }^{2}+\mathrm{a}_{12} \mathrm{x}_{1} \mathrm{x}_{2}+\ldots+\mathrm{a}_{1 \mathrm{n}} \mathrm{x}_{1} \mathrm{x}_{\mathrm{n}}+\mathrm{a}_{21} \mathrm{x}_{2} \mathrm{x}_{1}+\mathrm{a}_{22} \mathrm{x}_{2}{ }^{2}+\ldots+\mathrm{a}_{2 n} \mathrm{x}_{2} \mathrm{x}_{\mathrm{n}}+\ldots+\mathrm{a}_{\mathrm{n} 1} \mathrm{x}_{\mathrm{n}} \mathrm{x}_{1}+ \\
& \mathrm{a}_{\mathrm{n} 2} \mathrm{x}_{\mathrm{n}} \mathrm{x}_{2}+\ldots+\mathrm{a}_{\mathrm{nn}} \mathrm{x}_{\mathrm{n}}{ }^{2}=\sum_{j=1}^{n} \sum_{i=1}^{n} \mathrm{a}_{i j} \mathrm{x}_{i} \mathrm{x}_{j} .
\end{aligned}
$$

The matrix A is called the matrix of the quadratic form Q . This matrix A need not be symmetric. However, in the following theorem we show that every quadratic form corresponds to a unique symmetric matrix. Hence there is one to one correspondence between symmetric matrices of size $n$ and quadratic forms on $n$ variables.

Theorem 10.2.1: For every quadratic form $Q$ there is a unique symmetric matrix $B$ such that $\mathrm{Q}=\mathrm{x}^{\mathrm{T}} \mathrm{Bx}$.

Proof: We consider an arbitrary quadratic form $Q=x^{T} A x$, with $A=\left(a_{i j}\right)$. We construct a matrix $B=\left(b_{i j}\right)$, where $b_{i j}=\frac{a_{i j}+a_{j i}}{2}$. This matrix $B$ is symmetric and $x^{T}$ $A x=x^{T} B x$, i.e. quadratic forms associated with $A$ and $B$ are the same.

Example 10.2.1: For $A=\left(\begin{array}{lll}1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9\end{array}\right)$, the quadratic form associated with $A$ is
$\mathrm{Q}=\left(\begin{array}{lll}\mathrm{x}_{1} & \mathrm{x}_{2} & \mathrm{x}_{3}\end{array}\right)\left(\begin{array}{lll}1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9\end{array}\right)\left(\begin{array}{l}\mathrm{x}_{1} \\ \mathrm{x}_{2} \\ \mathrm{x}_{3}\end{array}\right)$.
$=x_{1}{ }^{2}+2 x_{1} x_{2}+3 x_{1} x_{3}+4 x_{2} x_{1}+5 x_{2}^{2}+6 x_{2} x_{3}+7 x_{3} x_{1}+8 x_{3} x_{2}+9 x_{3}{ }^{2}$.
$=x_{1}{ }^{2}+6 x_{1} x_{2}+10 x_{1} x_{3}+x_{2}{ }^{2}+14 x_{2} x_{3}+7 x_{3} x_{1}+9 x_{3}{ }^{2}$.

This quadratic form is equal to the quadratic form $\mathrm{x}^{\mathrm{T}} \mathrm{B} x$ where $\mathrm{B}=\left(\begin{array}{lll}5 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9\end{array}\right)$
which is a symmetric matrix.

If D is a diagonal matrix then the quadratic form associated with D is called a diagonal quadratic form, that is if $\mathrm{D}=\operatorname{diag}\left(\mathrm{a}_{11}, \mathrm{a}_{22}, \ldots, \mathrm{a}_{n n}\right)$ then $\mathrm{x}^{\mathrm{T}} \mathrm{D} x=\mathrm{a}_{11} \mathrm{x}_{1}{ }^{2}+$ $\mathrm{a}_{22} \mathrm{x}_{2}{ }^{2}+\ldots+\mathrm{a}_{\mathrm{nn}} \mathrm{x}_{\mathrm{n}}{ }^{2}$. This is also called the canonical representation of a quadratic
form. The theorem below says that every quadratic form has a canonical representation.

Theorem 10.2.2: every quadratic form $\mathrm{x}^{\mathrm{T}} \mathrm{A} \mathrm{x}$ can be reduced to a diagonal quadratic form $\mathrm{y}^{\mathrm{T}} \mathrm{D}$ y through a non-singular transformation $\mathrm{Px}=\mathrm{y}$, that is, P is a non-singular matrix.

The above theorem says that, for $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)^{T}$ and $y=\left(y_{1}, y_{2}, \ldots, y_{n}\right)^{T}$, variables $x_{1}, x_{2}, \ldots, x_{n}$ in $x^{T} A x$ can be changed to $y_{1}, y_{2}, \ldots, y_{n}$ through $P x=y$, $P$ is a non-singular matrix, so that $x^{T} A x=y^{T} D y$, where $D$ is a diagonal matrix.

We shall explain the above result through some examples.

Example 10.2.2: We reduce the quadratic forms (a) $4 x_{1}{ }^{2}+x_{2}{ }^{2}+9 x_{3}{ }^{2}-4 x_{1} x_{2}+$ $12 x_{1} x_{3}$ and (b) $x_{1} x_{2}+x_{2} x_{3}+x_{3} x_{1}$ to diagonal forms.

For (a), $4 x_{1}{ }^{2}+x_{2}{ }^{2}+9 x_{3}{ }^{2}-4 x_{1} x_{2}+12 x_{1} x_{3}$
$=4\left\{\mathrm{x}_{1}{ }^{2}+\mathrm{x}_{1}\left(3 \mathrm{x}_{3}-\mathrm{x}_{2}\right)\right\}+\mathrm{x}_{2}{ }^{2}+9 \mathrm{x}_{3}{ }^{2}$
$=4\left\{x_{1}{ }^{2}+2 \cdot x_{1} \cdot \frac{3 x_{3}-x_{2}}{2}+\left(\frac{3 x_{3}-x_{2}}{2}\right)^{2}\right\}+x_{2}{ }^{2}+9 x_{3}{ }^{2}-4\left(\frac{3 x_{3}-x_{2}}{2}\right)^{2}$
$=4\left(x_{1}+\frac{3 x_{3}-x_{2}}{2}\right)^{2}+x_{2}{ }^{2}+9 x_{3}{ }^{2}-9 x_{3}{ }^{2}+x_{2}{ }^{2}+6 x_{2} x_{3}$.
$=\left(2 x_{1}+3 x_{3}-x_{2}\right)^{2}+6 x_{2} x_{3}$.

We change the variables as: $x_{1}=y_{1}, x_{2}=y_{2}$, and $x_{3}=y_{2}+y_{3}$. Then the above expression $\left(2 x_{1}+3 x_{3}-x_{2}\right)^{2}+6 x_{2} x_{3}$

$$
\begin{aligned}
& =\left(2 y_{1}+3 y_{2}+3 y_{3}-y_{2}\right)^{2}+6 y_{2}\left(y_{2}+y_{3}\right) \\
& =\left(2 y_{1}+2 y_{2}+3 y_{3}\right)^{2}+6\left\{y_{2}{ }^{2}+2 y_{2} \frac{y_{3}}{2}+\left(\frac{y_{\mathrm{a}}}{2}\right)^{2}\right\}-6\left(\frac{y_{\mathrm{a}}}{2}\right)^{2} \\
& =\left(2 y_{1}+2 y_{2}+3 y_{3}\right)^{2}+6\left(y_{2}+\frac{y_{3}}{2}\right)^{2}-\frac{6}{4} y_{3}^{2} \\
& =\left(2 y_{1}+2 y_{2}+3 y_{3}\right)^{2}+\frac{6}{4}\left\{\left(2 y_{2}+y_{3}\right)^{2}-y_{3}^{2}\right\}
\end{aligned}
$$

Finally changing the variables as $2 \mathrm{y}_{1}+2 \mathrm{y}_{2}+3 \mathrm{y}_{3}=\mathrm{z}_{1}, 2 \mathrm{y}_{2}+\mathrm{y}_{3}=\mathrm{z}_{2}$ and $\mathrm{y}_{3}=\mathrm{z}_{3}$, we get the above quadratic form is $z_{1}{ }^{2}+\frac{3}{2}\left(z_{2}{ }^{2}-z_{3}{ }^{2}\right)$ which is in diagonal form. Here the transformation $\mathrm{Px}=\mathrm{z}$ is non-singular, because here P is the non-singular the matrix $\left(\begin{array}{ccc}2 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & -1\end{array}\right)$.

For the (b) part the quadratic form is $x_{1} x_{2}+x_{2} x_{3}+x_{3} x_{1}$. Here no square term is there and since the $1^{\text {st }}$ non-zero term is $x_{1} x_{2}$, we change the variables to $x_{1}=y_{1}$, $x_{2}=y_{1}+y_{2}$ and $x_{3}=y_{3}$. So this form is $y_{1}\left(y_{1}+y_{2}\right)+\left(y_{1}+y_{2}\right) y_{3}+y_{1} y_{3}$ $=y_{1}{ }^{2}+y_{1} y_{2}+y_{1} y_{3}+y_{2} y_{3}+y_{1} y_{3}$
$=y_{1}{ }^{2}+y_{1}\left(y_{2}+2 y_{3}\right)+y_{2} y_{3}$
$=\left\{\mathrm{y}_{1}{ }^{2}+2 \cdot \mathrm{y}_{1}\left(\frac{\mathrm{y}_{2}+2 \mathrm{y}_{\mathrm{a}}}{2}\right)+\left(\frac{\mathrm{y}_{2}+2 \mathrm{y}_{3}}{2}\right)^{2}\right\}-\left(\frac{\mathrm{y}_{2}+2 \mathrm{y}_{3}}{2}\right)^{2}+\mathrm{y}_{2} \mathrm{y}_{3}$.
$=\left(y_{1}+\frac{y_{2}+2 y_{3}}{2}\right)^{2}-\frac{1}{4}\left\{y_{2}{ }^{2}+4 y_{3}{ }^{2}+4 y_{2} y_{3}\right\}+y_{2} y_{3}$.
$=\frac{1}{4}\left(2 y_{1}+y_{2}+2 y_{3}\right)^{2}-\frac{1}{4} y_{2}{ }^{2} y_{3}{ }^{2}$.

Finally replacing $2 y_{1}+2 y_{2}+3 y_{3}=z_{1}$, and $y_{2}=z_{2}$ and $y_{3}=z_{3}$ the above form will reduce to $\frac{1}{4}\left(\mathrm{z}_{1}{ }^{2}-\mathrm{z}_{2}{ }^{2}\right)-\mathrm{z}_{3}{ }^{2}$. Here also the transformation $\mathrm{Px}=\mathrm{z}$ is non-singular as the matrix $P$ is $\left(\begin{array}{ccc}1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1\end{array}\right)$, which is non-singular.

### 10.3 Classification of Quadratic Forms

Quadratic forms are classified into several categories according to their range. These are given below.

Definition 10.3.1: A quadratic form $\mathrm{Q}=\mathrm{x}^{\mathrm{T}} \mathrm{A} \mathrm{x}$ is said to be
(i) Negative definite if $\mathrm{Q}<0$ for $\mathrm{x} \neq 0$.
(ii) Negative semi-definite if $\mathrm{Q} \leq 0$ for all x and $\mathrm{Q}=0$ for some $\mathrm{x} \neq 0$.
(iii) Positive definite if $\mathrm{Q}>0$ for $\mathrm{x} \neq 0$.
(iv) Positive semi-definite if $\mathrm{Q} \geq 0$ for all x and $\mathrm{Q}=0$ for some $\mathrm{x} \neq 0$.
(v) Indefinite if $\mathrm{Q}>0$ for some x and $\mathrm{Q}<0$ for other x .

Since there is one to one correspondence between real symmetric matrices and quadratic forms similar kind of classification is also there for the symmetric matrices. A real symmetric matrix A belongs to a class if the corresponding quadratic form $\mathrm{x}^{\mathrm{T}} \mathrm{A} \mathrm{x}$ belong to the same class.

Example 10.3.1: The form $Q_{1}=-x_{1}{ }^{2}-2 x_{2}{ }^{2}$ is a negative definite form where as: $Q_{2}=-x_{1}{ }^{2}+2 x_{1} x+x_{2}{ }^{2}$ is a negative semi-definite because $Q_{2}=-\left(x_{1}-x_{2}\right)^{2}$ which is always negative and also takes value zero for $x_{1}=x_{2} \neq 0$. The form $Q_{3}=2 x_{1}{ }^{2}+$
$3 x_{2}{ }^{2}$ is positive definite where as: $Q_{4}=x_{1}{ }^{2}-2 x_{1} x_{2}+x_{2}{ }^{2}$ is positive semi-definite. Finally $\mathrm{Q}_{5}=\mathrm{x}_{1}{ }^{2}-\mathrm{x}_{2}{ }^{2}$ is an indefinite form.

### 10.4 Rank and Signature of a Quadratic Form

To define rank and signature of a quadratic form we use its diagonal representation as given below.

For a real symmetric matrix $A$ let $P(A)$ and $N(A)$ be the numbers of positive and negative diagonal entries in any diagonal form to which $x^{T} A x$ is reduce through a non-singular transformation. The number $\mathrm{P}(\mathrm{A})-\mathrm{N}(\mathrm{A})$ is called the signature of the quadratic form $\mathrm{x}^{\mathrm{T}} \mathrm{A} x$. However rank of the matrix A is called the rank of the form $\mathrm{x}^{\mathrm{T}} \mathrm{A}$ x.

The quadratic form in example 10.2.2(a) has signature equal to 1 where as that in example 10.2.2(b) has signature -1 .

The classification of quadratic forms can also be done according to their rank and signatures as given in the theorem below.

Theorem 10.4.1: Let $\mathrm{Q}=\mathrm{x}^{\mathrm{T}} \mathrm{A} \mathrm{x}$ be an n variable quadratic form with rank r and signature s then Q is
(i) Positive definite if and only if $\mathrm{s}=\mathrm{n}$.
(ii) Positive semi-definite if and only if $\mathrm{r}=\mathrm{s}$.
(iii) Negative definite if and only if $\mathrm{s}=-\mathrm{n}$.
(iv) Negative semi-definite if and only if $\mathrm{r}=-\mathrm{s}$.
(v) Indefinite if and only if $|\mathrm{s}|<\mathrm{r}$.

The following is an important result on non-singular transformation of quadratic forms.

Theorem 10.4.2: Two quadratic forms on the same number of variables can be obtained from each other through a non-singular transformation if and only if they have the same rank and signature.

### 10.5 Hermitian Forms

The complex analogue of real quadratic form is known as Hermitian form. Here all vectors as well as matrices are taken as complex.

For a vector x in and a hermitian matrix A , the expression $\overline{\mathrm{x}}^{\mathrm{T}} \mathrm{A} \mathrm{x}$ is called a Hermitian form where $\overline{\mathrm{x}}$ is complex conjugate of x . Notice that if x and A are real then Hermitian form will be a quadratic form only.

Although the vector x and the matrix A are complex, the Hermitian form always takes real value that can be seen in the theorem below.

Theorem 10.5.1: A Hermitian form takes real values only.

Proof: Let $\mathrm{H}=\mathrm{x}^{\mathrm{T}} \mathrm{A} \mathrm{x}$ be a Hermitian form. Complex conjugate of H is $\bar{H}=\overline{\left(\bar{x}^{T} A x\right)}=\overline{\left(\bar{x}^{T}\right)} \bar{A} \bar{x}=x^{T} \bar{A} \bar{x}$.

Since $H$ is a scalar, $H=H^{T}=\left(\bar{x}^{T} A x\right)^{T}=x^{T} A^{T} \bar{x}$. Since $A$ is Hermitian $\bar{A}=\bar{A}$, so $\bar{H}=x^{T} \bar{A} \bar{x}=x^{T} A^{T} \bar{x}=H^{T}=H$. Therefore $A$ is real.

Example 10.5.1: Consider a Hermitian matrix $A=\left(\begin{array}{cc}2 & 3+i \\ 3-i & 1\end{array}\right)$. The Hermitian form associated with this is
$\mathrm{H}=\overline{\mathrm{x}}^{\mathrm{T}} \mathrm{Ax}=\left(\overline{\mathrm{x}_{1}}, \overline{\mathrm{x}_{2}}\right)\left(\begin{array}{cc}2 & 3+\mathrm{i} \\ 3-\mathrm{i} & 1\end{array}\right)\binom{\mathrm{x}_{1}}{\mathrm{x}_{2}}$.
$=2 x_{1} \overline{x_{1}}+(3+i) x_{2} \overline{x_{1}}+(3-i) \overline{x_{2}} x_{1}+x_{2} \overline{x_{2}}$.
$=2\left|\mathrm{x}_{1}\right|^{2}+2 \operatorname{Re}\left\{(3+\mathrm{i}) \mathrm{x}_{2} \overline{\mathrm{x}_{1}}\right\}+\left|\mathrm{x}_{2}\right|^{2}$.
which is a real number.

### 10.6 Conclusions

Vast literature is there on quadratic forms, to know them on should do further reading. Quadratic forms occur naturally in the study of conics and quadrics in geometry.

Keywords: Quadratic forms, positive definite matrix, negative definite matrix, rank and signature, Hermitian forms.

## Suggested Readings:

Linear Algebra, Kenneth Hoffman and Ray Kunze, PHI Learning pvt. Ltd., New Delhi, 2009.

Linear Algebra, A. R. Rao and P. Bhimasankaram, Hindustan Book Agency, New Delhi, 2000.

Linear Algebra and Its Applications, Fourth Edition, Gilbert Strang, Thomson Books/Cole, 2006.

Matrix Methods: An Introduction, Second Edition, Richard Bronson, Academic press, 1991.


## e-course Linear Algebra problems

1. Identify the following matrices as symmetric, skew-symmetric, Hermitian, skew-Hermitian or none?
(a) $\left(\begin{array}{lll}2 & 0 & 2 \\ 5 & 1 & 0 \\ 0 & 6 & 3\end{array}\right)$
(b) $\left(\begin{array}{ccc}2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3\end{array}\right)$
(c) $\left(\begin{array}{ccc}0 & 5 & -1 \\ -5 & 0 & -1 \\ 1 & 1 & 0\end{array}\right)$
(d) $\left(\begin{array}{ccc}1 & -1 & i \\ -1 & 0 & 1-i \\ -i & 1+i & 2\end{array}\right)$
(e) $\left(\begin{array}{ccc}i & -i & 3+i \\ -i & i & 0 \\ -3+i & 0 & 3\end{array}\right)$
2. Whether the system below is consistent? Justify.

$$
\begin{aligned}
& x+2 y-3 z=1 \\
& 3 x-y+2 z=5 \\
& 5 x+3 y-4 z=2
\end{aligned}
$$

3. Solve

$$
\begin{aligned}
x+2 y-3 z+2 w & =2 \\
2 x+5 y-8 z+6 w & =5 \\
3 x+4 y-5 z+2 w & =4
\end{aligned}
$$

4. Find rank of the matrix given below.

$$
\left(\begin{array}{llll}
1 & 2 & -3 & 0 \\
2 & 4 & -2 & 2 \\
3 & 6 & -4 & 3
\end{array}\right)
$$

5. Check whether the following are vector spaces?
(a) Let $V$ be the set of all real polynomials of degree $\geq 5$, with the usual addition and scalar multiplication.
(b) Let $V$ be the set of all nonzero real numbers with addition defined as $x+y=x y$ and scalar multiplication defined as $\alpha x=x$.
6. In the following, find out whether $S$ forms a subspace of $V$ ?
(a) $V=R^{3}, S=\left\{\left(x_{1}, x_{2}, x_{3}\right): x_{1}+5 x_{2}+3 x_{3}=0\right\}$
(b) $V=R^{3}, S=\left\{\left(x_{1}, x_{2}, x_{3}\right): x_{1}+5 x_{2}+3 x_{3}=1\right\}$
(c) $V=R^{3}, S=\left\{\left(x_{1}, x_{2}\right): x_{1} \geq 0, x_{2} \geq 0\right\}$
(d) $V=P(R)$, the set of all polynomials over reals and $S=\{p(x) \in P(R): P(5)=0\}$
(e) $V=R^{n}, S=\left\{\left(x_{1}, x_{2}, \ldots, x_{n}\right): x_{1}=x_{2}\right\}$
(f) $V=R^{3}, S=\left\{\left(x_{1}, x_{2}, \ldots, x_{n}\right): x_{1}^{2}=x_{2}^{2}\right\}$.
7. Prove or disprove:
(a) Union of two subspaces of $V$ is a subspace of $V$.
(b) Intersection of any number of subspaces is a subspace.
8. If $x, y, z$ are linearly independent vectors then whether $x+y, y+z, z+x$ are linearly independent?
9. For what values of $k$, do the vectors in the set $\{(0,1, k),(k, 1,0),(1, k, 1)\}$ form a basis for $R^{3}$ ?
10. Check whether the following set of vectors are linearly dependent or independent.
(a) $S=\{(1,2,-2,-1),(2,1,-1,4),(-3,0,3,-2)\}$
(b) $S=\{(1,3,-2,5,4),(1,4,1,3,5),(1,4,2,4,3),(2,7,-3,6,13)\}$
11. Determine whether or not the following form a basis for $\mathbb{R}^{3}$ ?
(a) $\{(1,1,1),(1,-1,5)\}$
(b) $\{(1,1,1),(1,2,3),(2,-1,1)\}$
(c) $\{(1,2,3),(1,0,-1),(3,-1,0),(2,1,-2)\}$
(d) $\{(1,1,2),(1,2,5),(5,3,4)\}$
12. Let $W$ be a subspace of $\mathbb{R}^{5}$ generated by the vectors in $10(\mathrm{~b})$. Find dimension and a basis for it.
13. Applying Gauss Jordan elimination method find inverse of the matrix
$A=\left(\begin{array}{ccc}2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3\end{array}\right)$
14. Whether $f$ is a linear transformation in each of the following? If yes then whether it is as isomorphism?
(a) $f: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}, f\left(x_{1}, x_{2}\right)=\left(x_{1}+x_{2}, x_{1} x_{2}\right)$.
(b) $f: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}, f\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{2}, x_{1}, 0\right)$.
(c) $f: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}, f\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{1}, x_{3}, x_{1}\right)$.
(d) $f: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}, f\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{1}-2, x_{2}-4, x_{3}\right)$.
15. Let $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2}$ be a linear transformation defined by $T\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{1}-x_{2}, x_{1}+x_{3}\right)$. Find the matrix of $T$ with respect to the basis $\left\{u_{1}, u_{2}, u_{3}\right\}$ of $\mathbb{R}^{3}$ and $\left\{u_{1}^{\prime}, u_{2}^{\prime}\right\}$ of $\mathbb{R}^{2}$ respectively, where $u_{1}=(1,-1,0), u_{2}=(2,0,1), u_{3}=(1,2,1), u_{1}^{\prime}=(-1,0)$ and $u_{2}^{\prime}=(0,1)$.
16. For the system

$$
\begin{array}{ll}
x+2 y-z & =0 \\
2 x+5 y+2 z & =0 \\
x+4 y+7 z & =0 \\
x+3 y+3 z & =0
\end{array}
$$

find the solution space as well as its dimension.
17. Consider the matrix $A=\left(\begin{array}{ccc}2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4\end{array}\right)$

For this find all eigenvalues and a basis for each eigenspace. Is $A$ diagonalizable?
18. Applying Cayley-Hamilton theorem find inverse of the matrix

$$
\left(\begin{array}{ccc}
1 & 2 & 0 \\
-1 & 1 & 2 \\
1 & 2 & 1
\end{array}\right)
$$

19. Find the symmetric matrix of the quadratic form $2 x_{1}^{2}+2 x_{1} x_{2}-6 x_{2} x_{3}-x_{2}^{2}$
20. Check whether the matrices below are positive definite or positive semi-definite?
(i) $\left(\begin{array}{ccc}10 & 2 & 0 \\ 2 & 4 & 6 \\ 0 & 6 & 10\end{array}\right)$.
(ii) $\left(\begin{array}{ccc}8 & 2 & -2 \\ 2 & 8 & -2 \\ -2 & -2 & 11\end{array}\right)$.
(iii) $\left(\begin{array}{ccc}3 & 10 & -2 \\ 10 & 6 & 8 \\ -2 & 8 & 12\end{array}\right)$.

## Answer and Hints

1. (a) none (b)symmetric (c) skew-symmetric (d) Hermitian (e) skew-Hermitian
2. Not consistent. Check that rank of the co-efficient matrix is 2 where as that of the augmented matrix is 3 .
3. Check that the system is consistent, where the rank of both the co-efficient matrix and augmented matrix is 2 . In echelon form the system is

$$
\begin{aligned}
x+2 y-3 z+2 w & =2 \\
y-2 z+2 w & =1
\end{aligned}
$$

Taking $z$ and $w$ as free variables, i.e., $z=\alpha, w=\beta$, we get the set of all solutions is $\{(-\alpha+2 \beta, 1+$ $2 \alpha-2 \beta, \alpha, \beta): \alpha, \beta \in \mathbb{R}\}$.
4. Making elementary row operations $R_{2} \rightarrow-2 R_{1}+R_{2}, R_{3} \rightarrow-3 R_{1}+R_{3}, R_{3} \rightarrow-5 R_{2}+4 R_{3}$, get an echelon form $\left(\begin{array}{cccc}1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2\end{array}\right)$

Thus rank of the given matrix is 3 .
5. (a) Not a vector space because zero vector is not there.
(b) Yes, it is a vector space as it satisfy all the axioms. Here 1 is the zero vector, and for any vector x its negative vector is $\frac{1}{x}$.
6. (a) yes, (b) neither closed under addition nor under scalar multiplication, (c) not closed under scalar multiplication, (d) yes, (e) yes, (f) not closed under addition.
7. (a) No, Counter Example: $V=\mathbb{R}^{2}, S_{1}=\left\{\left(x_{1}, x_{2}\right): x_{1}=x_{2}\right\}, S_{2}=\left\{\left(x_{1}, x_{2}\right): x_{1}+2 x_{2}=0\right\}$. $(1,1) \in S_{1},(-2,1) \in S_{2}$ but $(1,1)+(-2,1)=(-1,2) \in S_{1}, S_{2}$.
(b) Yes. Let $S_{i}(i=1,2, \cdots)$ be subspaces of $V$ and $S=\cap_{i=1}^{\infty} S_{i} . x, y \in S \Rightarrow x, y \in S_{i} \forall i$. Then $x+y \in S_{i} \forall i$ and so $x+y \in S$. Similarly $S$ is closed under scalar multiplication. So (b) is true.
8. Yes, linearly independent.
9. Taking scalar multiplication of the vectors and equating to 0 one gets the system in $\alpha, \beta, \gamma$, $\beta k+\gamma=0$
$\alpha+\beta+\gamma k=0$
$\alpha k+\gamma=0$. An echelon form of the system is $\left(\begin{array}{ccc}1 & 1 & k \\ 0 & k & 1 \\ 0 & 0 & 2-k^{2}\end{array}\right)$
The system should have unique solution and hence rank of this matrix is 3 . So $k^{2} \neq 2$. $k$ can not be equal to zero otherwise the set will have only two vectors. So $k$ can be any real number other than 0 and $\pm \sqrt{2}$.
10. (a) Echelon form of the corresponding matrix $\left(\begin{array}{cccc}1 & 2 & -2 & -1 \\ 2 & 1 & -1 & 4 \\ -3 & 0 & 3 & -2\end{array}\right)$ is $\left(\begin{array}{cccc}1 & 2 & -2 & -1 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & -3 & -7\end{array}\right)$. So the given set of vectors are linearly independent.
(b) Echelon form of the corresponding matrix

$$
\left(\begin{array}{ccccc}
1 & 3 & -2 & 5 & 4 \\
1 & 4 & 1 & 3 & 5 \\
1 & 4 & 2 & 4 & 3 \\
2 & 7 & -3 & 6 & 3
\end{array}\right) \text { is }\left(\begin{array}{ccccc}
1 & 3 & -2 & 5 & 4 \\
0 & -1 & -3 & 2 & -1 \\
0 & 0 & 1 & 1 & -2 \\
0 & 0 & 0 & 0 & 0
\end{array}\right) \text {. So the given set of vectors are linearly dependent. }
$$

11. (a) No, because $\operatorname{dim} \mathbb{R}^{3}=3$.
(b) Yes, because the set is linearly independent.
(c) No, because it contains more than 3 vectors.
(d) No, because it is linearly dependent.
12. First 3 rows of the echelon form in $10(\mathrm{~b})$ forms a basis for $W$. Therefore $\operatorname{dim} W=3$.
13. Consider $(A \mid I)=\left(\begin{array}{ccc|ccc}2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1\end{array}\right)$.

Apply each of these elementary row operations in the updated matrix $R_{1} \rightarrow \frac{1}{2} R_{1}, R_{2} \rightarrow R_{2}-5 R_{1}$, $R_{3} \rightarrow R_{3}+R_{2}, R_{1} \rightarrow R_{1}+R_{3}, R_{2} \rightarrow-R_{2}, R_{2} \rightarrow R_{2}-5 R_{3}, R_{3} \rightarrow 2 R_{3}$ and get
$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2\end{array}\right)$. So, $A^{-1}=\left(\begin{array}{ccc}3 & -1 & -1 \\ -15 & 6 & -5 \\ 5 & -2 & 2\end{array}\right)$.
14. (a) No. (b) Yes; not an isomorphism. (c)Yes; an isomorphism. (d) No
15. $T\left(u_{1}\right)=(2,1)=-2(-1,0)+1(0,1)$
$T\left(u_{2}\right)=(2,3)=-2(-1,0)+3(0,1)$
$T\left(u_{3}\right)=(-1,2)=1(-1,0)+2(0,1)$. So, answer is $\left(\begin{array}{ccc}-2 & -2 & 1 \\ 1 & 3 & 2\end{array}\right)$.
16. The system in echelon form is

$$
\begin{array}{r}
x+2 y-z=0 \\
y+4 z=0
\end{array}
$$

The solution space is $\{(9 \alpha,-4 \alpha, \alpha): \alpha \in \mathbb{R}\}$. It's dimension is 1 .
17. Eigenvalues are $2,2,3$. Basis for eigenspace corresponding to 2 and 3 are $\{(1,0,0)\}$ and $\{(1,1,-2)\}$ respectively. The matrix is not diagonalizable beacuse sum of dimension of eigenspaces is not equal to 3 .
18. Characteristic polynomial is $-\lambda^{3}+3 \lambda^{2}-\lambda+3$. So $A^{-1}=\frac{1}{3}\left(A^{2}-3 A+I\right)=\frac{1}{3}\left(\begin{array}{ccc}-3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3\end{array}\right)$
19. $\left(\begin{array}{ccc}2 & 1 & 0 \\ 1 & -1 & 3 \\ 0 & 3 & 0\end{array}\right)$
20. (i) Positive semi-definite.
(ii) Positive definite. (iii) Neither of them.

## Lesson 11

## Limit, Continuity, Derivative of Function of Complex Variable

### 11.1 Introduction

First we introduce some basic notations and terminology for the set of complex numbers as a metric space.

### 11.1.1 Circle, Disk and Annulus

Let $|z|=1$ be the unit circle and let $|z-a|=\rho$ denote the circle of radius $\rho$ and centre $a .|z-a|<\rho$ denotes the interior of the circle of radius $\rho$ and centre $a$. It is also called an open circular disk. Similarly $|z-a| \leq \rho$ is the closed circular disk and $|z-a|>\rho$ is the exterior of the circle.

The open circular disk $|z-a|<\rho$ is also called a neighbourhood of $a$.
Also $\rho_{1}<|z-a|<\rho_{2}$ denotes an open annulus or a circular ring.

### 11.1.2 Half-Planes

The following notations are used for half-planes:
(i) $\{z=x+i y: y>0\} \rightarrow$ upper half-plane
(ii) $\{z=x+$ iy: $y<0\} \rightarrow$ lower half-plane
(iii) $\{z=x+i y: x>0\} \rightarrow$ right half-plane
(iv) $\{z=x+$ iy: $x<0\} \rightarrow$ the left half-plane

### 11.1.3 Interior, Exterior and Boundary Points

A point $z_{0}$ is said to be an interior point of a set $D$ if there is a neighbourhood of $z_{0}$ that is entirely contained in $D$.

A point $z_{0}$ is called an exterior point of a set D if there is a neighbourhood of $z_{0}$ which does not have any point of $D$.

A point $z_{0}$ is called a boundary point of a set $D$, if every neighbourhood of $z_{0}$ contains points of $D$ as well as points of $D^{C}$.
11.1.3.1 Example: The boundary of the sets, $|z| \leq 1$ or $|z|<1$ is $|z|=1$.

### 11.1.4 Open and Closed Sets

A set $D$ is said to be an open set if all its points are interior points. For example, the open circular disk, the right half-plane etc. are open sets.

A set is closed if it contains all its boundary points. The closure of a set $D$ is the closed set consisting of all points in $D$ together with the boundary of $D$.
11.1.4.1 Example: The set $\{z:|z| \leq \rho\}$ is a closed set.
11.1.4.2 Example: The set $\{z: 0<|z| \leq 1\}$ is neither open nor closed.
11.1.4.3 Example: The set of all complex numbers is both open and closed.

### 11.1.5 Connected Sets, Bounded Sets, Domain

An open set $D$ is said to be connected if each pair of points $z_{1}$ and $z_{2}$ can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in $D$.
11.1.5.1 Example: The open set $\{z:|z|<1\}$ is connected.

Example: The open ring $\{z: 1<|z|<2\}$ is connected.
An open connected set is called a domain. Any neighbourhood is a domain. A domain together with some, none or all of its boundary points is called a region. A set D is closed if and only if its complement is open.

A set $D$ is bounded if every point of $D$ lies inside some circle $|z|=R$, otherwise it is unbounded.

A simple closed path is a closed path that does not intersect or touch itself. A simply connected domain D in the complex plane is a domain such that every simple closed path in D enclosed only points of D . A domain that is not simply connected is called multiply connected.
11.1.5.2 Example: The set $\{z: 1<|z|<2\}$ is bounded whereas right half plane is unbounded.

### 11.1.6 Examples

1. $|z-2+i| \leq 1$ closed, bounded
2. $|2 z+3|>4$ open, connected set, unbounded
3. $\operatorname{Im} z>1$ open, connected, unbounded
4. $\operatorname{Im} z=1$
5. $0 \leq \arg z \leq \frac{\pi}{4},(z \neq 0)$
6. $|z-4| \geq|z|$
7. $|\operatorname{Re} z| \leq|z|$
8. $\operatorname{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$
9. $\operatorname{Re}\left(z^{2}\right)>0$

### 11.2 Function

Let $D$ be a set of complex numbers. A function $f$ defined on $D$ is a rule that assigns to each $z$ in D a complex number $w$. The number $w$ is called the value of $f$ at $w$ and is denoted by $f(z)$; that is $w=f(z)$. The set $D$ is called the domain of definition of $f$. The set of all values of a function $f$ is called the range of $f$. Suppose that $w=u+i v$ is the value of a function $f$ at $z=x+i y$, so that $u+i v=f(x+i y)$.

Each of the real numbers $u$ and $v$ depends on real variable $x$ and $y$, and so it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables $x$ and $y$ :

$$
f(z)=u(x, y)+i v(x, y)
$$

Converse is not true, i.e., given two real functions $(x, y)$ we may not be able to define a complex function of $z=x+i y$ in an explicit form, for example, $w=(2 x+y)+i(6 x y)$.
11.2.1 Function in Polar Form: If the polar co-ordinates $r$ and $\theta$ are used then $u+i v=f\left(r e^{i \theta}\right)$, where $w=u+i v \quad$ and $\quad z=r e^{i \theta}$. So we may write $f(z)=u(r, \theta)+i v(r, \theta)$.
11.2.2 Example: If $f(z)=z^{2}$, then $f(x+i y)=(x+i y)^{2}=\left(x^{2}-y^{2}\right)+2 i x y$

Hence $u(x, y)=x^{2}-y^{2}, v(x, y)=2 x y$.
When polar co-ordinates are used,
$f\left(r e^{i \theta}\right)=\left(r e^{i \theta}\right)^{2}=r^{2} e^{2 i \theta}=r^{2} \cos 2 \theta+i r^{2} \sin 2 \theta$. Consequently,
$u(r, \theta)=r^{2} \cos 2 \theta, v(r, \theta)=r^{2} \sin 2 \theta$. If $v$ is always zero then $f$ is a real-valued function of a complex variable. For example, $f(z)=|z|^{2}=\left(x^{2}+y^{2}\right)$.

### 11.2.3 Polynomial and Rational Functions

If $a_{0}, a_{1}, \ldots, a_{n}$ are complex numbers, $a_{n} \neq 0, n \geq 0$, then $P(z)=a_{0}+a_{1} z+\ldots+a_{n} z^{n}$ is a polynomial of degree $n$. The domain of $z$ is the entire complex plane. For example, $P(z)=1+2 z-3 z^{2}$.

Quotients $\frac{P(z)}{Q(z)}$ of polynomials are called rational functions and are defined at each point $z$, where $Q(z) \neq 0$. For example, $g(z)=\frac{2-z^{2}}{z+4 z^{3}}$.

### 11.2.4 Examples

1. Domain of definition of $f(z)=\frac{1}{z}$ is the entire complex plane excluding the origin.
2. Domain of definition of $f(z)=\frac{1}{1-|z|^{2}}$ is the entire complex plane excluding the circle $|z|=1$.

### 11.2.5 Multiple-Valued Function

If to each value of $z$, there are several values of $f(z), f$ is called a multiplevalued function. For example, if $w=z^{\frac{1}{n}}$, then $w$ may take any of $n$ values:

$$
w_{k}=z^{\frac{1}{n}}\left[\cos \left(\frac{\theta+2 k \pi}{n}\right)+i \sin \left(\frac{\theta+2 k \pi}{n}\right)\right]
$$

for $k=0,1, \ldots,(n-1)$. In such cases, we consider those parts of the domain in which the multiple-valued function behaves like a single-valued function. Each one of these single valued functions is called a branch of the multiple-valued function.

### 11.3 Limit of a Function

Let a function $f$ be defined in some domain $D$ containing $z_{0}$. We say that $\lim _{z \rightarrow z_{0}} f(z)=s$, if for every $\in>0$ there exists $\delta>0$ such that $|f(z)-s|<\epsilon$ whenever $\left|z-z_{0}\right|<\delta$.

### 11.3.1 Examples

1. $\lim _{z \rightarrow 2} \frac{i z}{3}=\frac{2 i}{3}$

$$
\left|\frac{i}{3}(z-2)\right|=\frac{1}{3}|z-2|<\frac{\delta}{3}<\in \text { whenever }|z-2|<\delta \text { and } \delta<3 \in .
$$

2. $\lim _{z \rightarrow 0} \frac{Z}{Z}$ does not exists, as along $(x, 0), \frac{Z}{\bar{Z}}=\frac{x}{x}=1$ and along ( $0, y$ ), $\frac{z}{\bar{z}}=\frac{i y}{-i y}=-1$.
3. $\lim _{z \rightarrow \infty}[\sqrt{z-3 i}-\sqrt{z+i}]=\lim _{z \rightarrow \infty} \frac{[(z-3 i)-(z+i)]}{\sqrt{z-3 i}+\sqrt{z+i}}$

$$
=\lim _{z \rightarrow \infty} \frac{-4 i}{\sqrt{z-3 i}+\sqrt{z+i}}=\lim _{u \rightarrow 0} \frac{-4 i \sqrt{u}}{\sqrt{1-3 i u}+\sqrt{1+i u}}=0 .
$$

11.3.2 Theorem: Suppose that $f(z)=u(x, y)+i v(x, y), z_{0}=x_{0}+i y_{0}$, and $w_{0}=u_{0}+i v_{0}$. Then $\lim _{z \rightarrow z_{0}} f(z)=w_{0} \quad$ if $\quad$ and $\quad$ only if $\lim _{(x, y) \rightarrow\left(x_{0}, y_{0}\right)} u(x, y)=u_{0}$ and $\lim _{(x, y) \rightarrow\left(x_{0}, y_{0}\right)} v(x, y)=v_{0}$.
11.3.3 Theorem: Suppose that $\lim _{z \rightarrow z_{0}} f(z)=\alpha_{0}$ and $\lim g(z)=\beta_{0}$. Then $\lim _{z \rightarrow z_{0}}[f(z) \pm g(z)]=\alpha_{0}+\beta_{0} \quad, \quad \lim _{z \rightarrow z_{0}}[f(z) g(z)]=\alpha_{0} \beta_{0}, \quad$ and $\quad$ if $\quad \beta_{0} \neq 0, \quad$ then $\lim _{z \rightarrow z_{0}} \frac{f(z)}{g(z)}=\frac{\alpha_{0}}{\beta_{0}}$.

### 11.3.4 Infinite Limits and Limit at Infinity

We say that $\lim _{z \rightarrow z_{0}} f(z)=\infty$ if for every positive $\in>0$, these exists $\delta>0$ such that $|f(z)|>\frac{1}{\in}$ whenever $\left|z-z_{0}\right|<\delta$.
11.3.4.1 Example: $\lim _{z \rightarrow-1} \frac{(i z+3)}{z+1}=\infty$

We say that $\lim _{z \rightarrow \infty} f(z)=w_{0}$ if for every $\in>0$ there exists $\delta>0$ such that $\left|f(z)-w_{0}\right|<\in$ whenever $|z|>\frac{1}{\delta}$. Equivalently, we can say that $\lim _{z \rightarrow 0} f\left(\frac{1}{z}\right)=w_{0}$.
11.3.4.2 Examples: $\lim _{z \rightarrow \infty} \frac{2 z+i}{z+1}=2$ (ii) $\lim _{z \rightarrow \infty} \frac{z}{2-i z}=i$

We say that $\lim _{z \rightarrow \infty} f(z)=\infty$ if for every $\in>0$, there exists $\delta>0$ such that $|f(z)|>\frac{1}{\epsilon}$ whenever $|z|>\frac{1}{\delta}$. One can alternatively say $\lim _{z \rightarrow 0} \frac{1}{f\left(\frac{1}{z}\right)}=0$.
11.3.4.3 Example: $\lim _{z \rightarrow \infty} \frac{2 z^{3}-1}{z+1}=\infty$

### 11.4 Continuous Function

A function $f$ is continuous at a point $z_{0}$ if $\lim _{z \rightarrow z_{0}} f(z)=f\left(z_{0}\right)$. Using the definition of limit, we define $f$ is continuous at $z$ if for every $\in>0$, there exists $\delta>0$ such that $\left|f(z)-f\left(z_{0}\right)\right|<\in$ whenever $\left|z-z_{0}\right|<\delta$.

Compositions of continuous functions are again continuous.
11.4.1 Remark: If $f(z)$ is continuous, let $g(z)=\overline{f(z)}$. Now $\left|g(z)-g\left(z_{0}\right)\right|=\overline{\left|f(z)-f\left(z_{0}\right)\right|}=\left|f(z)-f\left(z_{0}\right)\right|<\epsilon$, whenever $\left|z-z_{0}\right|<\delta$. So $g(z)$ is also continuous.

### 11.4.2 Examples

1. $f(z)=z^{3}$ is continuous on the whole complex plane.
2. $f(z)=\frac{\sin z}{1+z^{2}}$ is continuous except at $z= \pm i$.
3. $f(z)=\left\{\begin{array}{l}\frac{\operatorname{Im}(z)}{|z|}, z \neq 0 \\ 0, z=0\end{array}\right.$
4. $f(z)=\left\{\begin{array}{l}\frac{\operatorname{Re}\left(z^{2}\right)}{|z|^{2}}, z \neq 0 \\ 0, z=0\end{array}\right.$
are not continuous at $z=0$
5. $f(z)=\left\{\begin{array}{l}\frac{z^{2}+1}{z+i}, z \neq-i \\ 0, z=-i\end{array}\right.$ is not continuous at $z=-i$ as $\lim _{z \rightarrow-i} f(z)=-2 i \neq f(-i)$.

### 11.5 Differentiability of a Function

The derivative of a complex function $f$ at a point $z_{0}$ is defined by

$$
\lim _{\Delta z \rightarrow 0} \frac{f\left(z_{0}+\Delta z\right)-f\left(z_{0}\right)}{\Delta z}=f^{\prime}\left(z_{0}\right)
$$

provided the limit exists. Then the function $f$ is said to be differentiable at $z_{0}$.
11.5.1 Example: $f(z)=z^{2}$

$$
\lim _{\Delta z \rightarrow 0} \frac{\left(z_{0}+\Delta z\right)^{2}-\left(z_{0}\right)^{2}}{\Delta z}=\lim _{\Delta z \rightarrow 0}(\Delta z+2 z)=2 z
$$

11.5.2 Remark: It can be easily seen that the differentiability of a function at a point implies its continuity at that point.

General differentiation rules are the same as in real calculus such as $(c f)^{\prime}=c f^{\prime},(f+g)^{\prime}=f^{\prime}+g^{\prime},(f g)^{\prime}=f^{\prime} g+f g^{\prime}$.
$\left(\frac{f}{g}\right)^{\prime}=\frac{f^{\prime} g-f g^{\prime}}{g^{2}}$ provided $g$ does not vanish.

### 11.5.3 Examples:

1. $f(z)=\bar{z}$.
$\frac{f\left(z_{0}+\Delta z\right)-f\left(z_{0}\right)}{\Delta z}=\frac{\overline{\left(z_{0}+\Delta z\right)}-\overline{\left(z_{0}\right)}}{\Delta z}=\frac{\overline{\Delta z}}{\Delta z}=\frac{\Delta x-i \Delta y}{\Delta x+i \Delta y}$
Now for $\Delta y=0$ this value is +1 and for $\Delta x=0$, it is -1 . Hence $\lim _{\Delta z \rightarrow 0} \frac{f\left(z_{0}+\Delta z\right)-f\left(z_{0}\right)}{\Delta z}$ does not exist for any $z$. That is, $f|z|=\bar{z}$ is not differentiable at any point.
2. $f(z)=|z|^{2}=z \bar{Z}$

$$
\frac{f(z+\Delta z)-f(z)}{\Delta z}=\frac{(z+\Delta z)(\bar{z}+\overline{\Delta z})-z \bar{z}}{\Delta z}
$$

$=z \frac{\overline{\Delta z}}{\Delta z}+\bar{z}+\overline{\Delta z}$

Now for $z=0, \frac{f(0+\Delta z)-f(0)}{\Delta z}=\overline{\Delta z}$
which has limit 0 as $\Delta z \rightarrow 0$. Hence $|z|^{2}$ is differentiable at $z=0$. However for any $z \neq 0, \lim _{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist. Consequently $|z|^{2}$ is not differentiable at any other point.
3. $f(z)=\operatorname{Re}(z)$ is not differentiable for any $z$.
4. $f(z)=\operatorname{Im}(z)$ is not differentiable for any $z$.
5. $f(z)=z^{n}$
$\frac{(z+\Delta z)^{n}-z^{n}}{\Delta z}=\frac{1}{\Delta z}\left[\binom{n}{1} z^{n-1} \Delta z+\binom{n}{2} z^{n-2}(\Delta z)^{2}+\ldots+\binom{n}{n}(\Delta z)^{n}\right]$
$\rightarrow n z^{n-1}$ as $\Delta z \rightarrow 0$.

Hence $\frac{d}{d z}\left(z^{n}\right)=n z^{n-1}$ for all $z$.

## Suggested Readings

Ahlfors, L.V. (1979). Complex Analysis, McGraw-Hill, Inc., New York.
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Conway, J.B. (1993). Functions of One Complex Variable, Springer-Verlag, New York.

Fisher, S.D. (1986). Complex Variables, Wadsworth, Inc., Belmont, CA.
Jain, R.K. and Iyengar, S.R.K. (2002). Advanced Engineering Mathematics, Narosa Publishing House, New Delhi.

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## Lesson 12

## Analytic Function, Cauchy-Riemann Equations, Harmonic Functions

### 12.1 Analytic Functions

A function $f(z)$ is said to be analytic at a point $z_{0}$ if it is differentiable at $z_{0}$ and also at each point in some neighbourhood of $z_{0}$. The function $f$ is said to be analytic in a domain $D$, if it is analytic at every point in $D$.

Analytic functions are also called holomorphic functions.

### 12.1.1 Examples:

1. $f(z)=z^{n}, n$ a positive integer, is analytic at every point in the complex plane.
2. $p(z)=a_{0}+a_{1} z+\ldots+a_{n} z^{n}$ where $a_{0}, a_{1}, \ldots, a_{n}$ are complex constants is analytic at every point in the complex plane.
3. $f(z)=\frac{P(z)}{Q(z)}$, where $P$ and $Q$ are polynomials, is analytic at all points except where $Q(z)$ vanishes.

### 12.1.2 Entire Function

A function which is analytic at all points in the complex plane is called an entire function.

### 12.1.3 Examples:

1. Every polynomial is an entire function.
2. $f(z)=|z|^{2}$ is not analytic anywhere as it is differentiable only at $z=0$.

A function $f(z)$ is said to be analytic at $Z=\infty$ if $f\left(\frac{1}{z}\right)$ is analytic at $z=0$.
Let us write the function
$f(z)=u(x, y)+i v(x, y)$
and let $u_{x}, u_{y}, v_{x}, v_{y}$ denote the partial derivatives of $u$ and $v$ with respect to $x$ and $y$ respectively.

### 12.2 Cauchy-Riemann Equations

$$
\begin{equation*}
u_{x}=v_{y}, u_{y}=v_{x} \tag{12.2.1}
\end{equation*}
$$

12.2.1 Theorem: Let $f(z)=u(x, y)+i v(x, y)$ be defined and continuous in some neighbourhood of a point $z=x+i y$ and differentiable at $z$ itself. Then at that point the first order partial derivatives of $u$ and $v$ exist and satisfy the Cauchy-Riemann equations (12.2.1).

Hence, if $f(z)$ is analytic in a domain $D$, then partial derivatives exist and satisfy (12.2.1) at all points of $D$.

Proof: Given that $f^{\prime}(z)=\lim _{\Delta z \rightarrow 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ exists. This implies that $\lim _{(\Delta x, \Delta y) \rightarrow(0,0)} \frac{\{[u(x+\Delta x, y+\Delta y)+i v(x+\Delta x, y+\Delta y)]-[u(x, y)+i v(x, y)]\}}{(\Delta x+i \Delta y)}$ exists.

Hence along $(\Delta x, 0)$ and $(0, \Delta y)$ the limit should be same. Now along $(\Delta x, 0)$ the limit is

$$
\begin{gather*}
\lim _{\Delta x \rightarrow 0} \frac{(u(x+\Delta x, y)-u(x, y))+i(v(x+\Delta x, y)-v(x, y))}{\Delta x} \\
=u_{x}(x, y)+i v_{x}(x, y), \tag{12.2.2}
\end{gather*}
$$

since limit is assumed to exist.

Similarly along $(0, \Delta y)$ the limit is

$$
\begin{gather*}
\lim _{\Delta y \rightarrow 0} \frac{(u(x, y+\Delta y)-u(x, y))+i(v(x, y+\Delta y)-v(x, y))}{i \Delta y} \\
=i u_{y}(x, y)+v_{y}(x, y) \tag{12.2.3}
\end{gather*}
$$

Equating the real and imaginary parts in (12.2.2) \& (12.2.3), we get the CauchyRiemann equations.

### 12.2.2 Example:

1. Let $f(z)=\bar{z}=x-i y, u=x, v=-y$. It can be easily seen that $u_{x}=1, v_{y}=-1, u_{y}=0, v_{x}=0$. Hence the Cauchy-Riemann equations are not satisfied. So $f$ cannot be differentiable at any point.
2. Let $f(z)=\frac{1}{z}=\frac{x}{x^{2}+y^{2}}-i \frac{y}{x^{2}+y^{2}}, z \neq 0$.
$u_{x}=\frac{y^{2}-x^{2}}{\left(x^{2}+y^{2}\right)^{2}}=v_{y}, u_{y}=-\frac{2 x y}{\left(x^{2}+y^{2}\right)^{2}}=-v_{x}$ except at $z=0$. The function is nalytic everywhere except at $z=0$.
12.2.3 Theorem: If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ of two real variables $x$ and $y$ have continuous first order partial derivatives that
satisfy the Cauchy-Riemann equations in some domain $D$, then the complex function $w=f(z)=u(x, y)+i v(x, y)$ is analytic in $D$.

Proof: Consider a neighbourhood of $z$. Now the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous. Therefore, we can write
$\Delta u=u(x+\Delta x, y+\Delta y)-u(x, y)=u_{x} \Delta x+u_{y} \Delta y+\epsilon_{1} \Delta x+\epsilon_{2} \Delta y$,
and $\quad \Delta v=v(x+\Delta x, y+\Delta y)-v(x, y)=v_{x} \Delta x+v_{y} \Delta y+\epsilon_{3} \Delta x+\epsilon_{4} \Delta y$,
where $\in_{1}, \in_{2}, \in_{3}, \in_{4} \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$.

Now $\Delta w=f(z+\Delta z)-f(z)=\Delta u+i \Delta v$

$$
=\left(u_{x}+i v_{x}\right) \Delta x+\left(u_{y}+i v_{y}\right) \Delta y+\left(\epsilon_{1}+i \in_{3}\right) \Delta x+\left(\epsilon_{2}+i \epsilon_{4}\right) \Delta y
$$

If we apply Cauchy-Riemann equations, the above expression reduces to

$$
\begin{aligned}
\Delta w=\left(u_{x}+i v_{x}\right) \Delta x+ & \left(v_{x}+i u_{x}\right) \Delta y+\left(\epsilon_{1}+i \epsilon_{3}\right) \Delta x+\left(\epsilon_{2}+i \epsilon_{4}\right) \Delta y \\
& =\left(u_{x}+i v_{x}\right)(\Delta x+i \Delta y)+\left(\epsilon_{1}+i \epsilon_{3}\right) \Delta x+\left(\epsilon_{2}+i \epsilon_{4}\right) \Delta y .
\end{aligned}
$$

So $\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-\left(u_{x}+i v_{x}\right)\right| \leq\left|\left(\epsilon_{1}+i \in_{3}\right)\right|\left|\frac{\Delta x}{\Delta z}\right|+\left|\left(\epsilon_{2}+i \epsilon_{4}\right)\right|\left|\frac{\Delta y}{\Delta z}\right|$.

Using the fact that $\left|\frac{\Delta x}{\Delta z}\right| \leq 1 \&\left|\frac{\Delta y}{\Delta z}\right| \leq 1$, we get

$$
\lim _{\Delta z \rightarrow 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}=u_{x}+i v_{x}=u_{y}+i v_{y} .
$$

This proves that $f$ is differentiable at an arbitrary point in $D$ and so it is analytic in $D$.

### 12.2.4 Examples:

1. $f(z)=z^{3}=x^{3}-3 x y^{2}+i\left(3 x^{2} y-y^{3}\right)$ is analytic in $D$.
2. $f(z)= \begin{cases}\frac{(\bar{z})^{2}}{z}, & z \neq 0 \\ 0, & z=0 .\end{cases}$

Then Cauchy-Riemann equations are satisfied at $(0,0)$ but $f$ is not differentiable at $(0,0)$.
3. Let $f(z)$ be analytic in a domain $D$ and $|f(z)|=k$ for all $z \in D$. So writing $f(z)=u(x, y)+i v(x, y)$, we get $u^{2}+v^{2}=k^{2}$. Differentiating with respect to $x$ and $y$ we get

$$
\begin{align*}
u u_{x}+v v_{x} & =0  \tag{12.2.4}\\
\text { and } \quad u u_{y}+v v_{y} & =0 \tag{12.2.5}
\end{align*}
$$

Using $v_{x}=-u_{y}$ in the first equation and $v_{y}=u_{x}$ in the second equation, we get

$$
u u_{x}-v u_{y}=0 \text { and } u u_{y}+v u_{x}=0
$$

$\Rightarrow\left(u^{2}+v^{2}\right) u_{x}=0,\left(u^{2}+v^{2}\right) u_{y}=0$

If $u^{2}+v^{2}=k^{2}=0$ then $u=0=v$ and hence $f=0$.

If $k \neq 0$ then $u_{x}=u_{y}=0$, then $v_{x}$ and $v_{y}$ are also zero. So $u=$ const. , $v=$ const. This proves that $f$ is constant.

### 12.2.5 Polar Co-ordinates

Let $x=r \cos \theta, y=r \sin \theta$. Consider the function $w=f(z)$. If we write $z=x+i y$, then the real and imaginary parts of $w=u+i v$ are expressed in terms of the variables $x$ and $y$. Similarly, if we write $z=r e^{i \theta},(z \neq 0)$, the real and imaginary parts of $w=u+i v$ are expressed in terms $r$ and $\theta$. Assume the existence and continuity of the first-order partial derivatives of $u$ and $v$ with respect to $x$ and $y$ everywhere in some neighbourhood of a given non zero point $z_{0}$. Then the first order partial derivatives with respect to $r$ and $\theta$ will also exist and be continuous in some neighbourhood. Using the chain rule for differentiating real-valued functions of two real variables we obtain

$$
\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x} \frac{\partial u}{\partial r}+\frac{\partial u}{\partial y} \frac{\partial y}{\partial r}, \frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x} \frac{\partial u}{\partial \theta}+\frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta}
$$

so that $u_{r}=u_{x} \cos \theta+u_{y} \sin \theta, \quad u_{\theta}=-u_{x} r \sin \theta+u_{y} r \cos \theta$.

Similarly $v_{r}=v_{x} \cos \theta+v_{y} \sin \theta, \quad v_{\theta}=-v_{x} r \sin \theta+v_{y} r \cos \theta$.

If the partial derivatives with respect to $x$ and $y$ also satisfy the CauchyRiemann equations $u_{x}=v_{y}, u_{y}=-v_{x}$ at $z_{0}$, then equation (12.2.7) becomes

$$
\begin{equation*}
v_{r}=-u_{y} \cos \theta+u_{x} \sin \theta, v_{\theta}=u_{y} r \sin \theta+u_{x} r \cos \theta \tag{12.2.8}
\end{equation*}
$$

Comparing (12.2.6) and (12.2.8), we get

$$
\begin{equation*}
u_{r}=\frac{1}{r} v_{\theta} \text { and } v_{r}=-\frac{1}{r} u_{\theta} . \tag{12.2.9}
\end{equation*}
$$

12.2.6 Theorem: Let the function $f(z)=u(r, \theta)+i v(r, \theta)$ be defined throughout some $\in$-neighbourhood of a non-zero point $z_{0}=r_{0} \exp \left(i \theta_{0}\right)$. Suppose that the first order partial derivatives of the functions $u$ and $v$ with respect to $r$ and $\theta$ exist anywhere in that neighbourhood and that they are continuous at $\left(r_{0}, \theta_{0}\right)$. Then if those partial derivatives satisfy the polar form (4) of the CauchyRiemann equations at $\left(r_{0}, \theta_{0}\right)$, the derivatives $f^{\prime}\left(z_{0}\right)$ exists and

$$
f^{\prime}\left(z_{0}\right)=e^{-i \theta}\left(u_{r}+i v_{r}\right),
$$

where the right hand side is evaluated at $\left(r_{0}, \theta_{0}\right)$.
12.2.7 Example: $f(z)=\frac{1}{z}=\frac{1}{r e^{i \theta}}=\frac{\cos \theta}{r}-i \frac{\sin \theta}{r}$

The conditions in the theorem are satisfied at every non-zero point $z=r e^{i \theta}$ in the plane. Hence the derivative of $f$ exists there and

$$
f^{\prime}(z)=e^{-i \theta}\left(-\frac{\cos \theta}{r^{2}}+i \frac{\sin \theta}{r^{2}}\right)=\frac{1}{\left(r e^{i \theta}\right)^{2}}=-\frac{1}{z^{2}} .
$$

### 12.2.8 Example:

$f(z)=z(\operatorname{Re} z)=x^{2}+i x y$. Then $u_{x}=2 x, u_{y}=0, v_{x}=y, v_{y}=x$.

So C. R. equations are satisfied only at the origin. Hence $f$ is not differentiable at any point $z \neq 0$. At $z=0$, partial derivatives are continuous. Hence $f$ is differentiable at $z=0$.

### 12.2.9 Example:

$$
\begin{aligned}
f(z) & =\frac{\bar{z}}{|z|^{2}}, \quad z \neq 0, \\
& =\frac{1}{z}, \quad z \neq 0 \\
u= & \frac{x}{x^{2}+y^{2}}, \quad v=\frac{-y}{x^{2}+y^{2}} .
\end{aligned}
$$

Here $f$ is differentiable everywhere except at $z=0$.

### 12.3 Harmonic Functions

A real valued function $\phi(x, y)$ of two variables $x$ and $y$ that has continuous second order partial derivatives in a domain $D$ and satisfies the Laplace equation

$$
\frac{\partial^{2} \phi}{\partial x^{2}}+\frac{\partial^{2} \phi}{\partial y^{2}}=0
$$

is said to be harmonic in $D$.
12.3.1 Theorem: If $f(z)=u(x, y)+i v(x, y)$ is analytic in a domain $D$, then $u$ and $v$ satisfy Laplace's equation

$$
\nabla^{2} u=u_{x x}+u_{y y}=0 \text { and } \nabla^{2} v=v_{x x}+v_{y y}=0
$$

respectively in $D$ and have continuous second order partial derivatives in $D$.

Proof: The function $f$ satisfies the Cauchy-Riemann equations

$$
\begin{array}{ll} 
& u_{x}=v_{y}, \\
\text { and } & u_{y}=v_{x} . \tag{12.3.2}
\end{array}
$$

Differentiating (12.3.1) with respect to $x$ and (12.3.2) with respect to $y$ we get

$$
\begin{align*}
& u_{x x}  \tag{12.3.3}\\
&=v_{x y}  \tag{12.3.4}\\
& \text { and } \quad u_{y y}=-v_{y x}
\end{align*}
$$

If $f$ is analytic in $D$ then $u$ and $v$ have continuous partial derivatives of all orders in $D$. Hence $v_{x y}=v_{y x}$. Hence adding equations (12.3.3) and (12.3.4), we get $u_{x x}+u_{y y}=0$. Similarly we can prove that $v_{x x}+v_{y y}=0$.

If two functions $u$ and $v$ are harmonic in a domain $D$ and their first order partial derivatives satisfy the Cauchy-Riemann equations throughout $D, v$ is said to be a harmonic conjugate of $u$.
12.3.2 Theorem: A function $f(z)=u(x, y)+i v(x, y)$ is analytic in a domain $D$ if and only if $v$ is a harmonic conjugate of $u$.

### 12.3.3 Example:

$$
\text { Let } u=x^{2}-y^{2}-y \text {. }
$$

Then

$$
u_{x}=2 x, u_{x x}=2, u_{y}=-2 y-1, u_{y y}=-2 .
$$

So $u_{x x}+u_{y y}=0$; that is, $u$ is harmonic.

To find the conjugate harmonic function $v$ of $u$, we should have $v_{y}=u_{x}=2 x$ and $v_{x}=-u_{y}=2 y+1$. Integrating the first equation with respect to $y$, we get $v=2 x y+h(x)$.

Differentiating with respect to $x$, we get $v_{x}=2 y+h^{\prime}(x)=2 y+1$, or, $h^{\prime}(x)=+1 \Rightarrow h(x)=+x+k$. Hence $v=2 x y+x+k$.

This $v$ is the general conjugate harmonic function of $u$ and $f(z)=u+i v=\left(x^{2}-y^{2}-y\right)+i(2 x y+x+k)=\left(z^{2}+i z+i k\right)$ is analytic.
12.3.4Remark: A conjugate of a given harmonic function is uniquely determined up to a constant.
12.3.5 Remark: If $u(x, y)$ and $v(x, y)$ are any two harmonic functions, then ( $u+i v$ ) need not be analytic in $D$. However, if second order partial derivatives of $u$ nad $v$ are continuous then $\left(u_{y}-v_{x}\right)+i\left(u_{x}+v_{y}\right)$ is analytic in $D$.
12.3.6 Example: Let $u=x^{2}-y^{2}, v=3 x^{2} y-y^{3}$. Then $u$ and $v$ are harmonic. But $u_{x} \neq v_{y}$ and so $f=u+i v$ is not analytic. Let $U=u_{x}-v_{x}$ and $V=u_{x}+v_{y}$. Then $U+i V$ is analytic.
12.3.7 Example: Let $u(x, y)=2 x+y^{3}-3 x^{2} y$. $u_{x}=2-6 x y, u_{x x}=-6 y, u_{y}=3 y^{2}-3 x^{2}, u_{y y}=+6 y$.

So $u_{x x}+u_{y y}=0$, that is, $u$ is harmonic.

For finding conjugate $v$,
$v_{y}=u_{x}=2-6 x y \Rightarrow v=2 y-3 x y^{2}+h(x)$
$\Rightarrow v_{x}=-3 y^{2}+h^{\prime}(x)=-u_{y}=3 x^{2}-3 y^{2}$
$\Rightarrow h^{\prime}(x)=3 x^{2} \Rightarrow h(x)=x^{3}+c$

Hence $v=2 y-3 x y^{2}+x^{3}+c . f=u+i v=2 z+i z^{3}+i c$ is analytic.

### 12.3.8 Laplace Equation in Polar Form

Consider the function $f$ in polar form $f(z)=u(r, \theta)+i v(r, \theta)$.

Cauchy-Riemann equations are

$$
\begin{equation*}
u_{r}=\frac{1}{r} v_{\theta} \tag{12.3.5}
\end{equation*}
$$

$$
\begin{align*}
\text { and } & \frac{1}{r} u_{\theta}=v_{r}  \tag{12.3.6}\\
& \begin{aligned}
(12.3 .5) & \Rightarrow v_{\theta}=r u_{r} \Rightarrow v_{r \theta}=u_{r}+r u_{r r} \\
& (12.3 .6) \Rightarrow v_{\theta r}=-\frac{1}{r} u_{\theta \theta}
\end{aligned} \tag{12.3.7}
\end{align*}
$$

Assuming $u_{r \theta}=v_{\theta r}$, we get $u_{r}+r u_{r r}=-\frac{1}{r} u_{\theta \theta}$

$$
\begin{equation*}
\Rightarrow u_{r}+r u_{r r}+\frac{1}{r} u_{\theta \theta}=0, \text { or, } u_{r r}+\frac{1}{r} u_{r}+\frac{1}{r^{2}} u_{\theta \theta}=0 \tag{12.3.9}
\end{equation*}
$$

Similarly, we will have

$$
\begin{equation*}
v_{r r}+\frac{1}{r} v_{r}+\frac{1}{r^{2}} v_{\theta \theta}=0 . \tag{12.3.10}
\end{equation*}
$$

Equations (12.3.9) and (12.3.10) are Laplace equations in polar form.

## Suggested Readings

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## Lesson 13

## Line Integral in the Complex Plane

### 13.1 Introduction

Complex definite integrals are called complex line integrals written as $\int_{C} f(z) d z$, where $C$ is a curve in the complex plane called the path of integration. We may represent such a curve $C$ by a parametric representation

$$
\begin{equation*}
z(t)=x(t)+i y(t), a \leq t \leq b \tag{13.1.1}
\end{equation*}
$$

The sense of increasing $t$ is called the positive sense on $C$. We assume $C$ to be smooth curve, that is, $C$ has a continuous and nonzero derivative $\dot{z}=d z / d t$ at each point. Geometrically this means that $C$ has a unique and continuously turning tangent. Consider the partition $a=t_{0}<t_{1}<\cdots<t_{n-1}<t_{n}=b$. Let $z_{i}=z\left(t_{i}\right)=x\left(t_{i}\right)+i y\left(t_{i}\right), i=0,1, \ldots, n$.

Further, we choose point $\zeta_{i}$ between $z_{i-1}$ and $z_{i, i} i=1,2, \ldots, n$; and consider the sum $S_{n}=\sum_{m=1}^{n} f\left(\zeta_{m}\right) \Delta \zeta_{m}$,
where $\Delta \zeta_{m}=\zeta_{m}-\zeta_{m-1}$.

The limit of $S_{n}$ as the maximum of $\left|\Delta t_{m}\right|=\left|t_{m}-t_{m-1}\right|$ approaches zero (consequently $\left|\Delta z_{m}\right|=\left|z_{m}-z_{m-1}\right|$ approaches zero) is called the line integral of $f$ over $C$ and denoted by $\int_{C} f(z) d z$ or, by $\oint f(z) d z_{,}$if $z_{n}$ coincides with $z_{0}$ (that is, $C$ is a closed curve).

In general all paths of integration for complex line integrals are assumed to be piecewise smooth. The following three properties are easily implied by the definition of the line integral.

1. Linearity: $\int_{C}\left(k_{1} f_{1}(z)+k_{2} f_{2}(z)\right) d z=k_{1} \int_{C} f_{1}(z)+k_{2} \int_{C} f_{2}(z)$.
2. Sense Reversal: $\int_{z_{0}}^{z} f(z) d z=-\int_{z}^{z_{0}} f(z) d z$.
3. Partitioning of Path: $\int_{C} f(z) d z=\int_{C} f(z) d z+\int_{C} f(z) d z$.

### 13.2 Existence of the Complex Line Integral

From our assumptions of the existence of the complex integral, $f$ is continuous and $C$ is piecewise smooth. Let us write $f(z)=u(x, y)+i v(x, y)$. Let us further take $\zeta_{m}=\xi_{m}+\eta_{m}$ and $\Delta \zeta_{m}=\Delta x_{m}+i \Delta y_{m}$. Then the sum $S_{n}$ in (13.1.2) becomes

$$
\begin{align*}
S_{n}=\sum_{m=1}^{m}(u+i v) & \left(\Delta x_{m}+i \Delta y_{m}\right) \\
& =\Sigma u \Delta x_{m}-\Sigma v \Delta y_{m}+i\left(\Sigma u \Delta y_{m}+\Sigma v \Delta x_{m}\right) \tag{13.2.1}
\end{align*}
$$

These sums are real. Since $f$ is continuous, $u$ and $v$ are continuous. As maximum of $\left|\Delta t_{n}\right| \rightarrow 0$, maximum of $\Delta x_{m}$ and $\Delta y_{m}$ also converges to zero and the sum on the right becomes a real line integral.

$$
\begin{equation*}
\lim _{n \rightarrow \infty} S_{n}=\int_{C} f(z) d z=\int_{C} u d x-\int_{C} v d y+i\left[\int_{C} u d y+\int_{C} v d x\right] \tag{13.2.2}
\end{equation*}
$$

This shows that under assumptions on $f$ and $C$, the line integral exists and its value is independent of the choice of subdivisions and intermediate points $\zeta_{m^{*}}$
13.2.1 Theorem: (Indefinite integration of analytic functions)

Let $f(z)$ be analytic in a simply connected domain $D$ (every simple closed curve in $D$ encloses only points of $D$ ). Then there exists an indefinite integral of $f(z)$ in the domain $D$, that is, an analytic function $F(z)$ such that $F^{\prime}(z)=f(z)$ in $D$, and for all paths in $D$ joining two points $z_{0}$ and $z_{1}$ in $D$ we have

$$
\begin{equation*}
\int_{z_{0}}^{z_{1}} f(z) d z=F\left(z_{1}\right)-F\left(z_{0}\right) \tag{13.2.3}
\end{equation*}
$$

### 13.2.2 Examples

1. $\int_{0}^{1+i} z^{2} d z=\left.\frac{z^{3}}{3}\right|_{0} ^{1+i}=\frac{1}{3}(1+i)^{3}=-\frac{2}{3}+\frac{2}{3} i$.
2. $\int_{-\pi \pi i}^{\pi i} \cos z d z=\left.\sin z\right|_{-\pi i} ^{\pi i}=2 \sin \pi i=-2 i \sin h \pi$
3. $\int_{8-\pi i}^{8-3 \pi i} e^{\frac{z}{2}} d z=\left.2 e^{\frac{z}{2}}\right|_{8+\pi i} ^{8-3 \pi i}=2\left(e^{4-\frac{3 \pi i}{z}}-e^{4+\frac{\pi i}{z}}\right)=0$,
(since $e^{z}$ is periodic with period $2 \pi i$ ).
4. $\int_{-i}^{i} \frac{d z}{z}=\operatorname{Ln} i-\operatorname{Ln}(-i)=\frac{i \pi}{2}-\left(-\frac{i \pi}{2}\right)=i \pi$.
13.2.3 Theorem (Integration by the use of the path)

Let $C$ be a piecewise smooth path, represented by $z=z(t)$, where $a \leq t \leq b$. Let
$f(z)$ be a continuous function on $c$. Then $\int_{C} f(z) d z=\int_{a}^{b} f(z(t)) z(t) d t$,
where $\dot{z}=\frac{d z}{d t}$.

Proof: The LHS of (13.2.4) is given by (13.2.2) in terms of real line integrals, and we show that the RHS of (13.2.4) also equals (13.2.2). We have $z=x+i y$, hence $\dot{z}=\dot{x}+i \dot{y}$. We simply write $u$ for $u[x(t), y(t)]$ and $v$ for $v[x(t), y(t)]$. We also have $d x=\dot{x} d t$ and $d y=\dot{y} d t$.

Consequently, in (13.2.4)

$$
\begin{aligned}
\int_{a}^{b} f(z(t)) \dot{z}(t) d t=\int_{a}^{b}(u+i v)(\dot{x}+i \dot{y}) d t & \\
& =\int_{C}[u d x-v d y+i(u d y+v d x)] \\
& =\int_{C}(u d x-v d y)+i \int_{C}(u d y+v d x) .
\end{aligned}
$$

### 13.2.4 Examples

1. $\int_{C} \frac{d z}{z}=2 \pi i$, where $C$ is a unit circle, counter clockwise.

Solution: $z(t)=\cos t+i \sin t=e^{i t}, 0 \leq t \leq 2 \pi$ (representation of unit circle)
$\dot{z}(t)=-\sin t+i \cos t=i e^{i t}$
$f(z(t))=\frac{1}{z(t)}=e^{-i t}$.

Thus from (13.2.4), we get
$\int_{C}^{1} \frac{1}{Z} d z=\int_{0}^{2 \pi} e^{-i t} \cdot i e^{i t} d t=i \int_{0}^{2 \pi} d z=2 \pi i$.
2. $f(z)=\left(z-z_{0}\right)^{m}, m$ is an integer, $z_{0}$ is a constant. $C$ is circle of radius $\rho$ with center at $z_{0}$, counter clockwise.

Solution: $C$ can be represented in the form
$z(t)=z_{0}+\rho(\cos t+i \sin t)=z_{0}+\rho e^{i t}, 0 \leq t \leq 2 \pi$.

Then $\quad f(z)=\left(z-z_{0}\right)^{m}=\rho^{m} e^{i m t}, \quad d z=i \rho e^{i t} d t$.
$\int_{C}\left(z-z_{0}\right)^{m} d z=\int_{0}^{2 \pi} \rho^{m} e^{i m t} i \rho e^{i t} d t=i \rho^{m+1} \int_{0}^{2 \pi} e^{i(m+1) t} d t$
$=i \rho^{m+1} \int_{0}^{2 \pi}[\cos (m+1) t+i \sin (m+1) t] d t$

When $m=-1, \rho^{m+1}=1, \cos 0=1, \sin 0=0$, so that the integral equals $i \int_{0}^{2 \pi} d t=2 \pi i$. For $m \neq-1$, the two integrals vanish. Hence

$$
\int_{C}\left(z-z_{0}\right)^{m}=\left\{\begin{array}{lc}
2 \pi i, & m=-1 \\
0, & m \neq-1 \text { and integer. }
\end{array}\right.
$$

3. Integrate $f(z)=\operatorname{Re} z=x$ from 0 to $1+2 i$
(a) along $C_{3}$, straight line joining origin to $1+2 \mathrm{i}$
(b) along $C$ containing of $C_{1}$ and $C_{2}$, straight lines from origin to 1 and 1 to 1 +2 i.

## Solution:

(a) $z(t)=t+2 i t, 0 \leq t \leq 1, \quad \dot{z}(t)=1+2 i, f(z(t))=x(t)=t$,

$$
\int_{C_{3}} \operatorname{Re} z d z=\int_{0}^{1} t(1+2 i) d t=\frac{1}{2}+i .
$$

(b) Along $C_{1}, z(t)=t, \dot{z}(t)=1, f(z(t))=x(t)=t, 0 \leq t \leq 1$.

Along $C_{2,} z(t)=1+i t, \dot{z}(t)=i, f(z(t))=x(t)=1,0 \leq t \leq 2$.

Hence $\int_{C} f(z) d z=\int_{C_{1}} f(z) d z+\int_{C_{2}} f(z) d z=\int_{0}^{1} t d t+\int_{0}^{2} i d t=\frac{1}{2}+2 i$.

Thus the integral is dependent on the path.

### 13.3 Bounds for the Absolute Value of the Integrals

### 13.3.1 ML- inequality

$\left|\int_{C} f(z) d z\right| \leq M L$, where $L$ is the length of $C$ and $M$ a constant such that $|f(z)| \leq M$ everywhere on $C$.

Proof: $\left|S_{n]}=\left|\sum_{m=1}^{n} f\left(\zeta_{m}\right) \Delta z_{m}\right| \leq \sum_{m=1}^{n}\right| f\left(\zeta_{m}\right)\left|\left|\Delta z_{m}\right| \leq M \sum_{m=1}^{n}\right| \Delta z_{m} \mid$

Now $\left|\Delta z_{m}\right|$ is the length of the chord whose endpoints are $z_{m-1}$ and $z_{m}$. Hence the sum on the right represents the length $L^{*}$ of the broken line of chords whose endpoints are $z_{0}, z_{1}, \ldots, z_{n}(=z)$. If $n$ approaches infinity such that max $\left|\Delta t_{m}\right|$ and so max $\left|\Delta z_{m}\right|$ tends to zero, then $L^{*}$ approaches the length $L$ of the curve C, by the definition of the length of the curve. This proves the ML- inequality.

### 13.3.2 Examples

1. Evaluate $\int_{C} \operatorname{Re}\left(z^{2}\right) d z$, where $C$ is from 0 to $2+4 i$ represents
(a) a line segment joining the points $(0,0)$ and $(2,4)$,
(b) x-axis from 0 to 2 , and then vertical line to $2+4 i$,
(c) parabola $y=x^{2}$.

Solution: (a) Equation of $c$ is $z(t)=t+2 i t=(1+2 i) t, \quad 0 \leq t \leq 2$
$z^{\prime}(t)=(1+2 i) f(z(t))=\operatorname{Re}\left(z^{2}(t)\right)=\operatorname{Re}\left(t^{2}(1+2 i)^{2}\right)=\operatorname{Re}\left((-3+4 i) t^{2}\right)=-3 t^{2}$
Hence, we obtain $I=\int_{C} f(z(t)) z^{\prime}(t) d t=\int_{0}^{2}\left(-3 t^{2}\right)(1+2 i) d t=-8(1+2 i)$.
(b) $C_{1}$ is $z(t)=t, 0 \leq t \leq 2$
$C_{2}$ is $z(t)=2+2 i t, 0 \leq t \leq 2$
For $C_{1}, z^{\prime}(t)=1, f(z(t))=\operatorname{Re}\left(z^{2}\right)=t^{2}$
For $C_{2} \quad z^{\prime}(t)=2 i, f(z(t))=\operatorname{Re}\left((2+2 i t)^{2}\right)=4-4 t^{2}$

Hence, we obtain

$$
\begin{array}{rl}
I=\int_{C_{1}} f(z(t)) z^{\prime}(t) d t+\int_{C_{2}} & f(z(t)) z^{\prime}(t) d t \\
& =\int_{0}^{2} t^{2} d t+\int_{0}^{2}\left(4-4 t^{2}\right) 2 i d t \\
& =\left[\frac{t^{3}}{3}+2 i\left(4 t-\frac{4}{3} t^{3}\right)\right]_{0}^{2}=(1-2 i)
\end{array}
$$

(c) The parametric form of the curve $y=x^{2}$ can be written as

$$
z=z(t)=t+i t^{2}, \quad 0 \leq t \leq 2
$$

So $z^{\prime}(t)=1+2 i t$, and

$$
f(z(t))=\operatorname{Re}\left(z^{2}(t)\right)=\operatorname{Re}\left(t+i t^{2}\right)^{2}=\left(t^{2}-t^{4}\right)
$$

Hence $I=\int_{C} f(z(t)) z^{\prime}(t) d t=\int_{0}^{2}\left(t^{2}-t^{4}\right)(1+2 i t) d t$

$$
=\left[\frac{t^{8}}{3}-\frac{t^{5}}{5}+2 i\left(\frac{t^{4}}{4}-\frac{t^{6}}{6}\right)\right]_{0}^{2}=-\left(\frac{56}{15}+\frac{40}{3} i\right) .
$$

## Suggested Readings

Ahlfors, L.V. (1979). Complex Analysis, McGraw-Hill, Inc., New York.
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## Lesson 14

## Cauchy's Integral Theorem and Cauchy's Integral Formula

### 14.1 Cauchy's Integral Theorem

A simple closed path is a closed path that does not intersect or touch itself. A simply connected domain D in the complex plane is a domain such that every simple closed path in D enclosed only points of D . A domain that is not simply connected is called multiply connected.
14.1.1 Theorem (Cauchy's Integral Theorem)

If $f(z)$ is analytic in a simply connected domain $D$, then for every simple closed path $C$ in $D, \int_{C} f(z) d z=0$

Proof: We have from (13.2.2),

$$
\oint_{C} f(z) d z=\oint_{C}(u d x-v d y)+i \int_{C}(u d y+v d x) .
$$

Since $f(z)$ is analytic in $D, u$ and $v$ have continuous partial derivatives in $D$. Hence by Green’s Theorem

$$
\int_{C}(u d x-v d y)=\iint_{R}\left(-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) d x d y
$$

where $R$ is the region bounded by $C$. By Cauchy-Riemann condition $v_{x}=-u_{y}$ the RHS vanishes. Similarly the second integral also vanishes.

### 14.1.2 Examples

1. $\oint_{C} e^{z} d z=0, \quad \oint_{C} \cos z d z=0, \quad \oint_{C} z^{n} d z=0, \quad n=0,1,2, \ldots \quad$ for any closed path $C$ as these are all entire functions.
2. $\oint_{C} \sec z d z=0, C$ is the unit circle, as $\sec z$ has singularities at $\pm \frac{\pi}{2}, \pm \frac{3 \pi}{2}, \ldots$ outside the unit circle.
3. $\oint_{C} \frac{1}{z^{2}+4} d z=0, C$ is unit circle, $z= \pm 2 i$ are outside the unit circle.
4. $\oint_{C} \bar{z} d z=\int_{0}^{2 \pi} e^{-i t} i e^{i t} d t=2 \pi i, \quad C: z(t)=e^{i t}$ is the unit circle. Here $\bar{z}$ is no analytic.
5. $\oint_{C} \frac{d z}{z^{2}}=\int_{0}^{2 \pi} e^{-2 i t} . i e^{i t} d t=0, C$ is the unit circle taken counter clockwise. $\frac{1}{z^{2}}$ is not analytic at $z=0$.
6. $\oint_{C} \frac{1}{Z} d z=2 \pi i, \quad C$ is the unit circle taken counter clockwise.
14.1.3 Theorem (Independence of Path): If $f(z)$ is analytic in a simply connected domain $D$, then the integral of $f(z)$ is independent of the path in $D$. Proof: Let $z_{1}$ and $z_{2}$ be any points in $D$. Consider two paths $C_{1}$ and $C_{2}$ in $D$ from $z_{1}$ to $z_{2}$ without further common points. Let $C_{2}$ * be the path $C_{2}$ with orientation reversed. Integrate from $z_{1}$ over $C_{1}$ to $z_{2}$ and over $C_{2} *$ back to $z_{1}$. This is a simple closed path, and Cauchy's theorem applies under our assumptions and gives zero:

$$
\begin{gathered}
\int_{C_{1}} f d z+\int_{C_{2}^{*}} f d z=0, \\
\Rightarrow \int_{C_{1}} f d z=-\int_{C_{2}^{*}} f d z=\int_{C_{2}} f d z .
\end{gathered}
$$

This proves the theorem for paths that have only the endpoints in common. For paths with finitely many further common points the above argument is applied to each loop.

### 14.2 Principle of Deformation of Path

The idea is related to path independence. We can imagine that path $C_{2}$ was obtained from $C_{1}$ by continuously moving $C_{1}$ (with ends fixed) until it coincides with $C_{2}$. As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value. This is called the principle of deformation of path.

### 14.2.1 Theorem (Existence of Indefinite Integral)

If $f(z)$ is analytic in a simply connected domain $D$, then there exists an indefinite integral $F(z)$ of $f(z)$ in $D$, thus $F^{\prime}(z)=f(z)$ which is analytic in $D$, and for all paths in $D$ joining any two points $z_{0}$ and $z_{1}$ in $D$, the integral of $f(z)$ from $z_{0}$ to $z_{1}$ can be evaluated by

$$
\int_{z_{0}}^{z_{1}} f(z) d z=F\left(z_{1}\right)-F\left(z_{0}\right) .
$$

Proof: Since $f$ is analytic in , the line integral of $f(z)$ from any $z_{0}$ in $D$ to any $z$ in $D$ is independent of path in $D$. We keep $z_{0}$ fixed. Then this integral becomes a function of $z$, say $F(z)$.

$$
F(z)=\int_{z_{0}}^{z} f(s) d s
$$

Now $F(z+\Delta z)-F(z)=\int_{z_{0}}^{z+\Delta z} f(s) d s-\int_{z_{0}}^{z} f(s) d s=\int_{z}^{z+\Delta z} f(s) d s$, where the path of integration from $z$ to $z+\Delta z$ may be selected as a line segment.

Since $\int_{z}^{z+\Delta z} d s=\Delta z$, we can write $f(z)=\frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z) d s$. So

$$
\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)=\frac{1}{\Delta z} \int_{z}^{z+\Delta z}[f(s)-f(z)] d s
$$

Since f is continuous at z , for each positive $\epsilon, \delta>0 \ni|f(s)-f(z)|<\epsilon$ whenever $\quad|s-z|<\delta$. Choosing $|\Delta z|<\delta$, we have

$$
\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right|<\frac{1}{|\Delta z|} \varepsilon|\Delta z|=\varepsilon
$$

that is, $\lim _{\Delta z \rightarrow 0} \frac{F(z+\Delta z)-F(z)}{\Delta z}=f(z)$ or, $F^{\prime}(z)=f(z)$.

Since z is arbitrary, F is analytic in D .
Further if $G^{\prime}(z)=f(z)$, then $F(z)-G(z)$ is constant in $D$. That is two independent integrals differ by a constant.

### 14.3 Cauchy's Theorem for Multiply Connected Domains

Consider a doubly connected domain $D$ with outer boundary curve $C_{1}$ and inner curve $C_{2}$. If $f$ is analytic in any domain $D^{*}$ that contains $D$ and its boundary curves, then $\int_{C_{1}} f(z) d z=\int_{C_{2}} f(z) d z$, both integrals being taken counter clockwise (or clockwise, full interior of $C_{2}$ may not belong to $\mathrm{D}^{*}$.

In general: let
(a) $C$ be a simple closed curve (counter clockwise)
(b) $C_{1}, \ldots, C_{n}$ are simple closed curves (all in counter clockwise directions) and interior to $C$ and whose interiors have no points in common.
14.3.1 Theorem: Let $C$ and $C_{1} \ldots C_{n}$ be simply closed curves as in (a) and (b). If a function $f$ is analytic throughout the closed region $D$. Then

$$
\int_{C} f(z) d z=\sum_{k=1}^{n} \int_{C_{n}} f(z) d z .
$$

As a consequence of the above results we have the following important observation:

$$
\int_{C}\left(z-z_{0}\right)^{m}=\left\{\begin{array}{lc}
2 \pi i, \quad m=-1 \\
0, \quad m \neq-1 \text { and integer }
\end{array}\right.
$$

for counter-clockwise integration around any simple closed path containing $z_{0}$ in its interior.

### 14.3.2 Examples

1. $\oint_{C} e^{\left(-z^{2}\right)} d z=0, C$ is unit circle, (Cauchy's Theorem is applicable), as $e^{-z^{2}}$ is analytic in the given domain.
2. $\oint_{C} \frac{1}{|z|^{2}} d z=\int_{0}^{2 \pi} i e^{i t} d t=\left.e^{i t}\right|_{0} ^{2 \pi}=0$. Here Cauchy's Theorem is not applicable.
3. $\oint_{C} \frac{1}{2 z-1} d z=\frac{1}{2} \oint_{C} \frac{1}{\left(z-\frac{1}{2}\right)} d z=\frac{1}{2} .2 \pi i=\pi i$, $C$ is unit circle, (Cauchy's Theorem is applicable)
4. $\oint_{C} \frac{d z}{z-3 i}=2 \pi i, C$ is the circle $|z|=\pi$, as $3 i$ is inside this circle.
5. $\oint_{C} \frac{e^{z}}{z} d z=0$, (using Cauchy's Theorem for doubly connected domain) $C$ is a circle $|z|=2$ counter clockwise and $|z|=1$ clockwise.
6. $C_{1}$ is upper semi-circle of $|z|=1$, clockwise.
$C_{2}$ is lower semi-circle counter-clockwise.

$$
I_{1}=\int_{C_{1}} \frac{1}{Z} d z=\int_{\pi}^{0} \frac{i e^{i t}}{e^{i t}} d t=-\pi i \quad \text { and } \quad I_{2}=\int_{C_{2}} \frac{1}{Z} d z=\int_{\pi}^{2 \pi} \frac{i e^{i t}}{e^{i t}} d t=\pi i
$$

$I_{1}$ and $I_{2}$ are not same, i.e., principle of deformation of paths is not applicable since the curve $C_{1}$ cannot be continuously deformed into $C_{2}$ without passing through $\mathrm{z}=0$ at which $f(z)$ is not analytic.
7. $I=\oint_{C} \frac{d z}{z(z+2)}$ where $C$ is any rectangle containing the points $z=0$ and $z=\tilde{=} 2$ inside it.

Solution: Enclose points $z=0$ and $z=\tilde{2}$ inside circles $C_{1}$ and $C_{2}$ respectively that do not intersect. Then applying Cauchy's integral theorem for triply connected domains, we get

$$
\begin{gathered}
\oint_{C} \frac{d z}{z(z+2)}=\int_{C_{1}} \frac{d z}{z(z+2)}+\int_{C_{2}} \frac{d z}{z(z+2)} \\
=\frac{1}{2}\left[\int_{C_{1}} \frac{d z}{z}-\int_{C_{1}} \frac{d z}{z+2}+\int_{C_{2}} \frac{d z}{z}-\int_{C_{2}} \frac{d z}{z+2}\right]=\frac{1}{2}(2 \pi i-0+0-2 \pi i)=0 .
\end{gathered}
$$

### 14.4 Cauchy's Integral Formula

14.4.1 Theorem: Let $f(z)$ be analytic in a simply connected domain $D$. Then for any point $z_{0}$ in $D$ and any simple closed path $C$ in $D$ that encloses $z_{0}$

$$
f\left(z_{0}\right)=\frac{1}{2 \pi i} \oint_{C} \frac{f(z)}{z-z_{0}} d z
$$

( $C$ is taken counter clockwise direction.)

### 14.4.2 Examples

1. $I=\int_{C} \frac{e^{z}}{z-2} d z= \begin{cases}2 \pi i e^{2}, & \text { for any } C \text { which has } z_{0}=2 \text { as interior point } \\ 0, & \text { for any } C \text { which has } z_{0}=2 \text { as exterior point }\end{cases}$
2. $I=\oint_{C} \frac{d z}{2-\bar{z}}, \quad C:|z|=1$

Now $2-\bar{z}=2-\frac{z \bar{Z}}{z}=2-\frac{1}{z}$ on $C$. Hence

$$
I=\oint_{C} \frac{z d z}{2 z-1}=\frac{1}{2} \oint_{C} \frac{z}{z-\frac{1}{2}} d z=\frac{1}{2} \cdot 2 \pi i \cdot \frac{1}{2}=\frac{\pi i}{2} .
$$

4. $I=\oint_{C} \frac{z^{2}+1}{z(2 z-1)} d z, C:|z|=1$.

The integrand is not analytic at $z=0$ and $z=\frac{1}{2}$. We write

$$
I=\oint_{C} \frac{z^{2}+1}{z-\frac{1}{2}}-\oint_{C} \frac{z^{2}+1}{z} d z=2 \pi i\left(\frac{1}{4}+1\right)-2 \pi i \cdot 1=\frac{\pi i}{2}
$$

14.4.3 Theorem (Derivatives of Analytic Function)

If $f(z)$ is analytic in a domain $D$, then it has derivatives of all orders in $D$, which are then analytic functions in $D$. The values of these derivatives at a point $z_{0}$ in $D$ are given by

$$
f^{\prime}\left(z_{0}\right)=\frac{1}{2 \pi i} \oint_{C} \frac{f(z)}{\left(z-z_{0}\right)^{2}} d z
$$

and in general

$$
f^{(n)}\left(z_{0}\right)=\frac{n!}{2 \pi i} \oint_{C} \frac{f(z)}{\left(z-z_{0}\right)^{n+1}} d z, n=1,2, \ldots
$$

Here $C$ is any simple closed path in $C$ that encloses $z_{0}$ and whose interior is a subset of $D$.

### 14.4.4 Examples

1. $\oint_{C} \frac{\cos z}{(z-\pi i)^{2}} d z=\left.2 \pi i(\cos z)^{\prime}\right|_{z=\pi i}=-2 \pi i \sin \pi i=2 \pi \sin h(\pi)$.
2. For any curve $C$ for which 1 lies inside and $\pm 2 i$ outside
$\oint_{C} \frac{e^{z}}{(z-1)^{2}\left(z^{2}+4\right)} d z=2 \pi i \frac{d}{d z}\left(\frac{e^{z}}{z^{2}+4}\right)_{z=1}$
$=2 \pi i\left[\frac{e^{z}\left(z^{2}+4\right)-e^{z} \cdot 2 z}{\left(z^{2}+4\right)^{2}}\right]_{z=1}=\frac{6 e \pi i}{25}$.
14.4.5 Cauchy's Inequality: Let $f(z)$ be analytic within and on $C:\left|z-z_{0}\right|=r$ and $|f(z)| \leq M$ on $C$, then $\left|f^{(n)}\left(z_{0}\right)\right| \leq \frac{M(n!)}{r^{n}}$.
14.4.6 Liouville's Theorem: If an analytic function $f(z)$ is bounded for all values of z in the complex plane, then $f(z)$ must be a constant.

Proof: Let $|f(z)| \leq K \forall z$. By Cauchy's inequality

$$
\left|f^{\prime}\left(z_{0}\right)\right| \leq \frac{K}{r} \text { for any r. }
$$

Taking $r \rightarrow \infty$, we get $f^{\prime}\left(z_{0}\right)=0$. Since $z_{0}$ is also arbitrary, $f^{\prime}\left(z_{0}\right)=0 \forall z$. So $f$ must be a constant.
14.4.7 Maximum Modulus Principle: If a function $f$ is analytic and not constant in a given domain $D$, then $|f(z)|$ has no maximum value in $D$.
14.4.8 Corollary: Suppose that a function f is continuous in a closed and bounded region $R$ and that it is analytic and not constant in the interior of $R$. Then the maximum value of $|f(z)|$ in $R$, which is always reached, occurs somewhere on the boundary of $R$ and never in the interior.

### 14.4.9 Examples

1. $I=\oint_{C} \frac{d z}{\left(z^{2}+4\right)^{2}}, \quad C:|z-i|=2$

The integrand is not analytic at $z=2 i$. The point $z=2 i$ lies inside the domain but $z=-2 i$ lies outside it. So

$$
\begin{aligned}
I & =\oint_{C} \frac{d z}{(z-2 i)^{2}(z+2 i)^{2}}=\oint_{C} \frac{f(z) d z}{(z-2 i)^{2}}, \text { where } f(z)=\frac{1}{(z+2 i)^{2}} \\
& =2 \pi i f^{\prime}(2 i)=\frac{\pi}{16}
\end{aligned}
$$

2. $I=\oint_{C} \frac{\left(3 z^{4}+5 z^{2}+2\right) d z}{(z+1)^{4}}$, where $C$ is any simple closed curve containing the point $z=\tilde{1}$ inside its interior.

$$
I=\frac{2 \pi i}{3!}\left[\frac{d^{3}}{d z^{3}}\left(3 z^{4}+5 z^{2}+2\right)\right]_{z=-1}=-24 \pi i .
$$

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## Lesson 15

## Infinite Series, Convergence Tests, Uniform Convergence

### 15.1 Infinite Series

Let $p_{k_{k}} k=1,2, \ldots$ be a set of real or complex numbers. Then

$$
\begin{equation*}
\sum p_{k}=\sum_{k=1}^{\infty} p_{k}=p_{1}+p_{2}+\cdots \tag{15.1.1}
\end{equation*}
$$

is an infinite series of numbers and $p_{k}$ is its $k$ th term. The partial sum $s_{n}$ of the series is defined by

$$
S_{n}=\sum p_{k}=\sum_{k=1}^{n} p_{k}=p_{1}+p_{2}+\cdots+p_{n} .
$$

The remainder of the series (after the nth term) is defined as

$$
R_{n}=\sum_{k=n+1}^{\infty} p_{k}=p_{n+1}+p_{n+2}+\cdots
$$

The series (15.1.1) is said to be convergent if the sequence $\left\{s_{n}\right\}$ of the partial sums is convergent. The limit $S$ of the sequence $\left\{S_{n}\right\}$ is called the sum of the series.
15.1.1 Theorem: A necessary condition for a series $\sum p_{k}$ to be convergent is $\lim _{n \rightarrow \infty} p_{n}=0$.

Proof: Suppose that the series $\sum p_{k}$ is convergent. Then

$$
\lim _{n \rightarrow \infty} S_{n}=S \text { and } \lim _{n \rightarrow \infty} S_{n-1}=S
$$

Since $p_{n}=S_{n}-S_{n-1}$, we get

$$
\lim _{n \rightarrow \infty} p_{n}=\lim _{n \rightarrow \infty} S_{n}-\lim _{n \rightarrow \infty} S_{n-1}=S-S=0
$$

15.1.2 Theorem (Cauchy's criterion for convergence): The series $\sum p_{n}$ is convergent if and only if for any given real positive number $\varepsilon>0$, there exists a natural number $N$ such that

$$
\left|S_{n}-S_{n-1}\right|<\varepsilon \text { for all } n, m \geq N
$$

15.1.3 Theorem: The series $\sum p_{k}$, where $p_{k}=x_{k}+i y_{k}$, of complex numbers converges to $S=X+i Y$ if and only if the series of the real parts $\sum x_{k}$ converges to $X$ and the series of the imaginary parts $\sum y_{k}$ converges to $Y$.

### 15.1.4 Geometric Series

Consider the geometric series $\sum_{n=1}^{\infty} r^{n}$, where r is any real number. We now find the conditions for the convergence of this series.

First consider the sequence of partial sums

$$
\begin{aligned}
& \quad S_{n}=1+r+r^{2}+\cdots+r^{n-1}=\frac{1-r^{n}}{1-r} \\
& \text { or, } \quad S_{n}-\frac{1}{1-r}=-\frac{r^{n}}{1-r^{n}} .
\end{aligned}
$$

Therefore, $\left|S_{n}-\frac{1}{1-r}\right|=\left|-\frac{r^{n}}{1-r}\right|=\frac{|r|^{n}}{|1-r|^{n}}$.

Note that $|r|^{n} \rightarrow 0$ when $|r|<1$, hence the geometric series converges to $1 /(1-r)$, when $|r|<1$.

When $|r|>1,|r|^{n} \rightarrow \infty$ as $n \rightarrow \infty$. So the geometric series diverges in this case.

For $r=1$, each term in the series is unity. Hence the partial sum $S_{n}=n \rightarrow \infty$ as $\rightarrow \infty$. Thus the series is divergent in this case.

For $r=-1$, the terms in the series are +1 and 1 alternatively. Now the sequence $\left\{s_{n}\right\}$ has two subsequences with limits 0 and 1 . Hence in this case, the sequence $\left\{S_{n}\right\}$ does not converge and consequently the series does not converge.
15.1.5 Example: Using the above argument, one can show that the series $\sum_{n=0}^{\infty} z^{n}$ converges to $1 /(1-z)$ if $|z|<1$. Here $z$ is complex variable.

### 15.1.6 Harmonic Series:

Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. We show that this series is divergent.

The sequence of partial sums is defined by
$S_{n}=1+\frac{1}{2}+\cdots+\frac{1}{n}$
and $\left|S_{n+p}-S_{n}\right|=\left|\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+p}\right|>\frac{1}{n+p}+\frac{1}{n+p}+\cdots+\frac{1}{n+p}=\frac{p}{n+p}$

Note that $\frac{p}{n+p}=\frac{1}{2}$ when $p=n$. Thus $\left|S_{2 n}-S_{n}\right|>\frac{1}{2}$. This shows that for $\varepsilon<\frac{1}{2}$,
one cannot satisfy the condition $\left|S_{n+p}-S_{n}\right|<\varepsilon, n \geq N, p=1,2, \ldots$. This violates the condition for Cauchy convergence. By Theorem 15.1.2 we conclude that the harmonic series is not convergent.

### 15.2 Tests for Convergence

The following results are frequently used to test the convergence of an infinite series.
15.2.1 Comparison Test: Let $\sum_{n=0}^{\infty} p_{n}$ and $\sum_{n=0}^{\infty} q_{n}$ be two real series with positive terms and $p_{n} \leq k q_{n}$ for any real positive k and $n=1,2, \ldots$. Then,
(i) convergence of the series $\sum_{n=0}^{\infty} q_{n}$ imply convergence of the series $\sum_{n=0}^{\infty} p_{n}$,
(ii) divergence of the series $\sum_{n=0}^{\infty} p_{n}$ implies the divergence of the series

$$
\sum_{n=0}^{\infty} q_{n}
$$

15.2.2 Limit comparison test $\sum_{n=0}^{\infty} p_{n}$ and $\sum_{n=0}^{\infty} q_{n}$ be two real series with positive terms and $\lim _{n \rightarrow \infty} \frac{p_{n}}{q_{n}}=l, 0<l<\infty$.

Then, both the series $\sum_{n=0}^{\infty} p_{n}$ and $\sum_{n=0}^{\infty} q_{n}$ converge or diverge together.
15.2.3 Theorem: The series $\sum_{n=1}^{\infty} \frac{1}{n^{p}}, p>0$ is convergent if $p>1$ and divergent if $p \leq 1$.

Proof: We write

$$
\begin{aligned}
S_{n} & =\frac{1}{1^{p}}+\frac{1}{2^{p}}+\ldots+\frac{1}{n^{p}}=\frac{1}{1^{p}}+\left(\frac{1}{2^{p}}+\frac{1}{3^{p}}\right)+\left(\frac{1}{4^{p}}+\frac{1}{5^{p}}+\frac{1}{6^{p}}+\frac{1}{7^{p}}\right)+\ldots \\
& <\frac{1}{1^{p}}+\frac{2}{2^{p}}+\frac{4}{4^{p}}+\ldots=\frac{1}{1^{p}}+\left(\frac{1}{2^{p-1}}\right)+\left(\frac{1}{2^{p-1}}\right)^{2}+\ldots
\end{aligned}
$$

The last series is a geometric series with common ratio $r=\frac{1}{2^{p-1}}$. Therefore, the series is convergent if $r=\frac{1}{2^{p-1}}<1$ or $p>1$. For $0<p \leq 1, \frac{1}{n} \leq \frac{1}{n^{p}}$. Since the harmonic series $\sum \frac{1}{n}$ is divergent, applying the comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{n^{p}}$ is also divergent for $0<p \leq 1$,
15.2.4 Example: Prove that the series $\sum_{n-1}^{\infty} \frac{1}{n(n+1)}$ is convergent. Also find its sum.

Solution: We can write
$S_{n}=\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{n(n+1)}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}$.
Now $\lim _{n \rightarrow \infty} S_{n}=1$, so the given series is convergent and the sum of the series is 1.
15.2.5 D' Alembert's test (Ratio test): Let $\sum p_{n}$ be a real series of positive terms or a complex series. Let $\lim _{n \rightarrow \infty}\left|\frac{p_{n+1}}{p_{n}}\right|=c$.

Then, the series $\sum p_{n}$ is (i) convergent if $\mathrm{c}<1$ and (ii) divergent if $c>1$. The ratio test does not give any information on convergence of the series when $c=1$.
15.2.6 Examples: Apply ratio test to the following series
(i) $\sum \frac{z^{n}}{2 n}$,
(ii) $\sum \frac{z^{n-1}}{(n+1)!}$,
(iii) $\sum n!z^{n}$.

## Solution:

(i) $\quad\left|\frac{a_{n+1}}{a_{n}}\right|=\left|\frac{2 n}{2 n+2} z\right|=\left(\frac{1}{1+\frac{1}{n}}\right)|z|$. Hence $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=|z|$.

Therefore, the series is convergent when $|z|<1$ and divergent when $|z|>1$. The test fails when $|z|=1$.
(ii) $\quad\left|\frac{a_{n+1}}{a_{n}}\right|=\left|\frac{1}{n+2} z\right|$. So $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=0<1$.

So the series is convergent for all $z$.
(iii) Here $\left|\frac{a_{n+1}}{a_{n}}\right|=|(n+1) z|=(n+1)|z|$ and so $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=\infty>1$.

So the series is divergent for all $z$.

### 15.2.6 Examples When Ratio Test Fails

(i) The series $\sum \frac{1}{n}$ is divergent. However, $\lim _{n \rightarrow \infty} \frac{a_{n+1}}{a_{n}}=1$
(ii)The series $\sum \frac{1}{n^{2}}$ is convergent. However, $\lim _{n \rightarrow \infty} \frac{a_{n+1}}{a_{n}}=1$
15.2.7 Cauchy's Root Test: Let $\sum p_{n}$ be a real series of positive terms or a complex series. Let $\lim _{n \rightarrow \infty}\left|p_{n}\right|^{1 / n}=c$. Then, the series $\sum p_{n}$ is (i) convergent if $c<1$ and (ii) divergent if $\mathrm{c}>1$. The root test does not give any information on the convergence if $c=1$.
15.2.7 Example: Let $p_{n}=\left(1+\frac{1}{n^{p}}\right)^{-n^{p+1}}, p>0$. Using the Cauchy root test we have
$\left(p_{n}\right)^{1 / n}=\left(1+\frac{1}{n^{p}}\right)^{-n^{p}}=\frac{1}{\left(1+\frac{1}{n^{p}}\right)^{n^{p}}}$

Now, $\lim _{n \rightarrow \infty}\left|p_{n}\right|^{\frac{1}{n}}=\lim _{n \rightarrow \infty} \frac{1}{\left(1+\frac{1}{n^{p}}\right)^{n p}}=\frac{1}{e}<1$.

So the series $\sum p_{n}$ is convergent.

### 15.3 Alternating Series

A real series in which the terms are alternatively positive and negative is called an alternative series and is of the form $\sum_{n=0}^{\infty}(-1)^{n} p_{n}, p_{n}>0$. The following theorem gives a sufficient condition for the convergence of an alternative series.
15.3.1 Theorem (Leibnitz theorem): Let $\sum_{n=0}^{\infty}(-1)^{n} p_{n}, p_{n}>0$ be an alternative series satisfying the following conditions
(i) The sequence $\left\{p_{n}\right\}$ is non-increasing, that is $p_{n+1} \leq p_{n}$ for all $n$, and
(ii) $\lim _{n \rightarrow \infty} p_{n}=0$.

Then, the series $\sum_{n=0}^{\infty}(-1)^{n} p_{n}$ is convergent.
15.3.2 Examples: Using Leibnitz Theorem, we can conclude that the following series are convergent:
(i) $\sum(-1)^{n} \frac{1}{n}$
(ii) $\sum(-1)^{n} \frac{1}{\sqrt{n}}$

### 15.3.3 Absolutely Convergent Series

Let $\sum p_{n}$ be an arbitrary series of real or complex numbers. If the series of positive terms $\sum\left|p_{n}\right|$ is convergent, then we say that the series $\sum p_{n}$ is absolutely convergent. If the series $\sum p_{n}$ is convergent but $\sum\left|p_{n}\right|$ is divergent, then the series is called conditionally convergent.
15.3.4 Example: The series $\sum(-1)^{n} \frac{1}{n}$ is conditionally convergent.

### 15.4 Uniform Convergence of the Series of Functions

Let $f_{1}(z)+f_{2}(z)+\cdots \quad$ be a series of single-valued complex functions defined in a domain D (or a series of real functions defined on a closed interval). Let $S_{n}(z)=f_{1}(z)+f_{2}(z)+\cdots+f_{n}(z)$ be the nth partial sum. If a point $z=z_{0}$ in D , the sequence $\left\{S_{n}(z)\right\}$ of partial sums converges to $f\left(z_{0}\right)_{\text {}}$ then we say that the series $\sum f_{k}\left(z_{0}\right)$ converges to $f\left(z_{0}\right)$. This convergence is called pointwise convergence of the series $\sum f_{n}(z)$.

We say that the series $\sum f_{k}(z)$ converges uniformly to $f(z)$, if, for a given real positive number $\varepsilon>0$, there exists a natural number $N$ independent of z , but dependent on $\varepsilon$ such that

$$
\left|S_{n}(z)-f(z)\right|<\varepsilon \text { for } n>N
$$

Thus, a series which is uniformly convergent is also pointwise convergent. Weierstrass's M-test gives sufficient conditions for the uniform convergence of a series.
15.4.1 Theorem: (Weierstrass's M-test) Let $\sum f_{n}(z)$ be an infinite series defined in some domain D of the complex plane and let $\left\{M_{n}\right\}$ be a sequence of positive terms, where $\left|f_{n}(z)\right| \leq M_{n}$ for all n and for all z in D . If the series $\sum M_{n}$ is convergent, then the series $\sum f_{n}(z)$ is uniformly and absolutely convergent.
15.4.2 Example: We discuss the uniform convergence of the series $\sum \frac{z^{n}-1}{n^{2}+|z|^{2}}$ on the disk $|z|<1$.

Note that

$$
\left|f_{n}(z)\right|=\left|\frac{z^{n}-1}{n^{2}+|z|^{2}}\right| \leq \frac{z^{n}+1}{n^{2}+|z|^{2}}<\frac{2}{n^{2}}
$$

for all z in $|z|<1$.

Since, the series $\sum 1 / n^{2}$ is convergent, the given series is uniformly convergent.
15.4.3 Example: We show that the geometric series $1+z+z^{2}+\cdots$ is
(i) uniformly convergent in any closed disk $|z| \leq r<1$.
(ii)not uniformly convergent in the open disk $|z|<1$.

We have
$S_{n}(z)=1+z+z^{2}+\cdots+z^{n-1}$
and

$$
f(z)=S(z)=\lim _{n \rightarrow \infty} S_{n}(z)=\frac{1}{1-z},|z|<1 .
$$

In the closed disk $|z| \leq r<1$, we have

$$
|1-z| \geq 1-|z| \geq 1-r \text { or } \frac{1}{1-|z|} \leq \frac{1}{1-r}
$$

Then

$$
\left|S_{n}(z)-s(z)\right|=\left|z^{n}+z^{n+1}+\cdots\right|=\left|\frac{z^{n}}{1-z}\right| \leq \frac{r^{n}}{1-r^{*}} .
$$

Using $r<1$, the right hand side can be made as small as necessary by choosing $n$ large enough.

Hence, $\left|S_{n}(z)-f(z)\right|<\varepsilon$ for $n>N$ and for all $z$. This shows that the given series is uniformly convergent.

If we consider the open disk $|z|<1$, we can find a $z$ for a given $n$ and a real number $k$ (no matter how large) such that

$$
\left|\frac{z^{n}}{1-z}\right|=\frac{|z|^{n}}{1-|z|}>k
$$

by taking $|z|$ sufficiently close to 1 . Thus, for no $N$ we can have $\left|S_{n}(z)-s(z)\right|<\varepsilon$ for every $z$ in the open disk $|z|<1$. Thus $N$ depends both on $z$ and $\varepsilon$. So the series is not uniformly convergent.

## Suggested Readings

Ahlfors, L.V. (1979). Complex Analysis, McGraw-Hill, Inc., New York.
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## Lesson 16

## Power Series

### 16.1 Introduction

A power series in powers of $\left(\boldsymbol{z}-\boldsymbol{z}_{\mathbf{0}}\right)$ is a series of the form

$$
\begin{equation*}
\sum_{n=0}^{\infty} c_{n}\left(z-z_{0}\right)^{n}=c_{0}+c_{1}\left(z-z_{0}\right)+c_{2}\left(z-z_{0}\right)^{2}+\cdots \tag{16.1.1}
\end{equation*}
$$

where $\boldsymbol{z}$ is a complex variable and $\boldsymbol{c}_{\mathbf{0}}, \boldsymbol{c}_{\mathbf{1}}, \ldots$ are complex (or real) constants, called the coefficients of the series, and $\mathbf{z}_{\mathbf{0}}$ is a complex (real) constant, called the center of the series.

If $\boldsymbol{z}_{\mathbf{0}}=\mathbf{0}$, we obtain a power series in the powers of $\boldsymbol{z}$ :

$$
\sum_{n=0}^{\infty} c_{n} z^{n}=c_{0}+c_{1} z+c_{2} z^{2}+\cdots
$$

### 16.1.1 Examples

1. It can be seen easily that the series $\sum_{n=0}^{\infty} z^{n}=\mathbf{1}+z+z^{2}+\cdots$, converges absolutely if $|\boldsymbol{z}|<\mathbf{1}$ and diverges for $|\boldsymbol{z}| \geq \mathbf{1}$.
2. The series $\sum_{n=0}^{\infty} \frac{z^{n}}{n!}=\mathbf{1}+\boldsymbol{z}+\frac{z^{2}}{2}+\cdots$ is absolutely convergent for every $\boldsymbol{z}$. In fact, by the ratio test, for any fixed $\mathbf{z}$,

$$
\left|\frac{z^{n+1} /(n+1)!}{z^{n} / n!}\right|=\frac{|z|}{n+1} \rightarrow 0 \text { as } n \rightarrow \infty
$$

### 16.1.2 Theorem: (Convergence of a Power Series)

(a) Every power series (16.1.1) converges at $z=z_{0}$.
(b) If (16.1.1) converges at $z=z_{1} \neq z_{0}$, it converges absolutely for every $z$ closer to $z_{0}$ than $z_{1}$, i.e., $\left|z-z_{0}\right|<\left|z_{1}-z_{0}\right|$.
(c) If (16.1.1) diverges at $z=z_{2}$ then it diverges at every $z$ further away from $z_{0}$ than $z_{2}$.

## Proof:

(a) The proof follows by observing that for $\mathbf{z}=\boldsymbol{z}_{\mathbf{0},}$ the series reduces to $\boldsymbol{a}_{\mathbf{0}}$.
(b) Since $\sum_{n=0}^{\infty} a_{n}\left(z_{1}-z_{0}\right)^{n}$ is convergent, the necessary condition for the convergence of a series implies that the $n$-th term $\boldsymbol{a}_{\boldsymbol{n}}\left(\mathbf{z}_{\mathbf{1}}-\boldsymbol{z}_{\mathbf{0}}\right) \rightarrow \mathbf{0}$ as $\boldsymbol{n} \rightarrow \infty$. Hence the terms $\boldsymbol{a}_{\boldsymbol{n}}\left(\boldsymbol{z}_{\mathbf{1}}-\boldsymbol{z}_{\mathbf{0}}\right)$ are bounded. So there exists $m$ such that $\left|\boldsymbol{a}_{n}\left(z_{1}-z_{0}\right)\right| \leq \boldsymbol{M}$ for all $n$. Thus we have

$$
\begin{aligned}
&\left|a_{n}\left(z-z_{0}\right)^{n}\right|=\left|a_{n}\right|\left|z-z_{0}\right|^{n}=\left|a_{n}\right|\left|z-z_{0}\right|^{n}\left|\frac{z-z_{0}}{z_{1}-z_{0}}\right|^{n} \leq M\left|\frac{z-z_{0}}{z_{1}-z_{0}}\right|^{n} \\
&=M k^{n} .
\end{aligned}
$$

Now for $\boldsymbol{k}<\mathbf{1}, \sum \boldsymbol{k}^{\boldsymbol{n}}$ is a convergent geometric series with common ratio $\boldsymbol{k}$. Therefore $\sum \boldsymbol{a}_{n}\left(z-z_{0}\right)^{n}$ converges absolutely for $\left|z-z_{0}\right|<\left|z_{1}-z_{0}\right|$.
(c) The proof follows assuming contrary to assumption.

### 16.2 Radius of Convergence

Let $\boldsymbol{R}$ be the radius of the circle with center at $\boldsymbol{z}_{\mathbf{0}}$ that contains all points at which the series is convergent and the series is divergent at all points outside it.

Then $\left|\boldsymbol{z}-z_{\mathbf{0}}\right|=\boldsymbol{R}$ is called the circle of convergence and $\boldsymbol{R}$ is the radius of convergence. The power series may or may not converge on the boundary. If $\boldsymbol{R}=\mathbf{0}$, the series is convergent only at $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ and if $\boldsymbol{R}=\infty$, the series converges for all $\mathbf{z}$.

### 16.2.1 Examples

1. The series $\sum \frac{z^{n}}{n^{2}}$ converges for $|z| \leq 1$. Here $R=1$.
2. The series $\sum \frac{z^{n}}{n}$ converges for $z=-1$ but diverges for $z=1$. Here $R=1$.
3. The series $\sum z^{n}$ diverges for $z=1$. Here $R=1$.
16.2.2 Theorem (Radius of Convergence) Let $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=\boldsymbol{L}$. Then the radius of convergence of the power series $\sum \boldsymbol{a}_{\boldsymbol{n}}\left(\boldsymbol{z}-\boldsymbol{z}_{0}\right)^{\boldsymbol{n}}$ is $\boldsymbol{R}=\frac{\mathbf{1}}{\boldsymbol{L}}$. (The case $\boldsymbol{L}=\mathbf{0}$ and $\boldsymbol{L}=\infty$ is included). (Cauchy-Hadamad formula)

Proof: By the ratio test, consider

$$
\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}\left(z-z_{0}\right)^{n}}{a_{n}\left(z-z_{0}\right)^{n}}\right|=\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|\left|z-z_{0}\right|=L\left|z-z_{0}\right| .
$$

If $\boldsymbol{L}=\mathbf{0}$ then for all $\boldsymbol{z}$, the power series will converge and so $\boldsymbol{R}=\infty$. If $\boldsymbol{L}=\infty$ then $\left|\frac{a_{n+1}}{a_{n}}\right|\left|z-z_{0}\right|>1$ for $\boldsymbol{z} \neq z_{0}$ and all $\boldsymbol{n}>\boldsymbol{N}$ (for some $\boldsymbol{N}$ ). Hence the series will not converge for any $\boldsymbol{z}$. So $\boldsymbol{R}=\mathbf{0}$. In all other cases, the series will converges for $\boldsymbol{L}\left|\boldsymbol{z}-z_{\mathbf{0}}\right|<\mathbf{1}$ or $\left|\boldsymbol{z}-\mathbf{z}_{\mathbf{0}}\right|<\frac{\mathbf{1}}{\boldsymbol{L}}$ and diverges for $\boldsymbol{L}\left|\boldsymbol{z}-\mathbf{z}_{\mathbf{0}}\right|>\mathbf{1}$ or $\left|\boldsymbol{z}-z_{\mathbf{0}}\right|>\frac{\mathbf{1}}{\boldsymbol{L}}$. Hence $\boldsymbol{R}=\frac{\mathbf{1}}{\boldsymbol{L}}$ is the radius of convergence.
16.2.3 Example: Consider the series $\sum_{n=0}^{\infty} \frac{2 n!}{(n!)^{2}}(z-3 i)^{n}$
$L=\lim _{n \rightarrow \infty}\left[\frac{(2 n+2)!}{\{(n+1)!\}^{2}} \frac{(n!)^{2}}{2 n!}\right]=\lim _{n \rightarrow \infty} \frac{(2 n+1)(2 n+2)}{(n+1)^{2}}=4$.

Hence $=\frac{\mathbf{1}}{\mathbf{4}}$. So the power series converges for $|\boldsymbol{z}-\mathbf{3 i}|<\frac{\mathbf{1}}{\mathbf{4}}$ and diverges for $|z-3 i|>\frac{1}{4}$.
16.2.4 Remark: We can also take $R=\frac{\mathbf{1}}{L^{*}}$ where $L^{*}=\lim _{n \rightarrow \infty}\left(\left|a_{n}\right|^{1 / n}\right)$.

### 16.2.5 Examples

1. For the series $\sum \frac{(n!)^{2} z^{n}}{2 n!}$, we find $R=\frac{1}{4}$.
2. Consider the series $\sum \frac{n!}{2^{n}}(z+1-i)^{n}$. Here $\frac{(n+1)!}{2^{n+1}} \frac{2^{n}}{n!}=\frac{n+1}{2} \rightarrow \infty$.

Hence $R=0$, that is, the series converges only at $z=-1+i$.
3. For the series $\sum\left(1+\frac{2}{n}\right)^{n^{2}} z^{n}$, note that $\lim \left(\left(1+\frac{2}{n}\right)^{n^{2}}\right)^{1 / n}=e^{2}$. Hence $R=e^{-2}$.
4. Take the series $\sum n^{\ln n} z^{n}$. Let $L=\lim \left(n^{\ln \ln n}\right)^{1 / n}$. So

$$
\log \log L=\lim \lim \frac{1}{n} \ln \ln \left(n^{\ln \ln n}\right)=\lim \lim \frac{1}{n}\left(\ln (\ln n)^{2}=0 .\right.
$$

Hence $L=e^{0}=1$, or, $R=1$.
5. For the series $\sum \frac{z^{2 n}}{4^{n_{n} \alpha}}, \quad \alpha>0$, let $p_{n}$ denote the $n^{\text {th }}$ term.

Then $\left|\frac{p_{n+1}}{p_{n}}\right|=\left|\frac{z^{2 n+2}}{4^{n+1}(n+1)^{\alpha}} \frac{4^{n} n^{\alpha}}{z^{2 n}}\right|=\frac{|z|^{2}}{4}\left(\frac{n}{n+1}\right)^{\alpha} \rightarrow \frac{|z|^{2}}{4}$.

Hence the series converges for $|z|<2$ and diverges for $|z|>2$.
6. $\sum \frac{n!}{n^{p}} z^{n}, \quad p$ is a positive integer. Here
$\frac{(n+1)!n^{p}}{(n+1)^{p} n!}=(n+1)\left(\frac{n}{n+1}\right)^{p} \rightarrow \infty$

Hence $R=0$.
7. $\sum \frac{z^{n^{2}}}{2^{n}}$,

$$
\left|\frac{z^{(n+1)^{2} 2^{n}}}{2^{n+1} z^{n^{2}}}\right|=\frac{|z|^{2 n+1}}{2} \rightarrow 0 \text { for }|z|<1
$$

Hence the series converges for $|z|<1$.

$$
\sum \frac{1}{n!}\left(\frac{i z-1}{2+i}\right)^{n}=\sum \frac{1}{n!} \frac{i^{n}}{(2+i)^{n}}\left(z-\frac{1}{i}\right)^{n}=\sum \frac{1}{n!} \frac{i^{n}}{(2+i)^{n}}(z+i)^{n}=\sum \frac{1}{n!} \frac{(1+2 i)^{n}}{5^{n}}(z+
$$

8. $i)^{n}$

$$
\left|\frac{(1+2 i)^{n+1}}{(n+1)!5^{n+1}} \frac{n!5^{n}}{(1+2 i)^{n}}\right|=\frac{|1+2 i|}{5(n+1)}=\frac{\sqrt{5}}{5(n+1)} \rightarrow 0 \text { as } n \rightarrow \infty .
$$

Hence $R=\infty$.
9. $\sum n^{n} z^{n}, \quad L=\infty, R=0$.
10. $\sum \frac{3^{n}}{4^{n}+5^{n}} z^{n}, \quad R=\frac{5}{3}$.

### 16.3 Results on Power Series

If any given power series $\sum_{n=0}^{\infty} \boldsymbol{a}_{n} \boldsymbol{z}^{\boldsymbol{n}}$ has a nonzero radius of convergence $\boldsymbol{R}(\boldsymbol{R}>\mathbf{0})$, we write its sum a function $\boldsymbol{f}(\mathbf{z})$;

$$
f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}, \quad|z|<R .
$$

We say that $\boldsymbol{f}(\boldsymbol{z})$ is represented by the power series or it is developed in the power series.
16.3.1 Theorem: The function $\boldsymbol{f}(\mathbf{z})$ in (1) with $\boldsymbol{R}>\boldsymbol{0}$ is continuous at $\boldsymbol{z}=\mathbf{0}$.

Proof: $\mathbf{f}(\mathbf{0})=\mathbf{a}_{\mathbf{0}}$. Now $\boldsymbol{f}(\mathbf{z})$ converges absolutely for $|\boldsymbol{z}| \leq \boldsymbol{r}$ for any $\boldsymbol{r}<\boldsymbol{R}$. Hence the series

$$
\sum_{n=0}^{\infty}\left|a_{n}\right| r^{n-1}=\frac{1}{r} \sum_{n=0}^{\infty}\left|a_{n}\right| r^{n} \text { with } r>0 \text { converges. }
$$

Let $\quad \sum_{n=0}^{\infty}\left|\boldsymbol{a}_{\boldsymbol{n}}\right| \boldsymbol{r}^{\boldsymbol{n - 1}}=\boldsymbol{S}(\neq \mathbf{0})$. Then for $\mathbf{0}<|\boldsymbol{z}| \leq \boldsymbol{r}$,

$$
\begin{aligned}
\left|f(z)-a_{0}\right| & =\left|\sum_{n=0}^{\infty} a_{n} z^{n}\right| \leq|z| \sum_{n=0}^{\infty}\left|a_{n}\right||z|^{n-1} \leq|z| \sum_{n=0}^{\infty}\left|a_{n}\right| r^{n-1} \\
& =|z| S<\epsilon,
\end{aligned}
$$

For $|\boldsymbol{z}|<\boldsymbol{\delta}\left(\boldsymbol{\delta}<\boldsymbol{r}, \boldsymbol{\delta}<\frac{\boldsymbol{\epsilon}}{\boldsymbol{s}}\right)$. Hence $\boldsymbol{f}$ is continuous at $\boldsymbol{z}=\mathbf{0}$.
16.3.2 Theorem: Suppose that the power series $\sum_{n=0}^{\infty} \boldsymbol{a}_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}}$ and $\sum_{n=0}^{\infty} \boldsymbol{b}_{\boldsymbol{n}} \mathbf{z}^{\boldsymbol{n}}$ both converge for $|\boldsymbol{z}|<\boldsymbol{R}(\boldsymbol{R}>\mathbf{0})$ and have the same sum for all these $\boldsymbol{z}$. Then these series are identical, i.e., $\boldsymbol{a}_{\boldsymbol{n}}=\boldsymbol{b}_{\boldsymbol{n}} \forall \boldsymbol{n}=\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$.

Proof: Given $\quad a_{0}+a_{1} z+a_{2} z^{2}+\cdots=b_{0}+b_{1} z+b_{2} z^{2}+\cdots \forall|z|<R$. Taking $\mathbf{z}=\mathbf{0}$, we get $\quad \boldsymbol{a}_{\mathbf{0}}=\boldsymbol{b}_{\mathbf{0}}$.

Assume $\boldsymbol{a}_{\boldsymbol{n}}=\boldsymbol{b}_{\boldsymbol{n}} \forall \boldsymbol{n} \leq \boldsymbol{k}$. Then
$a_{k+1} z^{k+1}+\cdots=b_{k+1} z^{k+1}+\cdots$

Dividing both sides by $\boldsymbol{z}^{\boldsymbol{k + 1}}$ and then taking $\boldsymbol{z} \rightarrow \infty$, we get $\boldsymbol{a}_{\boldsymbol{k}+\boldsymbol{1}}=\boldsymbol{b}_{\boldsymbol{k}+\boldsymbol{1}}$. Hence by Mathematical induction $\boldsymbol{a}_{n}=\boldsymbol{b}_{n} \forall \boldsymbol{n}$ and so the two power series are identical.

Term by term addition or subtraction of two power series with radii of convergence $\boldsymbol{R}_{\mathbf{1}}$ and $\boldsymbol{R}_{\mathbf{2}}$ yields a power series with radius of convergence at least equal to the smaller of $\boldsymbol{R}_{\mathbf{1}}$ and $\boldsymbol{R}_{\mathbf{2}}$.

Term by term multiplication of two power series

$$
f(z)=\sum_{n=0}^{\infty} a_{n} z^{n} \text { and } g(z)=\sum_{n=0}^{\infty} b_{n} z^{n}
$$

means the multiplication of each term of the first series by each term of the second series and the collection of like power of $\boldsymbol{z}$. This gives a power series and is given by

$$
\begin{aligned}
& a_{0} b_{0}+\left(a_{0} b_{1}+a_{1} b_{0}\right) z+\left(a_{0} b_{2}+a_{1} b_{1}+a_{2} b_{0}\right) z^{2}+\cdots \\
& =\sum_{n=0}^{\infty}\left(a_{0} b_{n}+\cdots+a_{n} b_{0}\right) z^{n}
\end{aligned}
$$

This power series converges absolutely for each $\mathbf{z}$ within the circle of convergence of each of the two given series and has the sum $s(z)=f(z) g(z)$.
16.3.3 Theorem: The derived series of a power series has the same radius of convergence as the original series.

Proof: $f(z)=\sum_{n=0}^{\infty} \boldsymbol{a}_{n} z^{n}$ have the same radius of convergence $R=\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|$.

The series after differentiation is
$f(z)=\sum_{n=1}^{\infty} n a_{n} z^{n-1}=a_{1}+2 a_{2} z+3 a_{3} z^{2}+\cdots$

Now
$\lim _{n \rightarrow \infty} \frac{n\left|a_{n}\right|}{(n+1)\left|a_{n+1}\right|}=\lim _{n \rightarrow \infty}\left(\frac{n}{n+1}\right) \lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=R$.
16.3.4 Example: Consider the series

$$
\sum_{n=1}^{\infty}\binom{n}{2} z^{n}=z^{2}+3 z^{3}+6 z^{4}+10 z^{5}+\cdots
$$

Then

$$
\left|\frac{\frac{(n+1) n}{2} z^{n+1}}{\frac{n(n-1)}{2} z^{n}}\right|=\frac{n+1}{n}|z| \rightarrow|z|
$$

So the series converges for $\boldsymbol{R}<\mathbf{1}$ and diverges for $\boldsymbol{R}>\mathbf{1}$.
16.3.5 Theorem: The power series $\sum_{n=0}^{\infty} \frac{a_{n}}{n+1} z^{n+1}=a_{0} z+\frac{a_{1}}{2} z^{2}+\cdots$ obtained by integrating $\boldsymbol{f}(\mathbf{z})=\sum_{n=0}^{\infty} \boldsymbol{a}_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}}$ term by term has the same radius of convergence as the orginal series.
16.3.6 Theorem: A power series with a nonzero radius of convergence $R$ represents an analytic function at every point interior to its circle of convergence. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence each of them is an analytic function.

Proof: Consider the two series $f(z)=\sum_{n=0}^{\infty} \boldsymbol{a}_{n} z^{n}, f_{1}(z)=\sum_{n=1}^{\infty} n \boldsymbol{a}_{n} z^{n-1}$ Let $\boldsymbol{f}(\mathbf{z})$ have the radius of convergence $\boldsymbol{R}$. We will show that the function $\boldsymbol{f}$ is analytic and has derivative $\boldsymbol{f}_{\mathbf{1}}(\boldsymbol{z})$ in the interior of the circle of convergence.

$$
\begin{aligned}
& \frac{f(\mathrm{z}+\Delta z)-f(z)}{\Delta z}-f_{1}(z)=\sum_{n=2}^{\infty} a_{n}\left[\frac{(\mathrm{z}+\Delta z)^{n}-z^{n}}{\Delta z}-n z^{n-1}\right] \\
& =\sum_{n=2}^{\infty} a_{n} \Delta z\left[(\mathrm{z}+\Delta z)^{n-2}+2 z(\mathrm{z}+\Delta z)^{n-3}+\cdots+(n-1) z^{n-2}\right]
\end{aligned}
$$

The bracket contains ( $\boldsymbol{n}-\mathbf{1}$ ) terms, and the largest coefficient is $(\boldsymbol{n}-\mathbf{1})$. For $|\boldsymbol{z}| \leq \boldsymbol{R}_{\mathbf{0}},|\mathbf{z}+\boldsymbol{\Delta z}| \leq \boldsymbol{R}_{\mathbf{0}}, \boldsymbol{R}_{\mathbf{0}}<\boldsymbol{R}$, the absolute value of the series is less than or equal to
$\leq|\Delta z| \sum_{n=2}^{\infty}\left|a_{n}\right|(n-1)^{2} R_{0}^{n-2}$
$\leq|\Delta z| \sum_{n=2}^{\infty}\left|a_{n}\right| n(n-1) R_{0}^{n-2}$

The series $\sum_{n=0}^{\infty} \boldsymbol{a}_{\boldsymbol{n}} \boldsymbol{n}(\boldsymbol{n}-\mathbf{1}) \boldsymbol{R}_{0}^{\boldsymbol{n}-\mathbf{2}}$ is the second derived series of $\boldsymbol{f}(\boldsymbol{z})$ at $\mathbf{z}=\boldsymbol{R}_{\mathbf{0}}\left(\boldsymbol{R}_{\mathbf{0}}<\boldsymbol{R}\right)$ and converges absolutely. Let $\boldsymbol{K}\left(\boldsymbol{R}_{\mathbf{0}}\right)$ be the sum then
$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-f_{1}(z)\right| \leq|\Delta z| K\left(R_{0}\right) \rightarrow 0$ as $\Delta z \rightarrow 0$.

This completes the proof of the theorem.

### 16.3.7 Examples:

1. $f(z)=\sum_{n=2}^{\infty} \frac{z^{n}}{n(n-1)}$. Differentiating twice, we get $\sum_{n=2}^{\infty} z^{n-2}$ which is convergent for $|z|<1$ and is divergent for $|z|>1$.
2. $\sum_{n=1}^{\infty} \frac{6^{n}}{n}(z-i)^{n}$. Differentiating, we get $\sum_{n=1}^{\infty} 6^{n}(z-i)^{n-1}$ whose radius of convergence is $\lim _{n \rightarrow \infty} \frac{6^{n}}{6^{n+1}}=\frac{1}{6}$.
3. $\sum_{n=2}^{\infty} n(n-1)\left(\frac{z}{5}\right)^{n}$. Consider the series $\sum_{n=0}^{\infty}\left(\frac{z}{5}\right)^{n}$. Differentiating this twice and multiplying by $z^{2}$, we get the original series. Now clearly the radius of convergence of this new series is 5 .
4. $\sum_{n=k}^{\infty}\binom{n}{k}\left(\frac{z}{\pi}\right)^{n}$. Differentiating the series $\sum\left(\frac{z}{\pi}\right)^{n}$ term by term $k$ times and multiplying by $\frac{z^{k}}{k!}$, we get the original series. Now the radius of convergence of $\sum\left(\frac{z}{\pi}\right)^{n}$ is 5 .

## Suggested Readings

Ahlfors, L.V. (1979). Complex Analysis, McGraw-Hill, Inc., New York.
Boas, R.P. (1987). Invitation to Complex Analysis, McGraw-Hill, Inc., New York.

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Conway, J.B. (1993). Functions of One Complex Variable, Springer-Verlag, New York.

Fisher, S.D. (1986). Complex Variables, Wadsworth, Inc., Belmont, CA.
Jain, R.K. and Iyengar, S.R.K. (2002). Advanced Engineering Mathematics, Narosa Publishing House, New Delhi.

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## Lesson 17

## Taylor Series and Laurent Series

### 17.1 Taylor Series

The following theorem shows that a Taylor series can be found for an analytic function.
17.1.1 Theorem: Let $\boldsymbol{f}(\boldsymbol{z})$ be analytic in a domain $\boldsymbol{D}$ and let $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ be any point in $\boldsymbol{D}$. Then there is a unique Taylor series

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n} \tag{17.1.1}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{n}=\frac{1}{n!} f^{(n)}\left(z_{0}\right)=\frac{1}{2 \pi i} \int_{C} \frac{f\left(z^{*}\right)}{\left(z^{*}-z_{0}\right)^{n+1}} \mathrm{dz}^{*} \tag{17.1.2}
\end{equation*}
$$

and $\boldsymbol{C}$ contains $\mathbf{z}_{\mathbf{0}}$. This representation is valid in the largest open disk with center $\mathbf{z}_{\mathbf{0}}$ in which $\boldsymbol{f}(\mathbf{z})$ is analytic. The remainder $\boldsymbol{R}_{\boldsymbol{n}}(\mathbf{z})$ of (17.1.1) can be represented as:

$$
\begin{equation*}
R_{n}(z)=\frac{\left(z-z_{0}\right)^{n}}{2 \pi i} \oint \frac{f\left(z^{*}\right)}{\left(z^{*}-z_{0}\right)^{n+1}\left(z^{*}-z\right)} d z^{*} \tag{17.1.3}
\end{equation*}
$$

The coefficient satisfy the inequality $\left|\boldsymbol{a}_{\boldsymbol{n}}\right| \leq \frac{\boldsymbol{M}}{\boldsymbol{r}^{n}}$, where $\boldsymbol{M}$ is the maximum of $|\boldsymbol{f}(\boldsymbol{z})|$ on a circle $\left|\boldsymbol{z}-z_{\mathbf{0}}\right|=\boldsymbol{r}$ in $\boldsymbol{D}$ whose interior is also in $\boldsymbol{D}$.

Proof: By Cauchy's integral formula, we have

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \int_{C} \frac{f\left(z^{*}\right)}{z^{*}-z_{0}} d z^{*} \tag{17.1.4}
\end{equation*}
$$

for $\boldsymbol{z}$ lying inside $\boldsymbol{C}$. Now

$$
\begin{gathered}
\frac{1}{z^{*}-z}=\frac{1}{z^{*}-z_{0}-\left(z-z_{0}\right)}=\frac{1}{\left(1-\frac{z-z_{0}}{z^{*}-z_{0}}\right)\left(z^{*}-z_{0}\right)} \\
=\frac{1}{z^{*}-z_{0}}\left[1+\frac{z-z_{0}}{z^{*}-z_{0}}+\left(\frac{z-z_{0}}{z^{*}-z_{0}}\right)^{2}+\cdots+\left(\frac{z-z_{0}}{z^{*}-z_{0}}\right)^{n}\right. \\
\left.+\left(\frac{z-z_{0}}{z^{*}-z_{0}}\right)^{n+1} \frac{1}{1-\frac{z-z_{0}}{z^{*}-z_{0}}}\right]
\end{gathered}
$$

This expansion is valid for $\left|\frac{z-z_{0}}{z^{*}-z_{0}}\right| \neq \mathbf{1}$, we can do so as $\mathbf{z}^{*}$ in on $\boldsymbol{C}$ and we choose $\mathbf{z}$ inside the circle of radius $\boldsymbol{r}$ with center $\boldsymbol{z}_{\mathbf{0}}$, so that $\left|\frac{\boldsymbol{z}-\mathbf{z}_{0}}{\mathbf{z}^{*}-\mathbf{z}_{0}}\right|<\mathbf{1}$.

## Thus

$$
\begin{align*}
\frac{1}{z^{*}-z}= & \frac{1}{z^{*}-z_{0}}\left[1+\frac{z-z_{0}}{z^{*}-z_{0}}+\left(\frac{z-z_{0}}{z^{*}-z_{0}}\right)^{2}+\cdots+\left(\frac{z-z_{0}}{z^{*}-z_{0}}\right)^{n}\right] \\
& +\frac{1}{z^{*}-z}\left(\frac{z-z_{0}}{z^{*}-z_{0}}\right)^{n+1} \tag{17.1.5}
\end{align*}
$$

Using (17.1.5) in (17.1.4), we get

$$
\begin{gathered}
f(z)=\frac{1}{2 \pi i} \int_{C} f\left(z^{*}\right)\left[\frac{1}{z^{*}-z_{0}}+\frac{z-z_{0}}{\left(z^{*}-z_{0}\right)^{2}}+\frac{\left(z-z_{0}\right)^{2}}{\left(z^{*}-z_{0}\right)^{3}}+\cdots\right. \\
\left.+\frac{\left(z-z_{0}\right)^{n}}{\left(z^{*}-z_{0}\right)^{n+1}}+\frac{1}{z^{*}-z} \frac{\left(z-z_{0}\right)^{n+1}}{\left(z^{*}-z_{0}\right)^{n+1}}\right] d z^{*}
\end{gathered}
$$

$$
\begin{align*}
&=\frac{1}{2 \pi i} \int_{C} \frac{f\left(z^{*}\right)}{z^{*}-z_{0}} d z^{*}+\frac{z-z_{0}}{2 \pi i} \int_{C} \frac{f\left(z^{*}\right)}{\left(z^{*}-z_{0}\right)^{2}} d z^{*}+\cdots \\
&+\frac{\left(z-z_{0}\right)^{n}}{2 \pi i} \int_{C} \frac{f\left(z^{*}\right)}{\left(z^{*}-z_{0}\right)^{n+1}} d z^{*} \\
&+\frac{\left(z-z_{0}\right)^{n+1}}{2 \pi i} \int_{C} \frac{f\left(z^{*}\right)}{\left(z^{*}-z\right)\left(z^{*}-z_{0}\right)^{n+1}} d z^{*} \tag{17.1.6}
\end{align*}
$$

This is Taylor's formula with remainder term.

Since analytic functions have derivatives of all orders, we can take $\boldsymbol{n}$ in (17.1.6) as large as possible. If we let $\boldsymbol{n} \rightarrow \infty$, we get (17.1.1). Clearly, (17.1.1) will convergence and represent $\boldsymbol{f}(\boldsymbol{z})$ if and only if

$$
\lim _{n \rightarrow \infty} R_{n}(z)=0 .
$$

Since $\boldsymbol{z}^{*}$ is on $\boldsymbol{C}$ and $\boldsymbol{z}$ is inside $\boldsymbol{C},\left|\boldsymbol{z}^{*}-\boldsymbol{z}\right|>0$. Since $\boldsymbol{f}(\mathbf{z})$ is analytic inside and on $C$, it is bounded, and so is the function $\frac{f\left(z^{*}\right)}{z^{*}-z}$, i.e.,

$$
\left|\frac{f\left(z^{*}\right)}{z^{*}-z}\right| \leq M^{*}, \forall z^{*} \text { on } C .
$$

Also $\boldsymbol{C}$ has the radius $\boldsymbol{r}=\left|\boldsymbol{z}^{*}-z_{0}\right|$ and the length $\mathbf{2 \pi r}$.
Hence by the ML-inequality, we get from (17.1.3)

$$
\begin{aligned}
& R_{n}=\frac{\left|z-z_{0}\right|^{n+1}}{2 \pi}\left|\int_{C} \frac{f\left(z^{*}\right)}{\left(z^{*}-z\right)\left(z^{*}-z_{0}\right)^{n+1}} d z^{*}\right| \\
\leq & \frac{\left|z-z_{0}\right|^{n+1}}{2 \pi} M^{*} 2 \pi r \frac{1}{r^{n+1}}
\end{aligned}
$$

$$
=M^{*} r\left|\frac{Z-z_{0}}{r}\right|^{n+1}
$$

Now $\left|\boldsymbol{z}-\boldsymbol{z}_{\mathbf{0}}\right|<\boldsymbol{r}$ as $\boldsymbol{z}$ lies inside $\boldsymbol{C}$. Hence the term on the right $\rightarrow \boldsymbol{0}$ as $\boldsymbol{n} \rightarrow \infty$. Hence the convergence of the Taylor series is proved. Uniqueness follows since power series have unique representation of functions.

## Finally

$\left|a_{n}\right|=\frac{1}{2 \pi}\left|\int_{C} \frac{f\left(z^{*}\right)}{\left(z^{*}-z_{0}\right)^{n+1}} \mathrm{dz}\right| \leq \frac{M}{r^{n}}$.

### 17.1.3 Maclaurin's Series

A Maclaurin's series is a Taylor series with center $\mathbf{z}_{\mathbf{0}}=\mathbf{0}$. That is,

$$
f(z)=\sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} z^{n},|z|<R_{0} .
$$

A point $\boldsymbol{z}=\boldsymbol{c}$ at which $\boldsymbol{f}(\boldsymbol{z})$ is not differentiable but such that every disk with center $\boldsymbol{c}$ contains points at which $\boldsymbol{f}(\boldsymbol{z})$ is differentiable. We say that $\boldsymbol{f}(\boldsymbol{z})$ is singular at $\boldsymbol{c}$ or has a singularity at $\boldsymbol{c}$.
17.1.3 Theorem: A power series with nonzero radius of convergence is the Taylor series of its sum.

Proof: Given the power series

$$
f(z)=a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\cdots
$$

Then $\boldsymbol{f}\left(\boldsymbol{z}_{0}\right)=\boldsymbol{a}_{\mathbf{0}}$. Now

$$
f^{\prime}(z)=a_{1}+2 a_{2}\left(z-z_{0}\right)+\cdots
$$

Thus $\boldsymbol{f}^{\prime}\left(\boldsymbol{z}_{0}\right)=\boldsymbol{a}_{1}$. Further,

$$
f^{\prime \prime}(z)=2 a_{2}+3 * 2 a_{3}\left(z-z_{0}\right)+\cdots
$$

Thus $\boldsymbol{f}^{\prime \prime}\left(\boldsymbol{z}_{0}\right)=\mathbf{2 a} \boldsymbol{a}_{2}$.

In general, $\boldsymbol{f}^{n}\left(\boldsymbol{z}_{\mathbf{0}}\right)=\boldsymbol{n}!\boldsymbol{a}_{\boldsymbol{n}}$. With these coefficients the given series becomes the Taylor's series of $\boldsymbol{f}(\mathbf{z})$.
17.1.4 Remark: Complex analytic functions have derivatives of all orders and they can always be represented by power series of the from (17.1.1). This is not true in general for real valued functions. In fact, there are real functions for which derivatives of all orders exist but it cannot be represented by a power series.

Consider for example, $f(x)=e^{-1 / x^{2}}, x \neq 0,=0, x=0$

This function cannot be represented by a Maclaurin's series since all its derivatives vanish at zero.

### 17.1.5 Examples:

1. $f(z)=\frac{1}{1-z}$. Then $f^{n}(z)=\frac{n!}{(1-z)^{n+1}}, f^{n}(0)=n$ !. Hence the Maclaurin's expansion of $f$ is the geometric series

$$
\frac{1}{1-z}=\sum_{n=0}^{\infty} z^{n}=1+z+z^{2}+\cdots, \quad|z| \leq 1 .
$$

$f(z)$ is singular at $z=1$. This point lies on the circle of convergence.
2. $f(z)=\mathrm{e}^{\mathrm{z}}=\sum_{n=0}^{\infty} \frac{\mathrm{z}^{\mathrm{n}}}{n!}=1+z+\frac{\mathrm{z}^{2}}{2}+\cdots$.
3. $\cos z=\sum_{n=0}^{\infty}(-1)^{n} \frac{z^{2 n}}{(2 n)!}=1-\frac{z^{2}}{2!}+\frac{z^{4}}{4!}+\cdots$.
4. $\sin z=\sum_{n=0}^{\infty}(-1)^{n} \frac{z^{2 n+1}}{(2 n+1)!}=z-\frac{z^{3}}{3!}+\frac{z^{5}}{5!}+\cdots$.
5. $\quad \sinh (z)=\sum_{n=0}^{\infty} \frac{z^{2 \mathrm{n}+1}}{(2 \mathrm{n}+1)!}=z+\frac{z^{3}}{3!}+\frac{z^{5}}{5!}+\cdots$.
6. $\quad \cosh (z)=\sum_{n=0}^{\infty} \frac{z^{2 n}}{(2 n)!}=1+\frac{z^{2}}{2!}+\frac{z^{4}}{4!}+\cdots$.
7. $\operatorname{Ln}(1+z)=z-\frac{z^{2}}{2}+\frac{z^{3}}{3}+\cdots, \quad|z| \leq 1$.
8. $\quad \operatorname{Ln}\left(\frac{1+z}{1-z}\right)=2\left(z+\frac{z^{3}}{3}+\frac{z^{5}}{5}+\cdots\right), \quad|z| \leq 1$.
9. $\frac{1}{1+z^{2}}=\frac{1}{1-\left(-z^{2}\right)}=\sum_{n=0}^{\infty}\left(-z^{2}\right)^{n}=1-z^{2}+z^{4}-z^{6}+\cdots, \quad|z| \leq 1$.
10. To find Maclaurin's series for $f(z)=\tan ^{-1} z$,
$f^{\prime}(z)=\frac{1}{1+z^{2}}=1-z^{2}+z^{4}-z^{6}+\cdots,|z| \leq 1$.

Integrating the power series term by term:

$$
\tan ^{-1} z=z-\frac{z^{3}}{3}+\frac{z^{5}}{5}-\cdots, \quad|z| \leq 1
$$

representing the principal value of $\tan ^{-1} z$.

### 17.2 Laurent Series

The following theorem gives the conditions for the existence of a Laurent's series.
17.2.1 Theorem: If $\boldsymbol{f}(\mathbf{z})$ is analytic on two concentric circles $\boldsymbol{C}_{\mathbf{1}}$ and $\boldsymbol{C}_{\mathbf{2}}$ with center $\mathbf{z}_{\mathbf{0}}$ and in the annulus between them, then $\boldsymbol{f}(\mathbf{z})$ can be represented by the Laurent series

$$
\begin{aligned}
& f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}+\sum_{n=0}^{\infty} \frac{b_{n}}{\left(z-z_{0}\right)^{n}} \\
& =a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\cdots+\frac{b_{1}}{z-z_{0}}+\frac{b_{2}}{\left(z-z_{0}\right)^{2}} \\
& +\cdots \quad(17.2 .1)
\end{aligned}
$$

consisting of nonnegative powers and the principal part (the negative powers).
The coefficients of this Laurent series are given by the integrals

$$
=\frac{1}{2 \pi i} \int_{C} \frac{f\left(z^{*}\right)}{\left(z^{*}-z_{0}\right)^{n+1}} \mathrm{dz}^{*}, \quad b_{n}=\frac{1}{2 \pi i} \int_{C}\left(z^{*}-z_{0}\right)^{n+1} f\left(z^{*}\right) \mathrm{dz}^{*}
$$

taken counter clockwise around any simple closed path $\boldsymbol{C}$ that lies in the annulus and encircles the inner circle.

This series converges and represents $\boldsymbol{f}(\mathbf{z})$ in the open annulus obtained from the given annulus by continuously increasing the outer circle $\boldsymbol{C}_{\mathbf{1}}$ and decreasing $\boldsymbol{C}_{\mathbf{2}}$ until each of the circles reaches a point where $\boldsymbol{f}(\mathbf{z})$ is singular.

In the special case that $\boldsymbol{z}_{\mathbf{0}}$ is the only singular point of $\boldsymbol{f}(\boldsymbol{z})$ inside $\boldsymbol{C}_{\mathbf{2}}$, this circle can be shrunk to the point $\mathbf{z}_{\mathbf{0}}$, giving convergence in a disk except at the center.

Proof: By Cauchy’s integral formula for multiply connected domains, we get

$$
\begin{equation*}
f(z)=g(z)+h(z)+\frac{1}{2 \pi i} \int_{C_{1}} \frac{f\left(z^{*}\right)}{z^{*}-z} d z^{*}-\frac{1}{2 \pi i} \int_{C_{2}} \frac{f\left(z^{*}\right)}{z^{*}-z} \mathrm{dz}^{*} \tag{17.2.3}
\end{equation*}
$$

where $\boldsymbol{z}$ is any point in the given annulus and both $\boldsymbol{C}_{\mathbf{1}}$ and $\boldsymbol{C}_{\mathbf{2}}$ are counterclockwise. Now $\boldsymbol{g}(\boldsymbol{z})$ integral is exactly the Taylor series so that

$$
g(z)=\frac{1}{2 \pi i} \int_{C_{1}} \frac{f\left(z^{*}\right)}{z^{*}-z} \mathrm{dz}^{*}=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}
$$

with coefficients $a_{n}=\frac{1}{2 \pi i} \int_{C} \frac{f\left(z^{*}\right)}{\left(z^{*}-z_{0}\right)^{n+1}} d z^{*}$.
Here $\boldsymbol{C}_{\boldsymbol{1}}$ can be replaced by $\boldsymbol{C}$ by the principal of deformation of path as $\boldsymbol{z}_{\mathbf{0}}$ is a point not in the annulus.

To get the expansion for $\boldsymbol{h}(\mathbf{z})$, we note that $\left|\frac{z^{*}-\mathbf{z}_{0}}{\boldsymbol{z}-\mathbf{z}_{0}}\right|<\mathbf{1}$ for $\boldsymbol{z}^{*}$ on $\boldsymbol{C}_{\mathbf{2}}$ and $\mathbf{z}$ is the annulus.

Now

$$
\frac{1}{z^{*}-z}=\frac{1}{z^{*}-z_{0}-\left(z-z_{0}\right)}=\frac{-1}{\left(z-z_{0}\right)\left(1-\frac{z^{*}-z_{0}}{z-z_{0}}\right)}
$$

$$
\begin{aligned}
=-\frac{1}{z-z_{0}}[ & \left.1+\frac{z^{*}-z_{0}}{z-z_{0}}+\left(\frac{z^{*}-z_{0}}{z-z_{0}}\right)^{2}+\cdots+\left(\frac{z^{*}-z_{0}}{z-z_{0}}\right)^{n}\right] \\
& -\frac{1}{z-z^{*}}\left(\frac{z^{*}-z_{0}}{z-z_{0}}\right)^{n+1}
\end{aligned}
$$

Multiplying by $\frac{-\boldsymbol{f}\left(\mathbf{z}^{*}\right)}{2 \boldsymbol{\pi} \boldsymbol{i}}$ and integrating over $\boldsymbol{C}_{\mathbf{2}}$ on both the sides, we get

$$
\begin{aligned}
h(z)=-\frac{1}{2 \pi i} & \int \frac{f\left(z^{*}\right)}{z^{*}-z} d z^{*}=\frac{1}{2 \pi i}\left[\frac{1}{z-z_{0}} \int_{C_{2}} f\left(z^{*}\right) \mathrm{dz}^{*}\right. \\
& +\frac{1}{\left(z-z_{0}\right)^{2}} \int_{C_{2}}\left(z^{*}\right. \\
& \left.\left.-z_{0}\right) f\left(z^{*}\right) d z^{*}+\cdots+\frac{1}{\left(z-z_{0}\right)^{n+1}} \int_{C_{2}}\left(z^{*}-z_{0}\right)^{n} f\left(z^{*}\right) d z^{*}\right]
\end{aligned}
$$

$+R_{n}^{*}(z)$,
where,

$$
R_{n}^{*}(z)=\frac{1}{2 \pi i\left(z-z_{0}\right)^{n+1}} \oint_{C_{2}} \frac{\left(z^{*}-z_{0}\right)^{n+1}}{\left(z-z^{*}\right)} f\left(z^{*}\right) \mathrm{dz}^{*}
$$

The integral over $\boldsymbol{C}_{\mathbf{2}}$ can be replaced by integrals over $\boldsymbol{C}$.

We see that on the right, the power $\frac{1}{\left(z-z_{0}\right)^{n}}$ is multiplied by $\boldsymbol{b}_{\boldsymbol{n}}$ as given in (17.2.2). This proves Laurent's theorem provided $\lim _{n \rightarrow \infty} \boldsymbol{R}_{n}^{*}(\mathbf{z})=\mathbf{0}$.

Now if the principal part consists of finitely many terms only, then there is nothing to prove. Otherwise, we note that $\frac{f\left(\boldsymbol{z}^{*}\right)}{\boldsymbol{z}-\boldsymbol{z}^{*}}$ in $\boldsymbol{R}_{n}^{*}(\boldsymbol{z})$ is bounded in the absolute value, say $\left|\frac{f\left(z^{*}\right)}{\boldsymbol{z}-\boldsymbol{z}^{*}}\right|<\boldsymbol{M}^{*} \forall \boldsymbol{z}^{*}$ on $\boldsymbol{C}_{\mathbf{2}}$ because $\boldsymbol{f}\left(\boldsymbol{z}^{*}\right)$ is analytic in the
annulus and on $\boldsymbol{C}_{\mathbf{2}}$, and $\boldsymbol{z}^{*}$ lies on $\boldsymbol{C}_{\mathbf{2}}$ and $\boldsymbol{z}$ outside, so that $\boldsymbol{z}-\boldsymbol{z}^{*} \neq \mathbf{0}$. From this and the ML-inequality, we get

$$
\begin{aligned}
& R_{n}^{*}(z) \leq \frac{1}{2 \pi\left|z-z_{0}\right|^{n+1}\left|z^{*}-z_{0}\right|^{n+1} M^{*} L \quad\left(L=\text { Length } C_{2}\right)} \\
& =\frac{M^{*} L}{2 \pi}\left|\frac{z^{*}-z_{0}}{z-z_{0}}\right|^{n+1} \rightarrow 0 \text { as } n \rightarrow \infty .
\end{aligned}
$$

The first series in (17.2.1) is a Taylor series $(\boldsymbol{g}(\boldsymbol{z}))$ and hence it converges in the disk $\boldsymbol{D}$ with center $\boldsymbol{z}_{\mathbf{0}}$ whose radius equals the distance of that singularity of $\boldsymbol{g}(\mathbf{z})$ which is closet to $\mathbf{z}_{\mathbf{0}}$. Also, $\boldsymbol{g}(\mathbf{z})$ must be singular at all points outside $\boldsymbol{C}_{\mathbf{1}}$ where $\boldsymbol{f}(\boldsymbol{z})$ is singular.

The second series in (17.2.1) representing $\boldsymbol{h}(\boldsymbol{z})$ is a power series in $\boldsymbol{z}=\frac{\mathbf{1}}{\boldsymbol{z}-\mathbf{z}_{0}}$. Let the given annulus be $\boldsymbol{r}_{\mathbf{2}}<\left|\boldsymbol{z}-\boldsymbol{z}_{\mathbf{0}}\right|<\boldsymbol{r}_{\mathbf{1}}$ where $\boldsymbol{r}_{\mathbf{1}}$ and $\boldsymbol{r}_{\mathbf{2}}$ are radii of $\boldsymbol{C}_{\mathbf{1}}$ and $\boldsymbol{C}_{\mathbf{2}}$ respectively. Then $\frac{\mathbf{1}}{r_{2}}>|\boldsymbol{z}|>\frac{\mathbf{1}}{r_{1}}$. Hence this power series in $\boldsymbol{z}$ must converge at least in the disk $|\boldsymbol{r}|<\frac{\mathbf{1}}{\boldsymbol{r}_{2}}$. This corresponds to the exterior $\left|\boldsymbol{z}-\boldsymbol{z}_{0}\right|>\boldsymbol{r}_{2}$ of $\boldsymbol{C}_{\mathbf{2}}$, so that $\boldsymbol{h}(\mathbf{z})$ is analytic for all $\mathbf{z}$ in the exterior $E$ of the circle with center $\mathbf{z}_{\mathbf{0}}$ and radius equal to the maximum distance from $\boldsymbol{z}_{\mathbf{0}}$ to the singularities of $\boldsymbol{f}(\boldsymbol{z})$ inside $\boldsymbol{C}_{\mathbf{2}}$. The domain common to $\boldsymbol{D}$ and $\boldsymbol{E}$ is the open annulus.
17.2.2 Remark: The Laurent series of a given analytic function $\boldsymbol{f}(\boldsymbol{z})$ is unique in its annulus of existence. However, $\boldsymbol{f}(\boldsymbol{z})$ may have different Laurent series in two annulus with the same center.

### 17.2.3 Examples:

1. $f(z)=z^{-5} \sin z$, with center 0 .

$$
\begin{aligned}
& z^{-5} \sin z=z^{-5}\left[z-\frac{z^{3}}{3!}+\frac{z^{5}}{5!}+\cdots\right] \\
& =\frac{1}{z^{4}}-\frac{1}{3!z^{2}}+\frac{1}{5!}-\frac{z^{7}}{7!}+\frac{z^{9}}{9!}-\cdots
\end{aligned}
$$

for $|z|>0$. Hence the annulus is the whole complex plane except the origin.
2. $f(z)=z^{2} e^{1 / z}=z^{2}\left[1+\frac{1}{1!z}+\frac{1}{2!z^{2}}+\cdots\right]$

$$
=z^{2}+\frac{z}{1!}+\frac{1}{2!}+\frac{1}{3!z}+\frac{1}{4!z^{2}}+\cdots, \quad|z|>0
$$

3. $f(z)=\frac{1}{1-z}=\sum_{n=0}^{\infty} z^{n}, \quad|z|<1$
and
$f(z)=-\frac{1}{z\left(1-z^{-1}\right)}=-\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}=-\frac{1}{z}-\frac{1}{z^{2}}-\cdots$
valid for $|\boldsymbol{z}|>1$.
4. $f(z)=\frac{1}{z^{3}-z^{4}}$ center 0

From the previous geometric series, we get by multiplying $\frac{1}{z^{3}}$,

$$
\begin{aligned}
& \frac{1}{z^{3}-z^{4}}=\frac{1}{z^{3}}+\frac{1}{z^{2}}+\frac{1}{z}+1+\cdots, \quad 0<|z|<1 . \\
& \frac{1}{z^{3}-z^{4}}=-\frac{1}{z^{4}}-\frac{1}{z^{5}}-\cdots, \quad|z|>1
\end{aligned}
$$

5. $f(z)=\frac{-2 z+3}{z^{2}-3 z+2}$ center 0

$$
\begin{aligned}
& \frac{-2 z+3}{z^{2}-3 z+2}=-\frac{1}{z-1}-\frac{1}{z-2} \\
& =\frac{1}{1-z}+\frac{1}{2\left(1-\frac{z}{2}\right)} \\
& =\sum_{n=0}^{\infty} z^{n}+\frac{1}{2} \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n} \text { for }|z|<1 \text { (first for }|z|<1 \text { and second for } \\
& |z|<2) \\
& =\sum_{n=0}^{\infty}\left(1+\frac{1}{2^{n+1}}\right) z^{n}, \text { for }|z|<1 \\
& =\frac{3}{2}+\frac{5}{4} z+\frac{9}{8} z^{2}+\cdots
\end{aligned}
$$

We can also write $f(z)=\frac{1}{1-z}+\frac{1}{2-z}=-\frac{1}{z\left(1-\frac{1}{z}\right)}-\frac{1}{z\left(1-\frac{2}{z}\right)}$

$$
\begin{aligned}
& =-\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}-\sum_{n=0}^{\infty} \frac{2^{n}}{z^{n+1}} \text { for }|z|>2 \text { (first for }|z|>1 \text { and second for }|z|>2 \text { ) } \\
& =\sum_{n=0}^{\infty}\left(1+2^{n}\right) \frac{1}{z^{n+1}}
\end{aligned}
$$

$$
=-\frac{2}{z}-\frac{3}{z^{2}}-\frac{5}{z^{3}}-\frac{9}{z^{4}}-\cdots
$$

$$
\text { for }|z|>2
$$

## Suggested Readings

Ahlfors, L.V. (1979). Complex Analysis, McGraw-Hill, Inc., New York.
Boas, R.P. (1987). Invitation to Complex Analysis, McGraw-Hill, Inc., New York.

Brown, J.W. and Churchill, R.V. (1996). Complex Variables and Applications. McGraw-Hill, Inc., New York.

Conway, J.B. (1993). Functions of One Complex Variable, Springer-Verlag, New York.

Fisher, S.D. (1986). Complex Variables, Wadsworth, Inc., Belmont, CA.
Jain, R.K. and Iyengar, S.R.K. (2002). Advanced Engineering Mathematics, Narosa Publishing House, New Delhi.

Ponnusamy, S. (2006). Foundations of Complex Analysis, Alpha Science International Ltd, United Kingdom.

## Lesson 18

## Zeros and Singularities

### 18.1 Singular Points

A function $\boldsymbol{f}(\boldsymbol{z})$ is singular or has a singularity at a point $\boldsymbol{z}=\mathbf{z}_{\mathbf{0}}$ if $\boldsymbol{f}(\boldsymbol{z})$ is not analytic at $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ but every neighbourhood of $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ contains points at which $\boldsymbol{f}(\boldsymbol{z})$ is analytic. Then we say that $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ is a singular point of $\boldsymbol{f}(\mathbf{z})$.

The point $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ is called an isolated singularity of $\boldsymbol{f}(\boldsymbol{z})$ if $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ has a neighbourhood without further singularites of $\boldsymbol{f}(\mathbf{z})$.
18.1.1 Example: The function $f(z)=\boldsymbol{\operatorname { t a n }}\left(\frac{1}{z}\right)$ has a non-isolated singularity at $\mathbf{z}=\mathbf{0}$.
18.1.2 Example: The function $\mathbf{f}(\mathbf{z})=\boldsymbol{\operatorname { t a n }}(\mathbf{z})$ has isolated singularities at $z= \pm \frac{\pi}{2}, \pm \frac{3 \pi}{2}, \ldots$. etc.

### 18.2 Poles

Isolated singularities of $\boldsymbol{f}(\mathbf{z})$ at $\mathbf{z}=\mathbf{z}_{\mathbf{0}}$ can be classified by the Laurent series

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}+\sum_{n=0}^{\infty} \frac{b_{n}}{\left(z-z_{0}\right)^{n}} \tag{18.2.1}
\end{equation*}
$$

valid in an immediate neighbourhood of the singular point $z=z_{0}$ at $z_{0}$ itself, that is, in a region of the form $\mathbf{0}<\left|\boldsymbol{z}-\mathbf{z}_{\mathbf{0}}\right|<\boldsymbol{R}$. The sum of the first series is
analytic at $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$. The second series, containing the negative powers, is called the principal part of (18.2.1). If it has only finitely many terms, it is of the form

$$
\begin{equation*}
\frac{b_{1}}{z-z_{0}}+\frac{b_{2}}{\left(z-z_{0}\right)^{2}}+\cdots+\frac{b_{m}}{\left(z-z_{0}\right)^{m}}, \quad b_{m} \neq 0 \tag{18.2.2}
\end{equation*}
$$

Then the singularity of $\boldsymbol{f}(\mathbf{z})$ at $\boldsymbol{z}=\mathbf{z}_{\mathbf{0}}$ is called a pole, and $\boldsymbol{m}$ is called the order of the pole. Poles of the first order are called simple poles. If the principal part of (18.2.1) has infinitely many terms, we say that $\boldsymbol{f}(\boldsymbol{z})$ has an isolated essential singularity at $\mathbf{z}=\mathbf{z}_{\mathbf{0}}$.

### 18.2.1 Examples:

1. $f(z)=\frac{1}{z(z-2)^{5}}+\frac{3}{(z-2)^{2}}$ has a simple pole at $z=0$ and a pole of fifth order $z=2$.
2. $f(z)=e^{1 / z}=1+\frac{1}{z}+\frac{1}{2!z^{2}}+\cdots$ has an isolated essential singularity at $z=0$.
3. $f(z)=\sin \left(\frac{1}{z}\right)$

$$
\begin{aligned}
& =\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n+1)!z^{2 n+1}} \\
& =\frac{1}{z}-\frac{1}{3!z^{3}}+\frac{1}{5!z^{5}}+\cdots
\end{aligned}
$$

has an isolated essential singularity at $z=0$.
4. $f(z)=z^{-5} \sin z=\frac{1}{z^{4}}-\frac{1}{6 z^{2}}+\frac{1}{120}-\frac{1}{5040} z^{2}+\cdots, \quad|z|>0$ has a pole of order four at $z=0$.
5. $f(z)=\frac{1}{z^{3}-z^{4}}=\left\{\begin{aligned} \frac{1}{z^{3}}+\frac{1}{z^{2}}+\frac{1}{z}+1+\cdots, & 0<|z|<1 \\ -\frac{1}{z^{4}}-\frac{1}{z^{5}}-\cdots, & |z|>1\end{aligned}\right.$

The first expansion shows that there is a pole of order 3 at $z=0$. The second expansion has infinitely many terms of negative power. But it is no contradiction as this later expansion is valid for $|z|>1$.
18.2.2 Theorem: If $\boldsymbol{f}(\boldsymbol{z})$ is analytic and has a pole at $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$, then $|\boldsymbol{f}(\mathbf{z})| \rightarrow \infty$ as $\mathbf{z} \rightarrow \mathbf{z}_{\mathbf{0}}$ in any manner.
18.2.3 Example: $\boldsymbol{f}(\boldsymbol{z})=\frac{\mathbf{1}}{\boldsymbol{z}^{2}}$ has a pole at $\boldsymbol{z}=\mathbf{0}$ and $|\boldsymbol{f}(\mathbf{z})| \rightarrow \infty$ as $\boldsymbol{z} \rightarrow \mathbf{0}$ in any manner.
18.2.4 Theorem (Picard's Theorem): If $\boldsymbol{f}(\boldsymbol{z})$ is analytic and has an isolated essential singularity at a point $\boldsymbol{z}_{\mathbf{0}}$, it takes on every value, with at most one exceptional value, in an arbitrarily small neighbourhood of $\mathbf{z}_{\mathbf{0}}$.
18.2.5 Example: The function $f(z)=e^{1 / z}$ has an isolated essential singularity at $\mathbf{z}=\mathbf{0}$. It has no limit for approach along the imaginary axis. It becomes infinite if $\mathbf{z} \rightarrow \mathbf{0}$ through negative real values. It takes on nay given value $\boldsymbol{c}=\boldsymbol{c}_{\mathbf{0}} \boldsymbol{e}^{\boldsymbol{i} \boldsymbol{\alpha}} \neq \mathbf{0}$ in an arbitrary small neighbourhood of $\boldsymbol{z}=\mathbf{0}$. Letting $\boldsymbol{z}=\boldsymbol{r} \boldsymbol{e}^{\boldsymbol{i \theta}}$, we must solve the equation

$$
e^{1 / z}=e^{(\cos \theta-i \sin \theta) / r}=c_{0} e^{i \alpha}
$$

for $\boldsymbol{r}$ and $\boldsymbol{\theta}$. Equating the absolute values and the arguments, we have

$$
e^{\cos \theta / r}=c_{0} \text {, i.e., } \cos \theta=r \ln c_{0} \text { and }-\sin \theta=\alpha r .
$$

From these two equations and $\cos ^{2} \theta+\sin ^{2} \theta=\boldsymbol{r}^{2}\left(\ln \boldsymbol{c}_{0}\right)^{2}+\alpha^{2} \boldsymbol{r}^{2}=1$, we obtain the formulae

$$
r^{2}=\frac{1}{\left(\ln c_{0}\right)^{2}+\alpha^{2}} \text { and } \tan \theta=-\frac{a}{\ln c_{0}} .
$$

Hence $\boldsymbol{r}$ can be made arbitrary small by adding multiples of $\mathbf{2 \pi}$ to $\boldsymbol{\alpha}$, leaving $c$ unaltered.

### 18.2.6 Removable Singularity

We say that a function $\boldsymbol{f}(\mathbf{z})$ has a removable singularity at $\mathbf{z}=\boldsymbol{z}_{\mathbf{0}}$ if $\boldsymbol{f}(\mathbf{z})$ is not analytic at $\mathbf{z}=\mathbf{z}_{0}$ but can be made analytic there by assigning a suitable value $\boldsymbol{f}\left(\mathbf{z}_{0}\right)$. Such singularities are of no interest as they can be removed.
18.2.7 Example: The function $f(z)=\frac{\sin z}{z}$ becomes analytic at $\boldsymbol{z}=\mathbf{0}$ if we define $\boldsymbol{f}(\mathbf{0})=1$.

### 18.3 Zeros

A zero of an analytic function $\boldsymbol{f}(\boldsymbol{z})$ in a domain $\boldsymbol{D}$ is a $\mathbf{z}=\boldsymbol{z}_{\mathbf{0}}$ in $\boldsymbol{D}$ such that $\boldsymbol{f}\left(\boldsymbol{z}_{0}\right)=\mathbf{0}$. A zero has order $\boldsymbol{n}$ if not only if $\boldsymbol{f}$ but the derivatives $\boldsymbol{f}^{\prime}, \boldsymbol{f}^{\prime \prime}, \ldots, \boldsymbol{f}^{(n-\mathbf{1})}$ are all 0 at $\mathbf{z}=\boldsymbol{z}_{\mathbf{0}}$ but $\boldsymbol{f}^{(n-\mathbf{1})}\left(\boldsymbol{z}_{\mathbf{0}}\right) \neq \mathbf{0}$. A first order zero is called a simple zero. For a second order zero $\boldsymbol{f}\left(\boldsymbol{z}_{0}\right)=\boldsymbol{f}^{\prime}\left(\boldsymbol{z}_{0}\right)=\mathbf{0}$ but $\boldsymbol{f}^{\prime \prime}\left(\mathrm{z}_{0}\right) \neq 0$.

### 18.3.1 Examples:

1. The function $\left(1+z^{2}\right)$ has simple zeros at $z= \pm i$.
2. The function $\left(1-z^{4}\right)^{2}$ has second-order zeros at $z= \pm 1$ and $\pm i$.
3. The function $(z-a)^{2}$ has a third order zeros at $z=a$.
4. The function $e^{z}$ has no zeros.
5. The function $\sin z$ has simple zeros at $z=0, \pm \pi, \pm 2 \pi, \ldots$ and $\sin ^{z} z$ has second-order zeros at these points.
6. The function $(1-\cos z)$ has second-order zeros at $0, \pm 2 \pi, \pm 4 \pi, \ldots$.

### 18.3.2 Taylor Series at a Zero

At an $\boldsymbol{n}^{\text {th }}$-order zero $\boldsymbol{z}=z_{0}$ of $\boldsymbol{f}(\mathbf{z})$, the terms $\boldsymbol{f}\left(\mathbf{z}_{\mathbf{0}}\right), \boldsymbol{f}^{\prime}\left(\mathbf{z}_{\mathbf{0}}\right), \ldots, \boldsymbol{f}^{(n-1)}\left(\mathbf{z}_{\mathbf{0}}\right)$ are all 0 and $\boldsymbol{f}^{(n)}\left(z_{0}\right) \neq \mathbf{0}$. Therefore, the Taylor series is of the form

$$
\begin{aligned}
f(z)=a_{n}(z & \left.-z_{0}\right)^{n}+a_{n+1}\left(z-z_{0}\right)^{n+1}+\cdots \\
& =\left(z-z_{0}\right)^{n}\left[a_{n}+a_{n+1}\left(z-z_{0}\right)+a_{n+2}\left(z-z_{0}\right)^{2}+\cdots\right]
\end{aligned}
$$

Conversely, if $\boldsymbol{f}(\boldsymbol{z})$ has a such a Taylor series then it has an $\boldsymbol{n}^{\text {th }}$-order zero at $z=z_{0}$.
18.3.3 Theorem: The zeros of an analytic function $\boldsymbol{f}(\mathbf{z})(\neq \mathbf{0})$ are isolated, i.e., each of them has a neighbourhood that contains no further zeros of $\boldsymbol{f}(\boldsymbol{z})$.

Proof: In (18.3.1), the factor $\left(\boldsymbol{z}-\mathbf{z}_{\mathbf{0}}\right)^{\boldsymbol{n}}$ is zero only at $\mathbf{z}=\mathbf{z}_{\mathbf{0}}$. The power series in the parenthesis represents an analytic function say $\boldsymbol{g}(\boldsymbol{z})$. Now $\boldsymbol{g}(\mathbf{z})=\boldsymbol{a}_{\boldsymbol{n}} \neq \mathbf{0}$. Since $\boldsymbol{g}(\mathbf{z})$ is also continuous, $\boldsymbol{g}(\mathbf{z}) \neq \mathbf{0}$ in some neighbourhood of $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$. Hence $\boldsymbol{f}(\boldsymbol{z}) \neq \mathbf{0}$ in some neighbourhood of $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$.
18.3.4 Theorem: Let $\boldsymbol{f}(\boldsymbol{z})$ be analytic at $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ and have a zero of $\boldsymbol{n}^{\boldsymbol{t h}}$-order at $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$. Then $\mathbf{1} / \boldsymbol{f}(\boldsymbol{z})$ has a pole of $\boldsymbol{n}^{\text {th }}$-order at $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$.

The same holds for $\frac{\boldsymbol{h}(\mathbf{z})}{\boldsymbol{f}(z)}$ if $\boldsymbol{h}(\mathbf{z})$ is analytic at $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ and $\boldsymbol{h}\left(\boldsymbol{z}_{\mathbf{0}}\right) \neq \mathbf{0}$.

### 18.3.5 Analytic or Singularity at Infinity

Infinity ( $\infty$ ) has been added to the complex plane resulting in the extended complex plane. The extended complex plane can be mapped into sphere of diameter 1 touching the plane at $\mathbf{z}=\mathbf{0}$. The image $\boldsymbol{A}^{*}$ of a complex number $\boldsymbol{A}$ is the intersection of the sphere with the segment from A to the "north pole" $\boldsymbol{N}$. The point $\infty$ is the image $\boldsymbol{N}$.

The sphere representing the extended complex plane in this way is called the Riemann number sphere. The mapping of the sphere onto the plane is called stereographic projection with center $\boldsymbol{N}$.

Thus for investigating a function $f(z)$ for large $|z|$, we set $z=\frac{1}{\omega}$ and investigate $\boldsymbol{f}(\mathbf{z})=\boldsymbol{f}\left(\frac{\mathbf{1}}{\boldsymbol{\omega}}\right)=\boldsymbol{g}(\boldsymbol{\omega})$ in the neighbourhood of $\boldsymbol{\omega}=\mathbf{0}$. We define $\boldsymbol{f}(\mathbf{z})$ to be analytic or singular at infinity if $\boldsymbol{g}(\boldsymbol{\omega})$ is analytic or singular at $\omega=0$.

We also define $\boldsymbol{g}(\mathbf{0})=\lim _{\omega \rightarrow \mathbf{0}} \boldsymbol{g}(\boldsymbol{\omega})$ if this limit exists. We say that $\boldsymbol{f}(\boldsymbol{z})$ has a $\boldsymbol{n}^{\boldsymbol{t h}}$-order zero at infinity if $\boldsymbol{f}\left(\frac{\mathbf{1}}{\boldsymbol{\omega}}\right)$ has such a zero at $\boldsymbol{\omega} \rightarrow \mathbf{0}$. Similarly we define poles and essential singularities.

### 18.3.6 Examples:

1. The function $\boldsymbol{f}(\boldsymbol{z})=\frac{\mathbf{1}}{\boldsymbol{z}^{2}}$ is analytic at $\infty$. Since $\boldsymbol{g}(\boldsymbol{\omega})=\boldsymbol{\omega}^{\mathbf{2}}$ is analytic at $\boldsymbol{\omega}=\mathbf{0}$ and $\boldsymbol{f}$ has a second-order zero at $\infty$.
2. The function $\boldsymbol{f}(\mathbf{z})=\boldsymbol{z}^{\mathbf{3}}$ is singular at $\infty$ and has third-order pole there since the function $\boldsymbol{g}(\boldsymbol{\omega})=\boldsymbol{f}\left(\frac{\mathbf{1}}{\omega}\right)=\frac{\mathbf{1}}{\boldsymbol{\omega}^{\mathbf{3}}}$ has such a pole at $\boldsymbol{\omega}=\mathbf{0}$.
3. The function $\boldsymbol{f}(\boldsymbol{z})=\boldsymbol{e}^{\boldsymbol{z}}$ has an essential singularity at $\infty$ since $\boldsymbol{e}^{\mathbf{1 / \omega}}$ has such a singularity at $\boldsymbol{\omega} \rightarrow \mathbf{0}$. Similarly, $\boldsymbol{\operatorname { c o s }} \boldsymbol{z}$ and $\boldsymbol{\operatorname { s i n }} \mathbf{z}$ have essential singularity at $\infty$.

By Liouville's theorem a bounded entire function is constant. Hence a nonconstant entire function must be unbounded. Hence it has a singularity at $\infty$, a pole if it is a polynomial or an essential singularity if it is not.

### 18.3.7 Meromorphic Function

Let $f(z)$ be analytic function and it has only singularities in the finite plane which are poles. Then $f(z)$ is called a meromorphic function. Some examples of meromorphic functions are rational functions with nonconstant denominator, trigonometric functions $\tan z, \cot z, \sec z$ and $\operatorname{cosec} z$.

### 18.3.7 Examples:

1. $f(z)=\operatorname{cosec} z=\frac{1}{\sin z}$.

Here $\mathrm{z}=0$ is a singular point of $f$. Now we can write
$f(z)=\frac{1}{\sin z}=\frac{1}{z-\frac{z^{3}}{3!}+\frac{z^{5}}{5!}+\cdots}$
$=\frac{1}{z}\left[1-\frac{\mathrm{z}^{2}}{3!}+\frac{\mathrm{z}^{4}}{5!}+\cdots\right]^{-1}$
$=\frac{1}{Z}+\frac{\mathrm{z}}{3!}+$ higher powers of $z$.

The principal part of the Laurent series is the single term $\frac{1}{z}$. Hence, $z=0$ is a simple pole.
2. $f(z)=\frac{1}{1-z^{2}}$

$$
\begin{aligned}
& =\frac{1}{2}\left[\frac{1}{1-z}+\frac{1}{1+z}\right] \\
& =-\frac{1}{2}(z-1)^{-1}+\frac{1}{2}\left[\frac{1}{2+(z-1)}\right]
\end{aligned}
$$

$$
=-\frac{1}{2}(z-1)^{-1}+\frac{1}{4}\left[1+\frac{z-1}{2}\right]^{-1}
$$

$$
=-\frac{1}{2}(z-1)^{-1}+\frac{1}{4}\left[1-\frac{z-1}{2}+\frac{(z-1)^{2}}{4}-\cdots\right]
$$

which is valid for $|\boldsymbol{z}-\mathbf{1}|<\mathbf{2}$.

Hence $\mathbf{z}=\mathbf{1}$ is a simple pole.
Alternatively, we can express

$$
\begin{aligned}
& f(z)=\frac{1}{2}(z+1)^{-1}+\frac{1}{4}\left[1-\frac{z+1}{2}\right]^{-1} \\
& =\frac{1}{2}(z+1)^{-1}+\frac{1}{4}\left[1+\frac{z+1}{2}+\frac{(z-1)^{2}}{4}+\text { higher powers of }(z+1)\right]
\end{aligned}
$$

for $|z+1|<2$

Hence $\mathbf{z}=\mathbf{- 1}$ is simple pole.

## Suggested Readings

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## Lesson 19

## Residue Theorem

### 19.1 Residues

If $\boldsymbol{f}(\mathbf{z})$ has a singularity at $\mathbf{z}=\mathbf{z}_{\mathbf{0}}$ inside a simple closed curve $\boldsymbol{C}$, but is otherwise analytic on $\boldsymbol{C}$ and inside $\boldsymbol{C}$, then we can expand the function $\boldsymbol{f}(\mathbf{z})$ in a Laurent series as

$$
f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}+\frac{b_{1}}{z-z_{0}}+\frac{b_{2}}{\left(z-z_{0}\right)^{2}}+\cdots
$$

This series is convergent for all points near $\mathbf{z}=\mathbf{z}_{\mathbf{0}}$ (except at $\mathbf{z}=\mathbf{z}_{\mathbf{0}}$ ) in the same domain of the form $\mathbf{0}<\left|\boldsymbol{z}-\boldsymbol{z}_{\mathbf{0}}\right|<\boldsymbol{R}$.

Now the coefficient $\boldsymbol{b}_{\mathbf{1}}$ of the first negative power $\frac{\mathbf{1}}{z-z_{0}}$ of this Laurent series is given by

$$
\begin{align*}
& b_{1}=\frac{1}{2 \pi i} \int_{C} f\left(z^{*}\right) \mathrm{dz}^{*} \\
& \int_{C} f\left(z^{*}\right) \mathrm{dz}^{*}=2 \pi i b_{1} \tag{19.1.1}
\end{align*}
$$

We define $\boldsymbol{b}_{\mathbf{1}}$ to be the residue of $\boldsymbol{f}(\boldsymbol{z})$ at $\mathbf{z}=\boldsymbol{z}_{\mathbf{0}}$ and denote it by

$$
=\lim _{z=z_{0}} f(z) \quad b_{1}
$$

### 19.1.1 Examples:

1. We want to integrate $f(z)=z^{-4} \sin z$ around the unit circle $C$. Consider the Laurent series expansion as

$$
f(z)=\frac{\sin z}{z^{4}}=\frac{1}{z^{3}}-\frac{1}{3!z}+\frac{z}{5!}+\cdots
$$

This is convergent for $|\boldsymbol{z}|>\mathbf{0}$. Hence $\boldsymbol{b}_{\mathbf{1}}=\frac{\mathbf{1}}{\mathbf{3 !}}$.
2. Here we integrate $f(z)=\frac{1}{z^{3}-z^{4}}$ clockwise around $|z|=\frac{1}{2}$. The function $f(z)$ has singularities at $z=0$ and $z=1$. However, $z=1$ lies outside the circle $C$. So we can expand $f(z)$ in Laurent series at $z=0$ as

$$
f(z)=\frac{1}{z^{3}}+\frac{1}{z^{2}}+\frac{1}{z}+1+\cdots, \quad 0<|z|<1
$$

Note that the residue is 1 and we get

$$
\int_{C} \frac{1}{z^{3}-z^{4}} \mathrm{dz}=-2 \pi i \lim _{z=0} \operatorname{Res} f(z)=-2 \pi i
$$

### 19.1.2 Residue at Simple Pole

For a simple pole at $\mathbf{z}=\boldsymbol{z}_{\mathbf{0}}$, the Laurent series is

$$
\begin{align*}
& f(z)=\frac{b_{1}}{z-z_{0}}+a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\cdots \\
& \left.\quad<\left|z-z_{0}\right|<R\right) .
\end{align*}
$$

This implies that

$$
\left(z-z_{0}\right) f(z)=b_{1}+a_{0}\left(z-z_{0}\right)+a_{1}\left(z-z_{0}\right)^{2}+\cdots
$$

So $\lim _{z=z_{0}}\left(z-z_{0}\right) f(z)=b_{1}=\lim _{z=z_{0}} \operatorname{Res} f(z)$.
19.1.3 Example: $\lim _{z=i} \frac{q z+i}{z\left(z^{2}+1\right)}=\lim (z-i) \frac{q z+i}{z(z+i)(z-i)}=\frac{10 i}{i(2 i)}=-5 i$.
19.1.3 Remark: Suppose we have $\boldsymbol{f}(\mathbf{z})=\frac{p(z)}{\boldsymbol{q}(z)}$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are analytic, $\boldsymbol{p}\left(\mathbf{z}_{\mathbf{0}}\right) \neq \mathbf{0}$ and $\boldsymbol{q}(\mathbf{z})$ has a simple pole $\mathbf{z}_{\mathbf{0}}$ so that $\boldsymbol{f}(\mathbf{z})$ has a simple pole $\mathbf{z}_{\mathbf{0}}$. So by Taylor series, we find

$$
\begin{aligned}
& \boldsymbol{q}(z)=\left(z-z_{0}\right) \boldsymbol{q}^{\prime}\left(z_{0}\right)+\frac{\left(z-z_{0}\right)^{2}}{2!} q^{\prime \prime}\left(z_{0}\right)+\cdots \\
& \qquad \begin{array}{l}
\text { So } \\
=\lim _{z=z_{0}}\left(z-z_{0}\right) f(z)=\lim _{z=z_{0}}\left(z-z_{0}\right) \frac{p(z)}{q(z)} \\
\qquad\left(z-z_{0}\right) q^{\prime}\left(z_{0}\right)+\frac{\left(z-z_{0}\right)^{2}}{2!} \boldsymbol{q}^{\prime \prime}\left(z_{0}\right)+\cdots \\
=\frac{p\left(z_{0}\right)}{q^{\prime}\left(z_{0}\right)}
\end{array}
\end{aligned}
$$

### 19.2 Residue at Pole of Any Order

If $\boldsymbol{f}(\boldsymbol{z})$ has a pole of any order $\boldsymbol{m}>\mathbf{1}$ at $\boldsymbol{z}=\boldsymbol{z}_{0}$, then its Laurent series can be written as

$$
f(z)=\frac{b_{m}}{\left(z-z_{0}\right)^{m}}+\frac{b_{m-1}}{\left(z-z_{0}\right)^{m-1}}+\cdots \frac{b_{1}}{z-z_{0}}+a_{0}+a_{1}\left(z-z_{0}\right)
$$

where $\boldsymbol{b}_{\boldsymbol{m}} \neq \mathbf{0}$.

The residue of $\boldsymbol{f}(\mathbf{z})$ at $\mathbf{z}=\mathbf{z}_{\mathbf{0}}$ is $\boldsymbol{b}_{\mathbf{1}}$. If we multiply both sides by $\left(\mathbf{z}-\mathbf{z}_{\mathbf{0}}\right)^{\boldsymbol{m}}$, we get

$$
\begin{gathered}
\left(z-z_{0}\right)^{m} f(z)=b_{m}+b_{m-1}\left(z-z_{0}\right)+\cdots+b_{1}\left(z-z_{0}\right)^{m-1} \\
+a_{0}\left(z-z_{0}\right)^{m}+a_{1}\left(z-z_{0}\right)^{m+1}+\cdots
\end{gathered}
$$

The residue $\boldsymbol{b}_{\mathbf{1}}$ of $\boldsymbol{f}(\boldsymbol{z})$ at $\boldsymbol{z}=\boldsymbol{z}_{\mathbf{0}}$ is now the coefficient of the power $\left(\boldsymbol{z}-\boldsymbol{z}_{\mathbf{0}}\right)^{\boldsymbol{m - 1}}$ in the Taylor series of the function $\boldsymbol{g}(\mathbf{z})=\left(\mathbf{z}-\mathbf{z}_{0}\right)^{m-1} \boldsymbol{f}(\mathbf{z})$ with center at $z=z_{0}$. So
$b_{1}=\frac{1}{(m-1)!} g^{(m-1)}\left(z_{0}\right)$
(by Taylor's Theorem). Hence, if $\boldsymbol{f}(\mathbf{z})$ has a pole of the $\boldsymbol{m}^{\text {th }}$-order at $\mathbf{z}=\boldsymbol{z}_{\mathbf{0}}$, the residue is given by
$\lim _{z \rightarrow z_{0}} \operatorname{Res} f(z)=\frac{1}{(m-1)!} \lim _{z \rightarrow z_{0}}\left\{\frac{d^{m-1}}{d z^{m-1}}\left[\left(z-z_{0}\right)^{m} f(z)\right]\right\}$
19.2.1 Example: The function $f(z)=\frac{50 z}{(z+4)(z-1)^{2}}$ has a pole of second order at $z=1$. So

$$
\begin{aligned}
& \lim _{z \rightarrow 1} \operatorname{Res} f(z)=\lim _{z \rightarrow 1} \frac{d}{d z}\left[\frac{50 z}{(z+4)}\right] \\
& =\lim _{z \rightarrow 1} \frac{d}{d z}\left[\frac{50(z+4)-50 z}{(z+4)^{2}}\right]=\lim _{z \rightarrow 1} \frac{200}{(z+4)^{4}}=8 .
\end{aligned}
$$

19.2.2 Residue Theorem: Let the function $\boldsymbol{f}(\mathbf{z})$ be analytic inside a simple closed path $\boldsymbol{C}$ and on $\boldsymbol{C}$, except for finitely many singular points $\mathbf{z}_{\mathbf{1}}, \mathbf{z}_{\mathbf{2}} \ldots, \mathbf{z}_{\boldsymbol{k}}$ inside $\boldsymbol{C}$. Then the integral of $\boldsymbol{f}(\mathbf{z})$ taken counter clockwise around $\boldsymbol{C}$ is given by

$$
\int_{C} f(\mathrm{z}) \mathrm{dz}=2 \pi i \sum_{j=1}^{k} \operatorname{ReS}_{z=z_{j}} f(\mathrm{z})
$$

Proof: We enclose each of the singular points $\boldsymbol{z}_{\boldsymbol{i}}$ in a circle $\boldsymbol{C}_{\boldsymbol{i}}$ with radius small enough that these $\boldsymbol{k}$ circles and $\boldsymbol{C}$ are all separated. Then $\boldsymbol{f}(\mathbf{z})$ is analytic in the domain $\boldsymbol{D}$ bounded by $\boldsymbol{C}$ and $\boldsymbol{C}_{\mathbf{1}}, \ldots, \boldsymbol{C}_{\boldsymbol{k}}$ and on the entire boundary of $\boldsymbol{D}$. From Cauchy's integral theorem, we thus have

$$
\begin{aligned}
& \int_{C} f(z) \mathrm{dz}=\sum_{j=1}^{k} \int_{C_{j}} f(z) \mathrm{dz} \\
= & \sum_{j=1}^{k} 2 \pi i \underset{z=z_{j}}{\operatorname{Res}} f(z)
\end{aligned}
$$

$$
=2 \pi i \sum_{j=1}^{k} \operatorname{Res}_{z=z_{j}} f(z)
$$

### 19.2.3 Examples:

1. Find $I=\oint_{C} \frac{4-3 z}{z^{2}-z} d z$, where $C$ is simple closed path that
(a) encloses 0 and 1 ,
(b) 0 is inside and 1 is outside,
(c) 0 and 1 are outside,
(d) 1 is inside, o is outside.
$\operatorname{Res}_{z=0} \frac{4-3 z}{z(z-1)}=\lim _{z \rightarrow 0} \frac{4-3 z}{z-1}=-4$
$\operatorname{Res}_{z=1} \frac{4-3 z}{z(z-1)}=\lim _{z \rightarrow 1} \frac{4-3 z}{z}=1$

Hence (a) $\boldsymbol{I}=\mathbf{2 \pi i}(-\mathbf{4}+\mathbf{1})=-\mathbf{6 \pi i}$, (b) $\boldsymbol{I}=-\mathbf{8} \boldsymbol{\pi} \boldsymbol{i}$, (c) 0 , (d) $\mathbf{2 \pi i}$.
2. $\oint_{C} \frac{\tan z}{z^{2}-1} d z=2 \pi i\left[\operatorname{Res}_{z=1}^{\tan z}+\underset{z=-1}{z^{2}-1} \frac{\tan z}{z^{2}-1}\right]$

$$
=2 \pi i\left[\frac{\tan z}{2}+\frac{\tan (-1)}{-2}\right]=2 \pi i \tan (1)
$$

3. Evaluate $I=\oint_{C}\left(\frac{z e^{z \pi}}{z^{4}-16}+z e^{\frac{\pi}{z}}\right) d z, C$ is ellipse $9 x^{2}+y^{2}=9$.

The first term in the integrand has simple poles at $z= \pm 2$ and $z= \pm 2 i$.
The poles at $\pm 2$ lie outside the curve $C$. So the first pole of $I$ is

$$
\begin{aligned}
& I_{1}=2 \pi i\left[\operatorname{Res}_{z=2 i} \frac{z e^{z \pi}}{z^{4}-16}+\operatorname{Res}_{z=-2 i} \frac{z e^{z \pi}}{z^{4}-16}\right] \\
& =2 \pi i\left[\lim _{z \rightarrow 2 i} \frac{z e^{z \pi}}{\left(z^{2}-4\right)(z+2 i)}+\lim _{z \rightarrow-2 i} \frac{z e^{z \pi}}{\left(z^{2}-4\right)(z+2 i)}\right] \\
& =2 \pi i\left[\frac{2 i}{-32 i}+\frac{-2 i}{32 i}\right]=\frac{-\pi i}{4}
\end{aligned}
$$

For the second term
$z e^{\frac{\pi}{z}}=z\left[1+\frac{\pi}{z}+\frac{\pi^{2}}{2!z^{2}}+\cdots\right]$
$=z+\pi+\frac{\pi^{2}}{2 z}+\cdots$

So $\underset{\boldsymbol{z}=\mathbf{0}}{\operatorname{Res}} \boldsymbol{z} \boldsymbol{e}^{\frac{\pi}{z}}=\frac{\boldsymbol{\pi}^{2}}{\mathbf{2}}$, and then the second term of the integral is $\mathbf{2 \pi i} \frac{\boldsymbol{\pi}^{2}}{\mathbf{2}}=\boldsymbol{\pi}^{\mathbf{3}} \boldsymbol{i}$.
Hence $I=\pi^{3} i-\frac{\pi i}{4}=\pi i\left(\pi^{2}-\frac{1}{4}\right)$.
4. Evaluate the integral $\oint_{C} \frac{z-23}{z^{4}-4 z-5} d z, \quad C:|z-2|=4$. We can write

$$
\begin{aligned}
& \oint_{C} \frac{z-23}{z^{4}-4 z-5} d z=\oint_{C} \frac{z-23}{(z-5)(z+1)} d z \\
& =2 \pi i\left[\operatorname{Res}_{z=5} \frac{z-23}{(z-5)(z+1)}+\operatorname{Res}_{z=-1} \frac{z-23}{(z-5)(z+1)}\right] \\
& =2 \pi i\left[\frac{-18}{6}+\frac{-24}{-6}\right]=2 \pi i
\end{aligned}
$$

5. Evaluate $I=\int_{|z|=1} \tan (\pi z) \mathrm{dz}$. The function $\tan (\pi z)$ has simple poles at $z= \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$ of which only $z= \pm \frac{1}{2}$ lie inside the contour. So

$$
\begin{aligned}
& I=2 \pi i\left[\operatorname{Res}_{z=\frac{1}{2}}^{\operatorname{Sen}} \tan (\pi z)+\underset{z=-\frac{1}{2}}{\operatorname{Res}} \tan (\pi z)\right] \\
& =2 \pi i\left[\lim _{z \rightarrow \frac{1}{2}}\left(z-\frac{1}{2}\right) \tan (\pi z)+\lim _{z \rightarrow-\frac{1}{2}}\left(z+\frac{1}{2}\right) \tan (\pi z)\right] \\
& =2 \pi i\left[-\frac{1}{\pi}-\frac{1}{\pi}\right]=-4 i \text {. }
\end{aligned}
$$

6. Evaluate $\oint_{C} \frac{\cosh (\pi z)}{z^{4}+5 z^{2}+4} d z, \quad C:|z|=4$.

The integral has simple poles at $z= \pm i, \pm 2 i$ and they all lie inside the contour.
Now for pole at $z=a$
$\operatorname{Res}_{z=a} f(z)=\frac{\phi(a)}{\psi^{\prime}(a)}=\frac{a \cosh (a \pi)}{4 a^{3}+10 a}=\frac{\cosh (a \pi)}{4 a^{2}+10}$.

So,

$$
\begin{aligned}
& I=2 \pi i\left[\operatorname{Res}_{z=i} f(z)+\operatorname{Res}_{z=-i} f(z)+\operatorname{Res}_{z=2 i} f(z)+\operatorname{Res}_{z=-2 i} f(z)\right] \\
& =2 \pi i\left[\frac{\cosh (\pi i)}{6}+\frac{\cosh (\pi i)}{6}-\frac{\cosh (2 \pi i)}{6}-\frac{\cosh (2 \pi i)}{6}\right]=-\frac{4 \pi i}{3}
\end{aligned}
$$

## Suggested Readings

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## All About Agriculture...

## Lesson 20

## Introduction

Before we start discussion on Fourier transform it is very important to discuss Fourier series firstly because it gives a pathway to understand Fourier transform. Fourier series has a wide range of applications, viz. in analysis of current flow, sound waves, image analysis and many more. They are also used to solve differential equations. In a general sense, we use Fourier series to represent a periodic functions. Indeed, not only periodic functions but also to represent and approximate functions defined on a finite interval.

### 20.1 Periodic Functions

If a function $f$ is periodic with period $T>0$ then $f(t)=f(t+T),-\infty<t<\infty$. The smallest of $T$, for which the equality $f(t)=f(t+T)$ is true, is called fundamental period of $f(t)$. However, if $T$ is the period of a function $f$ then $n T, n$ is any natural number, is also a period of $f$. Some familiar periodic functions are $\sin x, \cos x, \tan x$ etc.

### 20.1.1 Properties of Periodic Functions

We consider two important properties of periodic function. These properties will be used to discuss the Fourier series.

1. It should be noted that the sum, difference, product and quotient of two functions is also a periodic function. Consider for example:

$$
\begin{aligned}
& f(x)=\underbrace{\sin x}+\underbrace{\sin 2 x}_{2 \pi}+\underbrace{\cos 3 x}_{\frac{2 \pi}{3}} \\
& \text { period: } 2 \pi \quad \frac{2 \pi}{2}=\pi
\end{aligned}
$$

Period of $f=$ common period of $(\sin x, \sin 2 x, \cos 3 x)=2 \pi$
One can also confirms the period of the function $f(x)$ as

$$
\begin{aligned}
f(x+2 \pi) & =\sin (x+2 \pi)+\sin (2 x+2 \pi)+\cos (3 x+2 \pi) \\
& =\sin (x)+\sin (2 x)+\cos (3 x)=f(x)
\end{aligned}
$$

2. If a function is integrable on any interval of length $T$, then it is integrable on any other intervals of the same length and the value of the integral is the same, that is,

$$
\int_{a}^{a+T} f(x) \mathrm{d} x=\int_{b}^{b+T} f(x) \mathrm{d} x=\int_{0}^{T} f(x) \mathrm{d} x \text { for any value of } a \text { and } b
$$

This property has been depicted in Figure 20.1.1.


Figure 20.1: Area showing integral of a typical periodic function

### 20.2 Trigonometric Polynomials and Series

- Trigonometric polynomial of order $n$ is defined as

$$
S_{n}(x)=a_{0}+\sum_{k=1}^{n}\left(a_{k} \cos \frac{\pi k x}{l}+b_{k} \sin \frac{\pi k x}{l}\right)
$$

Here $a_{n}$ and $b_{n}$ are some constants. Since the sum of the periodic functions again represents a periodic function. Therefore $S_{n}$ will be a periodic function. What will be the period of the function $S_{n}$ ? The period can be identified simply by looking at the common period of the functions involved in the sum as

$$
\text { Period of } \begin{aligned}
S_{n}(x) & =\text { common period of }\left(\cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \cos \frac{2 \pi x}{l}, \ldots, \sin \frac{n \pi x}{l}, \cos \frac{n \pi x}{l}\right) \\
& =2 \pi /(\pi / l)=2 l
\end{aligned}
$$

- The infinite trigonometric series

$$
S(x)=a_{0}+\sum_{k=1}^{\infty}\left(a_{k} \cos \frac{\pi k x}{l}+b_{k} \sin \frac{\pi k x}{l}\right)
$$

if it converges, also represents a function of period $2 l$.

Now the question aries whether any function of period $T=2 l$ can be represented as the sum of a trigonometric series? The answer to this question is affirmative and it is possible for a very wide class of periodic functions. In the next lesson we will see how to obtain the constants $a_{n}$ and $b_{n}$ in order this trigonometric series to represent a given periodic function.

Remark 1: Though sine and cosine functions are quite simple in nature but their sum function may be quite complex. One can see the plot of $\sin x+\sin 2 x+\cos 3 x$ in Figure 20.2. However, the function has a period $2 \pi$ which is a common period of $\sin x, \sin 2 x, \cos 3 x$.


Figure 20.2: Plot of a trigonometric polynomial $f(x)=\sin x+\sin 2 x+\cos 3 x$

### 20.3 Orthogonality Property of Trigonometric System

We call two functions $\phi(x)$ and $\psi(x)$ to be orthogonal on the interval $[a, b]$ if

$$
\int_{a}^{b} \phi(x) \psi(x) d x=0
$$

With this definition we can say that the basic trigonometric system viz.

$$
1, \cos x, \sin x, \cos 2 x, \sin 2 x, \ldots
$$

is orthogonal on the interval $[-\pi, \pi]$ or $[0,2 \pi]$. In particular, we shall prove that any two distinct functions are orthogonal.

To show the orthogonality we take different possible combination as:
For any integer $n \neq 0$ : We have the following integrals to show the orthogonality of the function 1 with any member of sine or cosine family

$$
\int_{-\pi}^{\pi} 1 \cdot \cos (n x) \mathrm{d} x=\left.\frac{\sin (n x)}{n}\right|_{-\pi} ^{\pi}=0, \quad \int_{-\pi}^{\pi} 1 \cdot \sin (n x) \mathrm{d} x=-\left.\frac{\cos (n x)}{n}\right|_{-\pi} ^{\pi}=0
$$

We have also the following useful results

$$
\int_{-\pi}^{\pi} \cos ^{2}(n x) \mathrm{d} x=\int_{-\pi}^{\pi} \frac{1+\cos (2 n x)}{2}=\pi, \quad \int_{-\pi}^{\pi} \sin ^{2}(n x) \mathrm{d} x=\int_{-\pi}^{\pi} \frac{1-\cos (2 n x)}{2}=\pi
$$

For any integer $m$ and $n(\mathbf{m} \neq \mathbf{n})$ : Now we show that any two different members of the same family (sine or cosine) are orthogonal. For the cosine family we have

$$
\int_{-\pi}^{\pi} \cos (n x) \cos (m x) \mathrm{d} x=\frac{1}{2} \int_{-\pi}^{\pi}[\cos (n+m) x+\cos (n-m) x] \mathrm{d} x=0
$$

and for the sine family we have

$$
\int_{-\pi}^{\pi} \sin (n x) \sin (m x) \mathrm{d} x=\frac{1}{2} \int_{-\pi}^{\pi}[\cos (n-m) x-\cos (n+m) x] \mathrm{d} x=0
$$

For any integer $m$ and $n$ : Here we show that any two members of the two different family (sine and cosine) are orthogonal

$$
\int_{-\pi}^{\pi} \sin (n x) \cos (m x) \mathrm{d} x=0
$$

Note that the integrand is an odd function and therefore the integral is zero.
The above result can be summarized in a more general setting in the following theorem.

## Introduction

### 20.3.1 Theorem

## The trigonometric system

$$
1, \cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \cos \frac{2 \pi x}{l}, \sin \frac{2 \pi x}{l}, \ldots
$$

is orthogonal on the interval $[-l, l]$ or $[a, a+2 l]$, where $a$ is any real number.

Proof: Note that the common period of the trigonometric system

$$
1, \cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \cos \frac{2 \pi x}{l}, \sin \frac{2 \pi x}{l}, \ldots
$$

is $2 l$. Similar to the evaluation of the integral appeared above to show orthogonality of the basic trigonometric system, we have the following results:
a) $\quad \int_{-l}^{l} \cos \frac{m \pi x}{l} \cos \frac{n \pi x}{l} \mathrm{~d} x=\int_{a}^{a+2 l} \cos \frac{m \pi x}{l} \cos \frac{n \pi x}{l} \mathrm{~d} x= \begin{cases}0 & \text { if } m \neq n \\ l & \text { if } m=n \neq 0\end{cases}$
b) $\quad \int_{-l}^{l} \sin \frac{m \pi x}{l} \sin \frac{n \pi x}{l} \mathrm{~d} x=\int_{a}^{a+2 l} \sin \frac{m \pi x}{l} \sin \frac{n \pi x}{l} \mathrm{~d} x= \begin{cases}0 & \text { if } m \neq n \\ l & \text { if } m=n \neq 0\end{cases}$
All Abont Agriculture.
c) $\quad \int_{-l}^{l} \sin \frac{m \pi x}{l} \cos \frac{n \pi x}{l} \mathrm{~d} x=\int_{a}^{a+2 l} \sin \frac{m \pi x}{l} \cos \frac{n \pi x}{l} \mathrm{~d} x=0$

This completes the proof of the above theorem.

To summarize, the value of the integral over length of period of integrand is equal to zero if the integrand is a product of two different members of trigonometric system. If the integrand is product of two same member from sine or cosine family then the value of the integral will be half of the interval length on which the integral is performed. These results will be used to establish Fourier series of a function of period $2 l$ defined on the interval $[-l, l]$ or $[a, a+2 l]$. It should be noted that for $l=\pi$ we obtain results for standard trigonometric system of common period $2 \pi$.

## Suggested Readings

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## Lesson 21

## Construction of Fourier Series

In this lesson we shall introduce Fourier series of a piecewise continuous periodic function. First we construct Fourier series of periodic functions of standard period $2 \pi$ and then the idea will be extended for a function of arbitrary period.

### 21.1 Piecewise Continuous Functions

A function $f$ is piecewise continuous on $[a, b]$ if there are points

$$
a<t_{1}<t_{2}<\ldots<t_{n}<b
$$

such that $f$ is continuous on each open sub-interval $\left(a, t_{1}\right),\left(t_{j}, t_{j+1}\right)$ and $\left(t_{n}, b\right)$ and all the following one sided limits exist and are finite

$$
\lim _{t \rightarrow a+} f(t), \lim _{t \rightarrow t_{j}-} f(t), \lim _{t \rightarrow t_{j}+} f(t), \text { and } \lim _{t \rightarrow b-} f(t), j=1,2, \ldots, n
$$

This mean that $f$ is continuous on $[a, b]$ except possibly at finitely many points, at each of which $f$ has finite one sided limits. It should be clear that all continuous functions are obviously piecewise continuous.

### 21.1.1 Example 1

Consider the function

$$
f(x)= \begin{cases}3, & \text { for } x=-\pi \\ x^{2}, & \text { for }-\pi<x<1 \\ 1-x^{2}, & \text { for } 1 \leq x<2 \\ 2, & \text { for } 2 \leq x \leq \pi\end{cases}
$$

At each point of discontinuity the function has finite one sided limits from both sides. At the end points $x=-\pi$ and $\pi$ right and left sided limits exist, respectively. Therefore, the function is piecewise continuous.

### 21.1.2 Example 2

A simple example that is not piecewise continuous includes

$$
f(x)= \begin{cases}0, & x=0 \\ x^{-n}, & x \in(0,1], n>0\end{cases}
$$

Note that $f$ is continuous everywhere except at $x=0$. The function $f$ is also not piecewise continuous on $[0,1]$ because $\lim _{x=0+} f(x)=\infty$.

An important property of piecewise continuous functions is boundedness and integrability over closed interval. A piecewise continuous function on a closed interval is bounded and integrable on the interval. Moreover, if $f_{1}$ and $f_{2}$ are two piecewise continuous functions then their product, $f_{1} f_{2}$, and linear combination, $c_{1} f_{1}+c_{2} f_{2}$, are also piecewise continuous.

### 21.2 Fourier Series of a $2 \pi$ Periodic Function

Let $f$ be a periodic piecewise continuous function on $[-\pi, \pi]$ and has the following trigonometric series expansion

$$
\begin{equation*}
f \sim \frac{a_{0}}{2}+\sum_{k=1}^{\infty}\left[a_{k} \cos (k x)+b_{k} \sin (k x)\right] \tag{21.1}
\end{equation*}
$$

The aim is to determine the coefficients $a_{k}, k=0,1,2, \ldots$ and $b_{k}, k=1,2, \ldots \ldots$ First we assume that the above series can be integrated term by term and its integral is equal to the integral of the function $f$ over $[-\pi, \pi]$, that is,

$$
\int_{-\pi}^{\pi} f(x) \mathrm{d} x=\int_{-\pi}^{\pi} \frac{a_{0}}{2} \mathrm{~d} x+\sum_{k=1}^{\infty}\left(a_{k} \int_{-\pi}^{\pi} \cos (k x) \mathrm{d} x+b_{k} \int_{-\pi}^{\pi} \sin (k x) \mathrm{d} x\right)
$$

This implies

$$
a_{0}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \mathrm{d} x
$$

Multiplying the series by $\cos (n x)$, integrating over $[-\pi, \pi]$ and assuming its value equal to the integral of $f(x) \cos (n x)$ over $[-\pi, \pi]$, we get

$$
\int_{-\pi}^{\pi} f(x) \cos (n x) \mathrm{d} x=0+\sum_{k=1}^{\infty}\left(a_{k} \int_{-\pi}^{\pi} \cos (n x) \cos (k x) \mathrm{d} x+b_{k} \int_{-\pi}^{\pi} \cos (n x) \sin (k x) \mathrm{d} x\right)
$$

Note that the first term on the right hand side is zero because $\int_{-\pi}^{\pi} \cos (k x) \mathrm{d} x=0$. Further, using the orthogonality of the trigonometric system we obtain

$$
a_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos (n x) \mathrm{d} x
$$

Similarly, by multiplying the series by $\sin (n x)$ and repeating the above steps we obtain

$$
b_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin (n x) \mathrm{d} x
$$

The coefficients $a_{n}, n=0,1,2, \ldots$ and $b_{n}, n=1,2, \ldots$ are called Fourier coefficients and the trigonometric series (21.1) is called the Fourier series of $f(x)$. Note that by writing the constant $a_{0} / 2$ instead of $a_{0}$, one can use a single formula of $a_{n}$ to calculate $a_{0}$.

Remark 1: In the series (21.1) we can not, in general, replace $\sim b y=$ sign as clear from the determination of the coefficients. In the process we have set two integrals equal which does not imply that the function $f(x)$ is equal to the trigonometric series. Later we will discuss conditions under which equality holds true.

Remark 2: (Uniqueness of Fourier Series) If we alter the value of the function $f$ at a finite number of points then the integral defining Fourier coefficients are unchanged. Thus function which differ at finite number of points have exactly the same Fourier series. In other words we can say that if $f, g$ are piecewise continuous functions and Fourier series of $f$ and $g$ are identical, then $f(x)=g(x)$ except at a finite number of points.

### 21.3 Fourier Series of a $2 l$ Periodic Function

Let $f(x)$ be piecewise continuous function defined in $[-l, l]$ and it is $2 l$ periodic. The Fourier series corresponding to $f(x)$ is given as

$$
\begin{equation*}
f \sim \frac{a_{0}}{2}+\sum_{k=1}^{\infty}\left[a_{n} \cos \frac{k \pi x}{l}+b_{n} \sin \frac{k \pi x}{l}\right] \tag{21.2}
\end{equation*}
$$

where the Fourier coefficients, derived exactly in the similar manner as in the previous case, are given as

$$
a_{k}=\frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{k \pi x}{l} \mathrm{~d} x, \quad k=0,1,2, \ldots
$$

$$
b_{k}=\frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{k \pi x}{l} \mathrm{~d} x \quad k=1,2, \ldots
$$

In must be noted that just for simplicity we will be discussing Fourier series of $2 \pi$ periodic function. However all discussions are valid for a function of an arbitrary period.

Remark 3: It should be noted that piecewise continuity of a function is sufficient for the existence of Fourier series. If a function is piecewise continuous then it is always possible to calculate Fourier coefficients. Now the question arises whether the Fourier series of a function $f$ converges and represents $f$ or not. For the convergence we need additional conditions on the function $f$ to ensure that the series converges to the desired values. These issues on convergence will be taken in the next lesson.

### 21.4 Example Problems

### 21.4.1 Problem 1

Find the Fourier series to represent the function

$$
f(x)= \begin{cases}-\pi, & -\pi<x<0 \\ x, & 0<x<\pi\end{cases}
$$

Solution: The Fourier series of the given function will represent a $2 \pi$ periodic function and the series is given by

$$
f(x) \sim \frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left(a_{n} \cos (n x)+b_{n} \sin (n x)\right)
$$

with

$$
a_{0}=\int_{-\pi}^{\pi} f(x) \mathrm{d} x=\frac{1}{\pi}\left[-\int_{-\pi}^{0} \pi \mathrm{~d} x+\int_{0}^{\pi} x \mathrm{~d} x\right]=-\frac{\pi}{2}
$$

and the coefficients $a_{n}, n=1,2, \ldots$ as

$$
\begin{aligned}
a_{n} & =\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos (n x) \mathrm{d} x=\frac{1}{\pi}\left[-\int_{-\pi}^{0} \pi \cos (n x) \mathrm{d} x+\int_{0}^{\pi} x \cos (n x) \mathrm{d} x\right] \\
& =-\left[\frac{\sin (n x)}{n}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[\left\{x \frac{\sin (n x)}{n}\right\}_{0}^{\pi}-\int_{0}^{\pi} \frac{\sin (n x)}{n} \mathrm{~d} x\right]
\end{aligned}
$$

It can be further simplified to give

$$
a_{n}=\frac{1}{n^{2} \pi}\left[(-1)^{n}-1\right]= \begin{cases}0, & n \text { is even } \\ -\frac{2}{n^{2} \pi}, & n \text { is odd }\end{cases}
$$

Similarly $b_{n}, n=1,2, \ldots$ can be calculated as

$$
\begin{aligned}
b_{n} & =\frac{1}{\pi}\left[-\int_{-\pi}^{0} \pi \sin (n x) \mathrm{d} x+\int_{0}^{\pi} x \sin (n x) \mathrm{d} x\right] \\
& =\left[\frac{\cos (n x)}{n}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[-\left\{x \frac{\cos (n x)}{n}\right\}_{0}^{\pi}+\int_{0}^{\pi} \frac{\cos (n x)}{n} \mathrm{~d} x\right]
\end{aligned}
$$

After simplification we get

$$
b_{n}=\frac{1}{n}\left[1-2(-1)^{n}\right]= \begin{cases}-\frac{1}{n}, & n \text { is even } \\ \frac{3}{n}, & n \text { is odd }\end{cases}
$$

Substituting the values of $a_{n}$ and $b_{n}$, we get

$$
f(x) \sim-\frac{\pi}{4}-\frac{2}{\pi}\left[\cos x+\frac{\cos 3 x}{3^{2}}+\frac{\cos 5 x}{5^{2}}+\ldots\right]+\left[3 \sin x-\frac{\sin 2 x}{2}+\frac{3 \sin 3 x}{3}-\ldots\right] .
$$

Remark 4: Let a function is defined on the interval $[-l, l]$. It should be noted that the periodicity of the function is not required for developing Fourier series. However, the Fourier series, if it converges, defines a $2 l$-periodic function on $\mathbb{R}$. Therefore, this is sometimes convenient to think the given function as $2 l$-periodic defined on $\mathbb{R}$.

### 21.4.2 Problem 2

Expand $f(x)=|\sin x|$ in a Fourier series.
Solution: There are two possibilities to work out this problem. This may be treated as a function of period $\pi$ and we can work in the interval $(0, \pi)$ or we treat this function as of period $2 \pi$ and work in the interval $(-\pi, \pi)$.

Case I: First we treat the function $|\sin x|$ as $\pi$ periodic we have $2 l=\pi \Rightarrow l=\frac{\pi}{2}$. The coefficient $a_{0}$ is given as

$$
a_{0}=\frac{1}{\frac{\pi}{2}} \int_{0}^{\pi} f(x) \mathrm{d} x=\frac{2}{\pi} \int_{0}^{\pi} \sin x \mathrm{~d} x=\frac{2}{\pi}[-\cos x]_{0}^{\pi}=\frac{4}{\pi} .
$$

The other coefficient $a_{n}, n=1,2, \ldots$ are given by

$$
a_{n}=\frac{2}{\pi} \int_{0}^{\pi} \sin x \cos (2 n x) \mathrm{d} x=\frac{1}{l} \int_{0}^{\pi}[\sin (2 n+1) x-\sin (2 n-1) x] \mathrm{d} x
$$

It can be further simplified to have

$$
a_{n}=\frac{1}{\pi}\left[-\left.\frac{\cos (2 n+1) x}{2 n+1}\right|_{0} ^{\pi}+\left.\frac{\cos (2 n-1) x}{2 n-1}\right|_{0} ^{\pi}\right]=\frac{1}{\pi}\left[\frac{2}{2 n+1}-\frac{2}{2 n-1}\right]=-\frac{4}{\pi\left(4 n^{2}-1\right)}
$$

Now we compute the coefficients $b_{n}, n=1,2, \ldots$ as

$$
\begin{aligned}
b_{n}=\frac{2}{\pi} \int_{0}^{\pi} \sin x \sin (2 n x) \mathrm{d} x & =\frac{2}{\pi} \int_{0}^{\pi} \frac{1}{2}[\cos (2 n-1) x-\cos (2 n+1) x] \mathrm{d} x \\
& =\frac{1}{\pi}\left[-\left.\frac{\sin (2 n-1) x}{2 n-1}\right|_{0} ^{\pi}+\left.\frac{\sin (2 n+1) x}{2 n+1}\right|_{0} ^{\pi}\right]=0
\end{aligned}
$$

Hence the Fourier series is given by

$$
f(x) \sim \frac{2}{\pi}+\sum_{n=1}^{\infty} \frac{-4}{\pi\left(4 n^{2}-1\right)} \cos (2 n x)=\frac{2}{\pi}-\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos (2 n x)}{4 n^{2}-1}, \quad 0 \leq x \leq 1
$$

Case II: If we treat $f(x)$ as $2 \pi$ periodic then

$$
\begin{aligned}
a_{n} & =\frac{2}{\pi} \int_{0}^{\pi} \sin x \cos (n x) \mathrm{d} x=\frac{1}{\pi} \int_{0}^{\pi}[\sin (n+1) x-\sin (n-1) x] \mathrm{d} x \\
& =\frac{1}{\pi}\left[-\left.\frac{\cos (n+1) x}{n+1}\right|_{0} ^{\pi}+\left.\frac{\cos (n-1) x}{n-1}\right|_{0} ^{\pi}\right] \mathrm{d} x=\frac{1}{\pi}\left[\frac{-(-1)^{n+1}+1}{n+1}+\frac{(-1)^{n-1}-1}{n-1}\right]
\end{aligned}
$$

Thus, for $n \neq 1$ we have

$$
a_{n}= \begin{cases}0, A B & \text { when } n \text { is odd } \\ -\frac{1}{\pi} \frac{4}{n^{2}-1}, & \text { when } n \text { is even }\end{cases}
$$

The coefficient $a_{1}$ needs to calculated separately as

$$
a_{1}=\frac{2}{\pi} \int_{0}^{\pi} \sin x \cos x \mathrm{~d} x=\frac{1}{\pi} \int_{0}^{\pi} \sin 2 x \mathrm{~d} x=\left.\frac{1}{\pi}\left[-\frac{\cos 2 x}{2}\right]\right|_{0} ^{\pi}=\frac{1}{2 \pi}[-1+1]=0
$$

Clearly, the coefficients $b_{n}$ 's are zero because

$$
b_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin (n x) \mathrm{d} x=\frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{|\sin x| \sin (n x)}_{\text {odd function }} \mathrm{d} x=0
$$

The Fourier series can be written as

$$
f(x) \sim \frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 2 x}{3}+\frac{\cos 4 x}{15}+\frac{\cos 6 x}{35}+\ldots\right]=\frac{2}{\pi}-\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos (2 n x)}{4 n^{2}-1}
$$

Therefore we ended up with the same series.

Remark 5: If we develop the Fourier series of a function considering its period as any integer multiple of its fundamental period, we shall end up with the same Fourier series.

Remark 6: Note that in the above example the given function is an even function and therefore the Fourier series is simpler as we have seen that the coefficient $b_{n}$ is zero in this case. The determination of the Fourier series of a given function becomes simpler if the function is odd or even. More detail of this we shall see in the Lesson 23.

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## Lesson 22

## Convergence Theorems

We have seen that piecewise continuity of a function is sufficient for the existence of the Fourier series. We have not yet discussed the convergence of the Fourier series. Convergence of the Fourier series is a very important topic to be explored in this lesson.

In order to motivate the discussion on convergence, let us construct the Fourier series of the function

$$
f(x)=\left\{\begin{array}{ll}
-\cos x, & -\pi / 2 \leq x<0 ; \\
\cos x, & 0 \leq x \leq \pi / 2 .
\end{array} \quad f(x+\pi)=f(x)\right.
$$

In this case the function is an odd function and therefore $a_{n}=0, n=0,1,2, \ldots$. We compute the Fourier coefficient $b_{n}$ by

$$
b_{n}=\frac{2}{\pi} \int_{-\pi / 2}^{\pi / 2} f(x) \sin (2 n x) \mathrm{d} x=\frac{4}{\pi} \int_{0}^{\pi / 2} \cos x \sin (2 n x) \mathrm{d} x=\frac{8}{\pi} \frac{n}{\left(4 n^{2}-1\right)}
$$

The Fourier series is given by

$$
f(x) \sim \sum_{n=1}^{\infty} b_{n} \sin (2 n x)=\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \sin (2 n x)}{4 n^{2}-1} .
$$

Note that the Fourier series at $x=0$ converges to 0 . So the Fourier series of $f$ does not converge to the value of the function at $x=0$.

With this example we pose the following questions in connection to the convergence of the Fourier series

1. Does the Fourier series of a function $f(x)$ converges at a point $x \in[-L, L]$.
2. If the series converges at a point $x$, is the sum of the series equal to $f(x)$.

The answers of these questions are in the negative because

1. There are Lebesgue integrable functions on $[-L, L]$ whose Fourier series diverge everywhere on $[-L, L]$.
2. There are continuous functions whose Fourier series diverge at a countable number of points.
3. We have already seen in the above examples that the Fourier series converges at a point but the sum is not equal to the the value of the function at that point.

We need some additional conditions to ensure that the Fourier series of a function $f(x)$ converges and it converges to the function $f(x)$. Though, we have several notions of convergence like pointwise, uniform, mean square, etc. we first stick to the most common notion of convergence, that is, pointwise convergence. Let $\left\{f_{m}\right\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$. We say that $\left\{f_{m}\right\}_{m=1}^{\infty}$ converges pointwise to $f$ on $[a, b]$ if for each $x \in[a, b]$ we have $\lim _{m \rightarrow \infty} f_{m}(x)=f(x)$. A more formal definition of pointwise convergence will be given later.

### 22.1 Convergence Theorem (Dirichlet's Theorem, Sufficient Conditions)

Theorem Statement: Let $f$ be a piecewise continuous function on $[-L, L]$ and the one sided derivatives of $f$, that is,

$$
\begin{equation*}
\lim _{h \rightarrow 0+} \frac{f(x+h)-f(x+)}{h} \text { in } x \in[-L, L) \quad \& \quad \lim _{h \rightarrow 0+} \frac{f(x-)-f(x-h)}{h} \text { in } x \in(-L, L] \tag{22.1}
\end{equation*}
$$

exist (and are finite), then for each $x \in(-L, L)$ the Fourier series converges and we have

$$
\frac{f(x+)+f(x-)}{2}=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left[a_{n} \cos \frac{k \pi x}{L^{n}}+b_{n} \sin \frac{k \pi x}{L}\right]
$$

At both endpoints $x= \pm L$ the series converges to $[f(L-)+f((-L)+)] / 2$, thus we have

$$
\frac{f(L-)+f((-L)+)}{2}=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}(-1)^{n} a_{n}
$$

Remark 1: If the function is continuous at a point $x$, that is, $f(x+)=f(x-)$ then we have

$$
\begin{equation*}
f(x)=\frac{a_{0}}{2}+\sum_{k=1}^{\infty}\left[a_{n} \cos \frac{k \pi x}{L}+b_{n} \sin \frac{k \pi x}{L}\right] \tag{22.2}
\end{equation*}
$$

In other words, if $f$ is continuous with $f(-L)=f(L)$ and one sided derivatives (22.1) exist then equality (22.2) holds for all $x$.

Remark 2: In the above theorem condition on $f$ are sufficient conditions. One may replace these conditions (piecewise continuity and one sided derivatives) by slightly more restrictive conditions of piecewise smoothness. A function is said to be piecewise smooth on $[-L, L]$ if it is piecewise continuous and has a piecewise continuous derivative. The difference between the two similar restrictions on $f$ will be clear from the example of the function

$$
f(x)= \begin{cases}x^{2} \sin (1 / x), & x \neq 0 \\ 0, & x=0\end{cases}
$$

It can easily easily be shown that derivative of the function exist everywhere and thus the function has one sided derivatives and satisfy the conditions of the convergence Theorem (22.1). However the function is not piecewise smooth because the $\lim _{x \rightarrow 0} f^{\prime}(x)$ does not exist as

$$
f^{\prime}(x)= \begin{cases}2 x \sin (1 / x)-\cos (1 / x), & x \neq 0 \\ 0, & x=0\end{cases}
$$

If a function is piecewise smooth then it can easily be shown that left and right derivatives exist. Let $f$ be a piecewise smooth function on $[-L, L]$ then $\lim _{x \rightarrow a \pm} f^{\prime}(x)$ exists for all $a \in[-L, L]$. This implies

$$
\lim _{x \rightarrow a+} f^{\prime}(x)=\lim _{x \rightarrow a+}\left(\lim _{h \rightarrow 0+} \frac{f(x+h)-f(x)}{h}\right)
$$

Interchanging the two limits on the right hand side we obtain

$$
\lim _{x \rightarrow a+} f^{\prime}(x)=\lim _{h \rightarrow 0+}\left(\lim _{x \rightarrow a+} \frac{f(x+h)-f(x)}{h}\right)=\lim _{h \rightarrow 0+} \frac{f(a+h)-f(a+)}{h \text { LOUU }}
$$

Similarly one can shown the existence of left derivative. This example confirms that piecewise smoothness is stronger condition than piecewise continuity with existence of one sided derivatives.

### 22.2 Different Notions of Convergence

### 22.2.1 Mean Square Convergence

Let $\left\{f_{m}\right\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$. Let $f$ be defined on $[a, b]$. We say that the sequence $\left\{f_{m}\right\}_{m=1}^{\infty}$ converges in the mean square sense to $f$ on $[a, b]$ if

$$
\lim _{m \rightarrow \infty} \int_{a}^{b}\left|f(x)-f_{m}(x)\right|^{2} \mathrm{~d} x=0
$$

### 22.2.2 Pointwise Convergence

Let $\left\{f_{m}\right\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$ and let $f$ be defined on $[a, b]$. We say that $\left\{f_{m}\right\}_{m=1}^{\infty}$ converges pointwise to $f$ on $[a, b]$ if for each $x \in[a, b]$ we have $\lim _{m \rightarrow \infty} f_{m}(x)=f(x)$. That is, for each $x \in[a, b]$ and $\varepsilon>0$ there is a natural number $N(\varepsilon, x)$ such that

$$
\left|f_{n}(x)-f(x)\right|<\varepsilon \text { for all } n \geq N(\varepsilon, x)
$$

### 22.2.3 Uniform Convergence

Let $\left\{f_{m}\right\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$ and let $f$ be defined on $[a, b]$. We say that $\left\{f_{m}\right\}_{m=1}^{\infty}$ converges uniformly to $f$ on $[a, b]$ if for each $\varepsilon>0$ there is a natural number $N(\varepsilon)$ such that

$$
\left|f_{n}(x)-f(x)\right|<\varepsilon \text { for all } n \geq N(\varepsilon), \text { and for all } x \in[a, b]
$$

There is one more interesting fact about the uniform convergence. If $\left\{f_{m}\right\}_{m=1}^{\infty}$ is a sequence of continuous functions which converge uniformly to a function to $f$ on $[a, b]$, then $f$ is continuous.

### 22.2.4 Example 1

Let $u_{n}=x^{n}$ on $[0,1)$. Clearly, the sequence $\left\{u_{n}\right\}_{n=1}^{\infty}$ converges pointwise to 0 , that is, for fixed $x \in[0,1)$ we have $\lim _{n \rightarrow \infty} u_{n}=0$. But it does not converge uniformly to 0 as we shall show that for given $\varepsilon$ there does not exist a natural number $N$ independent of $x$ such that $\left|u_{n}-0\right|<\varepsilon$. Suppose that the series converges uniformly, then for a given $\varepsilon$ with

$$
\begin{equation*}
\left|u_{n}-0\right|<\varepsilon, \tag{22.3}
\end{equation*}
$$

we seek for a natural number $N(\varepsilon)$ such that relation (22.3) holds for $n>N$. Note that relation (22.3) holds true if

$$
x^{n}<\varepsilon \Longleftrightarrow n>\frac{\ln \varepsilon}{\ln x}
$$

It should be evident now that for given $x$ and $\varepsilon$ one can define

$$
N:=\left[\frac{\ln \varepsilon}{\ln x}\right], \quad \text { where }[] \text { gives integer rounded towards infinity }
$$

It once again confirms pointwise convergence. However if $x$ is not fixed then $\ln \varepsilon / \ln x$ grows without bounds for $x \in[0,1)$. Hence it is not possible to find $N$ which depends only on $\varepsilon$ and therefore the sequence $u_{n}$ does not converge uniformly to 0 .

### 22.2.5 Example 2

Let $u_{n}=\frac{x^{n}}{n}$ on $[0,1)$. This sequence converges uniformly and of course pointwise to 0 . For given $\varepsilon>0$ take $n>N:=\left[\frac{1}{\varepsilon}\right]$ then noting $\left[\frac{1}{\varepsilon}\right]>\frac{1}{\varepsilon}$ we have $\left|u_{n}-0\right|<x^{n} / n<1 / n<\varepsilon$ for all $n>N$ Hence the sequence $u_{n}$ converges uniformly.

Now we discuss these three types of convergence for the Fourier series of a function.

- Let $f$ be a piecewise continuous function on $[-\pi, \pi]$ then the Fourier series of $f$ convergence to $f$ in the mean square sense. That is

$$
\lim _{m \rightarrow \infty} \int_{\pi}^{\pi}\left|f(x)-\left[\frac{a_{0}}{2}+\sum_{k=1}^{m}\left(a_{k} \cos k x+b_{k} \sin k x\right)\right]\right|^{2} \mathrm{~d} x=0
$$

- Let $f$ be a piecewise continuous function on $[-\pi, \pi]$ and the appropriate one sided derivatives of $f$ at each point in $[-\pi, \pi]$ exists then for each $x \in[-\pi, \pi]$ the Fourier series of $f$ converges pointwise to the value $(f(x-)+f(x+)) / 2$.
- If $f$ is continuous on $[-\pi, \pi], f(-\pi)=f(\pi)$, and $f^{\prime}$ is piecewise continuous on $[-\pi, \pi]$, then the Fourier series of $f$ converges uniformly (and also absolutely) to $f$ on $[-\pi, \pi]$.


### 22.3 Best Trigonometric Polynomial Approximation

An interesting property of the partial sums of a Fourier series is that among all trigonometric polynomials of degree $N$, the partial sum of Fourier Series yield the best approximation of $f$ in the mean square sense. This result has been summarized in the following lemma.

### 22.3.1 Lemma

Let $f$ be piecewise continuous function on $[-\pi, \pi]$ and let the mean square error is defined by the following function

$$
E\left(c_{0}, \ldots, c_{N}, d_{1}, \ldots, d_{N}\right)=\int_{-\pi}^{\pi}\left|f-\left[\frac{c_{0}}{2}+\sum_{k=1}^{N}\left(c_{k} \cos k x+d_{k} \sin k x\right)\right]\right|^{2} \mathrm{~d} x
$$

then $E\left(a_{0}, \ldots, a_{N}, b_{1}, \ldots, b_{N}\right) \leq E\left(c_{0}, \ldots, c_{N}, d_{1}, \ldots, d_{N}\right)$ for any real numbers $c_{0}, c_{1}, \ldots, c_{N}$ and $d_{1}, d_{2}, \ldots, d_{N}$. Note that $a_{k}$ and $b_{k}$ are the Fourier coefficients of $f$.

### 22.4 Example Problems

### 22.4.1 Problem 1

Let the function $f(x)$ be defined as

$$
f(x)= \begin{cases}-\pi, & -\pi<x<0 \\ x, & 0<x<\pi\end{cases}
$$

Find the sum of the Fourier series for all point in $[-\pi, \pi]$.
Solution: At $x=0$, the Fourier series will converge to

$$
\frac{f(0+)+f(0-)}{2}=\frac{0+(-\pi)}{2}=-\frac{\pi}{2}
$$

Again, $x= \pm \pi$ are another points of discontinuity and the value of the series at these point will be

$$
\frac{f(\pi-)+f((-\pi)+)}{2}=\frac{\pi+(-\pi)}{2}=0
$$

At all other points the series will converge to functional value $f(x)$.

### 22.4.2 Problem 2

Let the Fourier series of the function $f(x)=x+x^{2},-\pi<x<\pi$ be given by

$$
x+x^{2} \sim \frac{\pi^{2}}{3}+\sum_{n=1}^{\infty}(-1)^{n}\left[\frac{4}{n^{2}} \cos n x-\frac{2}{n} \sin n x\right]
$$

Find the sum of the Fourier series for all point in $[-\pi, \pi]$. Applying the result on convergence of the Fourier series find the value of

$$
1+\frac{1}{2^{2}}+\frac{1}{3^{2}}+\frac{1}{4^{2}}+\ldots \quad \text { and } \quad 1-\frac{1}{2^{2}}+\frac{1}{3^{2}}-\frac{1}{4^{2}}+\ldots
$$

Solution: Clearly the required series may be obtained by substituting $x= \pm \pi$ and $x=0$. At the points of discontinuity $x= \pm \pi$ the series converges to

$$
\frac{f(\pi-)+f((-\pi)+)}{2}=\frac{\left(\pi+\pi^{2}\right)+\left(-\pi+\pi^{2}\right)}{2}=\pi^{2}
$$

Substituting $x= \pm \pi$ into the series we get

$$
\frac{\pi^{2}}{3}+\sum_{n=1}^{\infty}(-1)^{(2 n)} \frac{4}{n^{2}}=\pi^{2} \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^{2}}=\frac{\pi^{2}}{6}
$$

At the point $x=0$ is a point of continuity and therefore the series will converge to 0 . Substituting $x=0$ into the series we obtain

$$
\frac{\pi^{2}}{3}+\sum_{n=1}^{\infty}(-1)^{(n)} \frac{4}{n^{2}}=0 \Longrightarrow \sum_{n=1}^{\infty}(-1)^{(1+n)} \frac{1}{n^{2}}=\frac{\pi^{2}}{12}
$$

## Suggested Readings

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## Lesson 23

## Half Range Sine and Cosine Series

In this chapter, we start discussion on even and odd function. As mentioned earlier if the function is odd or even then the Fourier series takes a rather simple form of containing sine or cosine terms only. Then we discuss a very important topic of developing a desired Fourier series (sine or cosine) of a function defined on a finite interval by extending the given function as odd or even function.

### 23.1 Even and Odd Functions

A function is said to be an even about the point $a$ if $f(a-x)=f(a+x)$ for all $x$ and odd about the point $a$ if $f(a-x)=-f(a+x)$ for all $x$. Further, note the following properties of even and odd functions:
a) The product of two even or two odd functions is again an even function.
b) The product of and even function and an odd function is an odd function.

Using these properties we have the following results for the Fourier coefficients

$$
\begin{aligned}
& a_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos (n x) \mathrm{d} x=\frac{2}{\pi} \begin{cases}\int_{0}^{\pi} f(x) \cos (n x) \mathrm{d} x, & \text { when } f \text { is even function about } 0 \\
0, & \text { when } f \text { is odd function about } 0\end{cases} \\
& b_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin (n x) \mathrm{d} x=\frac{2}{\pi} \begin{cases}0, & \text { when } f \text { is even function about } 0 \\
\int_{0}^{\pi} f(x) \sin (n x) \mathrm{d} x, & \text { when } f \text { is odd function about } 0\end{cases}
\end{aligned}
$$

From these observation we have the following results

### 23.1.1 Proposition

Assume that $f$ is a piecewise continuous function on $[-\pi, \pi]$. Then
a) If $f$ is an even function then the Fourier series takes the simple form

$$
f(x) \sim \frac{a_{0}}{2}+\sum_{n=1}^{\infty} a_{n} \cos (n x) \quad \text { with } \quad a_{n}=\frac{2}{\pi} \int_{0}^{\pi} f(x) \cos (n x) \mathrm{d} x, n=0,1,2, \ldots
$$

Such a series is called a cosine series.
b) If $f$ is an odd function then the Fourier series of $f$ has the form

$$
f(x) \sim \sum_{n=1}^{\infty} b_{n} \sin (n x) \quad \text { with } \quad b_{n}=\frac{2}{\pi} \int_{0}^{\pi} f(x) \sin (n x) \mathrm{d} x, n=1,2, \ldots
$$

Such a series is called a sine series.

### 23.2 Example Problems

### 23.2.1 Problem 1

Obtain the Fourier series to represent the function $f(x)$

$$
f(x)= \begin{cases}x, & \text { when } 0 \leq x \leq \pi \\ 2 \pi-x, & \text { when } \pi<x \leq 2 \pi\end{cases}
$$

Solution: The given function is an even function about $x=\pi$ and therefore

$$
b_{n}=\frac{1}{\pi} \int_{0}^{2 \pi} f(x) \sin (n x) \mathrm{d} x=0
$$

The coefficient $a_{0}$ will be calculated as

$$
a_{0}=\frac{1}{\pi} \int_{0}^{2 \pi} f(x) \mathrm{d} x=\frac{1}{\pi}\left[\int_{0}^{\pi} x \mathrm{~d} x+\int_{p i}^{2 \pi}(2 \pi-x) \mathrm{d} x\right]=\frac{1}{\pi}\left[\frac{\pi^{2}}{2}+\frac{\pi^{2}}{2}\right]=\pi
$$

The other coefficients $a_{n}$ are given as

$$
a_{n}=\frac{1}{\pi} \int_{0}^{2 \pi} f(x) \cos (n x) \mathrm{d} x=\frac{1}{\pi}\left[\int_{0}^{\pi} x \cos (n x) \mathrm{d} x+\int_{p i}^{2 \pi}(2 \pi-x) \cos (n x) \mathrm{d} x\right]
$$

It can be further simplified as

$$
a_{n}=\frac{2}{n^{2} \pi}\left[(-1)^{n}-1\right]= \begin{cases}0, & \text { when } n \text { is even } \\ \frac{4}{n^{2} \pi}, & \text { when } n \text { is odd }\end{cases}
$$

Therefore, the Fourier series is given by

$$
\begin{equation*}
f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\cos x+\frac{\cos 3 x}{3^{2}}+\frac{\cos 5 x}{5^{2}}+\ldots\right] \text { where } 0 \leq x \leq 2 \pi \tag{23.1}
\end{equation*}
$$

In this case as the function is continuous and $f^{\prime}$ is piecewise continuous, the series converges uniformly to $f(x)$ and we can write the equality (23.1).

### 23.2.2 Problem 2

Determine the Fourier Series of $f(x)=x^{2}$ on $[-\pi, \pi]$ and hence find the value of the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{n^{2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{2}}$.

Solution: The function $f(x)=x^{2}$ is even on the interval $[-\pi, \pi]$ and therefore $b_{n}=0$ for all n . The coefficient $a_{0}$ is given as

$$
a_{0}=\frac{1}{\pi} \int_{\pi}^{\pi} x^{2} \mathrm{~d} x=\left.\frac{x^{3}}{3 \pi}\right|_{-\pi} ^{\pi}=\frac{2 \pi^{2}}{3}
$$

The other coefficients can be calculated by the general formula as

$$
a_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} x^{2} \cos (n x) \mathrm{d} x=\frac{2}{\pi} \int_{0}^{\pi} x^{2} \cos (n x) \mathrm{d} x=\frac{2}{\pi}\left[\left.x^{2} \frac{\sin (n x)}{n}\right|_{0} ^{\pi}-\frac{1}{n} \int_{0}^{\pi} 2 x \sin (n x) \mathrm{d} x\right]
$$

Again integrating by parts we obtain

$$
a_{n}=\frac{4}{n \pi}\left[\left.x \frac{\cos (n x)}{n}\right|_{0} ^{\pi}-\int_{0}^{\pi} \frac{\cos (n x)}{n} \mathrm{~d} x\right]=\frac{4}{n \pi}\left[\frac{\pi(-1)^{n}}{n}-0\right]=\frac{4(-1)^{n}}{n^{2}}
$$

Therefore the Fourier series is given as

$$
\begin{equation*}
x^{2}=\frac{\pi^{2}}{3}+\sum_{n=1}^{\infty} \frac{4(-1)^{n}}{n^{2}} \cos (n x) \quad \text { for } \quad x \in[-\pi, \pi] \tag{23.2}
\end{equation*}
$$

If we substitute $x=0$ in the equation (23.2) we get

$$
0=\frac{\pi^{2}}{3}+\sum_{n=1}^{\infty} \frac{4(-1)^{n}}{n^{2}} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2}}=\frac{\pi^{2}}{12}
$$

If we now substitute $x=\pi$ in the equation (23.2) we get

$$
\pi^{2}=\frac{\pi^{2}}{3}+\sum_{n=1}^{\infty} \frac{4(-1)^{2 n}}{n^{2}} \Rightarrow \frac{1}{4} \frac{2 \pi^{2}}{3}=\sum_{n=1}^{\infty} \frac{1}{n^{2}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{2}}=\frac{\pi^{2}}{6}
$$

### 23.3 Half Range Series

Suppose that $f(x)$ is a function defined on $(0, l]$. Suppose we want to express $f(x)$ in the cosine or sine series. This can be done by extending $f(x)$ to be an even or an odd function on $[-l, l]$. Note that there exists an infinite number of ways to express the function in the interval $[-l, 0]$. Among all possible extension of $f$ there are two, even and odd extensions, that lead to simple and useful series:
a) If we want to express $f(x)$ in cosine series then we extend $f(x)$ as an even function in the interval $[-l, l]$.
b) On the other hand, if we want to express $f(x)$ in sine series then we extend $f(x)$ as an odd function in $[-l, l]$.

We summarize the above discussion in the following proposition

### 23.3.1 Proposition

Let $f$ be a piecewise continuous function defined on $[0, l]$. The series

$$
f(x) \sim \frac{a_{0}}{2}+\sum_{n=1}^{\infty} a_{n} \cos \frac{n \pi x}{l} \text { with } a_{n}=\frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n \pi x}{l} \mathrm{~d} x
$$

is called half range cosine series of $f$. Similarly, the series

$$
f(x) \sim \sum_{n=1}^{\infty} b_{n} \sin \frac{n \pi x}{l} \quad \text { with } \quad b_{n}=\frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n \pi x}{l} \mathrm{~d} x
$$

is called half range sine series of $f$.

Remark: Note that we can develop a Fourier series of a function $f$ defined in $[0, l]$ and it will, in general, contain all sine and cosine terms. This series, if converges, will represent a l-periodic function. The idea of half range Fourier series is entirely different where we extend the function $f$ as per our desire to have sine or cosine series. The half range series of the function $f$ will represent a $2 l$-periodic function.

### 23.4 Example Problems

### 23.4.1 Problem 1

Obtain the half range sine series for $e^{x}$ in $0<x<1$.
Solution: Since we are developing sine series of $f$ we need to compute $b_{n}$ as

$$
\begin{aligned}
b_{n} & =\frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n \pi x}{l} \mathrm{~d} x=2 \int_{0}^{1} e^{x} \sin n \pi x=2\left[\left.e^{x} \sin n \pi x\right|_{0} ^{1}-n \pi \int_{0}^{1} e^{x} \cos n \pi x d x\right] \mathrm{d} x \\
& =2\left[-n \pi\left\{\left.e^{x} \cos n \pi x\right|_{0} ^{1}+n \pi \int_{0}^{1} e^{x} \sin n \pi x \mathrm{~d} x\right\}\right]=-2 n \pi\left(e(-1)^{n}-1\right)-n^{2} \pi^{2} b_{n}
\end{aligned}
$$

Taking second term on the right side to the left side and after simplification we get

$$
b_{n}=\frac{2 n \pi\left[1-e(-1)^{n}\right]}{1+n^{2} \pi^{2}}
$$

Therefore, the sine series of $f$ is given as

$$
e^{x}=2 \pi \sum_{n=1}^{\infty} \frac{n\left[1-e(-1)^{n}\right]}{1+n^{2} \pi^{2}} \sin n \pi x \quad \text { for } \quad 0<x<1
$$

### 23.4.2 Problem 2

Let $f(x)=\sin \frac{\pi x}{l}$ on $(0, l)$. Find Fourier cosine series in the range $0<x<l$.
Solution: Sine we want to find cosine series of the function $f$ we compute the coefficients $a_{n}$ as

$$
a_{n}=\frac{2}{l} \int_{0}^{l} \sin \frac{\pi x}{l} \cos \frac{n \pi x}{l} \mathrm{~d} x=\frac{1}{l} \int_{0}^{l}\left[\sin \frac{(n+1) \pi x}{l}+\sin \frac{(1-n) \pi x}{l}\right] \mathrm{d} x
$$

For $n \neq 1$ we can can compute the integrals to get

$$
a_{n}=\frac{1}{l}\left[-\frac{\cos \frac{(n+1) \pi x}{l}}{\frac{(n+1) \pi}{l}}+\frac{\cos \frac{(1-n) \pi x}{l}}{\frac{(n-1) \pi}{l}}\right]_{0}^{l}=\frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{1}{n+1}+\frac{(-1)^{n-1}}{n-1}-\frac{1}{n-1}\right]
$$

It can be further simplified as

$$
a_{n}= \begin{cases}0, & \text { when } n \text { is odd } \\ -\frac{4}{\pi(n+1)(n-1)}, & \text { when } n \text { is even }\end{cases}
$$

The coefficient $a_{1}$ needs to calculated separately as

$$
a_{1}=\frac{1}{l} \int_{0}^{l} \sin \frac{2 \pi x}{l} \mathrm{~d} x=\frac{1}{l}\left[\cos \frac{2 \pi x}{l} \frac{l}{2 \pi}\right]_{0}^{l}=\frac{1}{2 \pi}(1-1)=0
$$

The Fourier cosine series of $f$ is given as

$$
\sin \frac{\pi x}{l}=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos \frac{2 \pi x}{l}}{1 \cdot 3}+\frac{\cos \frac{4 \pi x}{l}}{3 \cdot 5}+\frac{\cos \frac{6 \pi x}{l}}{5 \cdot 7}+\ldots\right]
$$

### 23.4.3 Problem 3

Expand $f(x)=x, 0<x<2$ in a (i) sine series and (ii) cosine series.
Solution: (i) To get sine series we calculate $b_{n}$ as

$$
b_{n}=\frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n \pi x}{L} \mathrm{~d} x=\frac{2}{2} \int_{0}^{2} x \sin \frac{n \pi x}{2} \mathrm{~d} x
$$

Integrating by parts we obtain

$$
b_{n}=\left[x \cos \frac{n \pi x}{2}\left(-\frac{2}{n \pi}\right)\right]_{0}^{2}+\frac{2}{n \pi} \int_{0}^{2} \cos \frac{n \pi x}{2} \mathrm{~d} x=-\frac{4}{n \pi} \cos n \pi .
$$

Then for $0<x<2$ we have the Fourier sine series

$$
x=-\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos n \pi}{n} \sin \frac{n \pi x}{2}=\frac{4}{\pi}\left(\sin \frac{\pi x}{2}-\frac{1}{2} \sin \frac{2 \pi x}{2}+\frac{1}{3} \sin \frac{3 \pi x}{2}+\ldots\right) .
$$

(ii) Now we express $f(x)=x$ in cosine series. We need to calculate $a_{n}$ for $n \neq 0$ as

$$
a_{n}=\frac{2}{2} \int_{0}^{2} x \cos \frac{n \pi x}{2} \mathrm{~d} x=\left[x \sin \frac{n \pi x}{2}\left(\frac{2}{n \pi}\right)\right]_{0}^{2}-\int_{0}^{2} \sin \frac{n \pi x}{2}\left(\frac{2}{n \pi}\right) \mathrm{d} x
$$

After simplifications we obtain

$$
a_{n}=\frac{2}{n \pi}\left(\frac{2}{n \pi}\right)\left[\cos \frac{n \pi x}{2}\right]_{0}^{2}=\frac{4}{n^{2} \pi^{2}}(\cos n \pi-1)=\frac{4}{n^{2} \pi^{2}}\left[(-1)^{n}-1\right]
$$

The coefficient $a_{0}$ is given as

$$
a_{0}=\int_{0}^{2} x \mathrm{~d} x=2
$$

Then the Fourier sine series of $f(x)=x$ for $0<x<2$ is given as

$$
x=1+\frac{4}{\pi^{2}} \sum_{n=1}^{\infty} \frac{\left[(-1)^{n}-1\right]}{n^{2}} \cos \frac{n \pi x}{2}=1-\frac{8}{\pi^{2}}\left(\cos \frac{\pi x}{2}+\frac{1}{3^{2}} \cos \frac{3 \pi x}{2}+\frac{1}{5^{2}} \cos \frac{5 \pi x}{2}+\ldots\right) .
$$

It is interesting to note that the given function $f(x)=x, \quad 0<x<2$ is represented by two entirely different series. One contains only sine terms while the other contains only cosine terms.

Note that we have used series equal to the given function because the series converges for each $x \in(0,2)$ to the function value. It should also be pointed out that one can deduce sum of several series by putting different values of $x \in(0,2)$ in the above sine and cosine series.

## Suggested Readings

Davis, H.F. (1963). Fourier Series and Orthogonal Functions. Dover Publications, Inc. New York.

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## Lesson 24

## Integration and Differentiation of Fourier Series

In this lesson we discuss differentiation and integration of the Fourier series of a function. We can get some idea of the complexity of the new series if looking at the terms of the series. In the case of differentiation we get terms like $n \sin (n x)$ and $n \cos (n x)$, where presence of $n$ as product makes the magnitude of the terms larger then the original and therefore convergence of the new series becomes more difficult. This is exactly other way round in the case of integration where $n$ appears in division and new terms become smaller in magnitude and thus we expect better convergence in this case. We shall deal these two case separately in next sections.

### 24.1 Differentiation

We first discuss term by term differentiation of the Fourier series. Let $f$ be a piecewise continuous with the Fourier series

$$
\begin{equation*}
f(x) \sim \frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left[a_{n} \cos (n x)+b_{n} \sin (n x)\right] \tag{24.1}
\end{equation*}
$$

Can we differentiate term by term the Fourier series of a function $f$ in order to obtain the Fourier series of $f^{\prime}$ ? In other words, is it true that

$$
\begin{equation*}
f^{\prime}(x) \sim \sum_{n=1}^{\infty}\left[-n a_{n} \sin (n x)+n b_{n} \cos (n x)\right] ? \tag{24.2}
\end{equation*}
$$

In general the answer to this question is no.
Let us consider the Fourier series of $f(x)=x$ in $[-\pi, \pi]$. This is an odd function and therefore Fourier series will be $x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin (n x)$. If we differentiate the series term by term we get $\sum_{n=1}^{\infty} 2(-1)^{n+1} \cos (n x)$. Note that this is not the Fourier series of $f^{\prime}(x)=1$ since the Fourier series of $f(x)=1$ is simply 1.

We consider one more simple example to illustrate this fact. Consider the half range sine series for $\cos x$ in $(0, \pi)$

$$
\cos x \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \sin (2 n x)}{\left(4 n^{2}-1\right)}
$$

If we differentiate this series term by term then we obtain the series

$$
\frac{16}{\pi} \sum_{n=1}^{\infty} \frac{n^{2} \cos (2 n x)}{\left(4 n^{2}-1\right)}
$$

This series can not be the Fourier series of $-\sin x$ because it diverges as

$$
\lim _{n \rightarrow \infty} \frac{16}{\pi} \frac{n^{2} \cos (2 n x)}{\left(4 n^{2}-1\right)} \neq 0
$$

For the term by term differentiation we have the following result

### 24.1.1 Theorem

If $f$ is continuous on $[-\pi, \pi], f(-\pi)=f(\pi), f^{\prime}$ is piecewise continuous on $[-\pi, \pi]$, and if

$$
f(x) \sim \frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left[a_{n} \cos (n x)+b_{n} \sin (n x)\right]
$$

(in fact in this case we can replace $\sim b y=$ ) is the Fourier series of $f$, then the Fourier series of $f^{\prime}$ is given by

$$
f^{\prime}(x) \sim \sum_{n=1}^{\infty}\left[-n a_{n} \sin (n x)+n b_{n} \cos (n x)\right] .
$$

Moreover, if the function $f^{\prime}$ has appropriate left and right derivatives at a point $x$, then we have

$$
\frac{f^{\prime}(x+)+f^{\prime}(x-)}{2}=\sum_{n=1}^{\infty}\left[-n a_{n} \sin (n x)+n b_{n} \cos (n x)\right]
$$

If $f^{\prime}$ is continuous at a point $x$ then

$$
f^{\prime}(x)=\sum_{n=1}^{\infty}\left[-n a_{n} \sin (n x)+n b_{n} \cos (n x)\right] .
$$

Proof: Since $f^{\prime}$ is piecewise continuous and this is sufficient condition for the existence of Fourier series of $f^{\prime}$. So we can write Fourier series of as

$$
\begin{equation*}
f^{\prime}(x) \sim \frac{\bar{a}_{0}}{2}+\sum_{n=1}^{\infty}\left[\bar{a}_{n} \cos (n x)+\bar{b}_{n} \sin (n x)\right] \tag{24.3}
\end{equation*}
$$

where

$$
\bar{a}_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f^{\prime}(x) \cos (n x) \mathrm{d} x, \quad \bar{b}_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f^{\prime}(x) \sin (n x) \mathrm{d} x
$$

Now we simplify coefficients $\bar{a}_{n}$ and $\bar{b}_{n}$ and write them in terms of $a_{n}$ and $b_{n}$. Using the condition $f(-\pi)=f(\pi)$, we can easily show that

$$
\bar{a}_{0}=0, \quad \bar{a}_{n}=n b_{n}, \quad \bar{b}_{n}=-n a_{n}
$$

Now the Fourier series of $f^{\prime}(24.3)$ reduces to

$$
f^{\prime}(x) \sim \sum_{n=1}^{\infty}\left[n b_{n} \cos (n x)-n a_{n} \sin (n x)\right]
$$

Convergence of this series to $\frac{f^{\prime}(x+)+f^{\prime}(x-)}{2}$ or $f^{\prime}(x)$ is a direct consequence of convergence theorem of Fourier series.

### 24.2 Integration

In general, for an infinite series uniform convergence is required to integrate the series term by term. In the case of Fourier series we do not even have to assume the convergence of the Fourier series to be integrated. However, integration term by term of a Fourier series does not, in general, lead to a Fourier series. The main results can be summarize as:

### 24.2.1 Theorem

Let $f$ be piecewise continuous function and have the following Fourier series

$$
\begin{equation*}
f(x) \sim \frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left[a_{n} \cos (n x)+b_{n} \sin (n x)\right] \tag{24.4}
\end{equation*}
$$

Then no matter whether this series converges or not we have for each $x \in[-\pi, \pi]$,

$$
\begin{equation*}
\int_{-\pi}^{x} f(t) d t=\frac{a_{0}(x+\pi)}{2}+\sum_{n=1}^{\infty}\left[\frac{a_{n}}{n} \sin (n x)-\frac{b_{n}}{n}(\cos (n x)-\cos n \pi)\right] \tag{24.5}
\end{equation*}
$$

and the series on the right hand side converges uniformly to the function on the left.
Proof: We define

$$
g(x)=\int_{-\pi}^{x} f(t) d t-\frac{a_{0}}{2} x
$$

Since $f$ is piecewise continuous function, it is easy to prove that $g$ is continuous. Also

$$
\begin{equation*}
g^{\prime}(x)=f(x)-\frac{a_{0}}{2} \tag{24.6}
\end{equation*}
$$

at each point of continuity of $f$. This implies that $g^{\prime}$ is piecewise continuous and further we see that

$$
g(-\pi)=\frac{a_{0} \pi}{2}
$$

and

$$
g(\pi)=\int_{-\pi}^{\pi} f(t) d t-\frac{a_{0}}{2} \pi=\pi a_{0}-\frac{a_{0}}{2} \pi=\frac{a_{0} \pi}{2}
$$

Hence, the Fourier series of the function $g$ converges uniformly to $g$ on $[-\pi, \pi]$. Thus we have

$$
g(x)=\frac{\alpha_{0}}{2}+\sum_{n=1}^{\infty}\left[\alpha_{n} \cos (n x)+\beta_{n} \sin (n x)\right]
$$

Using Theorem 24.1.1 we have the following result for the Fourier series of $g^{\prime}$ as

$$
g^{\prime}(x) \sim \sum_{n=1}^{\infty}\left[-n \alpha_{n} \sin (n x)+n \beta_{n} \cos (n x)\right]
$$

Fourier series of $f$ and the relation (24.6) gives

$$
g^{\prime}(x)=f(x)-\frac{a_{0}}{2} \sim \sum_{n=1}^{\infty}\left[a_{n} \cos (n x)+b_{n} \sin (n x)\right]
$$

Now comparing the last two equations we get

$$
n \beta_{n}=a_{n} \quad-n \alpha_{n}=b_{n} \quad n=1,2, \ldots
$$

Substituting these values in the Fourier series of $g$ we obtain

$$
g(x)=\int_{-\pi}^{x} f(t) d t-\frac{a_{0}}{2} x=\frac{\alpha_{0}}{2}+\sum_{n=1}^{\infty}\left[\frac{a_{n}}{n} \sin (n x)-\frac{b_{n}}{n} \cos (n x)\right]
$$

We can rewrite this to get

$$
\begin{equation*}
\int_{-\pi}^{x} f(t) d t=\frac{a_{0}}{2} x+\frac{\alpha_{0}}{2}+\sum_{n=1}^{\infty}\left[\frac{a_{n}}{n} \sin (n x)-\frac{b_{n}}{n} \cos (n x)\right] \tag{24.7}
\end{equation*}
$$

To obtain $\alpha_{0}$ we set $x=\pi$ in the above equation

$$
\alpha_{0}=a_{0} \pi+\sum_{n=1}^{\infty} \frac{2 b_{n}}{n} \cos (n x)
$$

Substituting $\alpha_{0}$ in the equation (24.7) we obtain the required result (24.5).

Remark 1: Note that the series on the right hand side of (24.5) is not a Fourier series due to presence of $x$.

Remark 2: The above Theorem on integration can be established in a more general sense as:
If $f$ be piecewise continuous function in $-\pi \leq x \leq \pi$ and if

$$
\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left[a_{n} \cos (n x)+b_{n} \sin (n x)\right]
$$

is its Fourier series then no matter whether this series converges or not, it is true that

$$
\int_{a}^{x} f(t) d t=\frac{a_{0}}{2} \int_{a}^{x} a_{0} \mathrm{~d} x+\sum_{n=1}^{\infty} \int_{a}^{x}\left[a_{n} \cos (n x)+b_{n} \sin (n x)\right] \mathrm{d} x
$$

where $-\pi \leq a \leq x \leq \pi$ and the series on the right hand side of converges uniformly in $x$ to the function on the left for any fixed value of $a$.

## Suggested Readings

Davis, H.F. (1963). Fourier Series and Orthogonal Functions. Dover Publications, Inc. New York.

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## Lesson 25

## Bessel's Inequality and Parseval's Identity

In this lesson some properties of the Fourier coefficients will be given. We will mainly derive two important inequalities related to Fourier series, in particular, Bessel's inequality and Parseval's identity. One of the applications of Parseval's identity for summing certain infinite series will be discussed.

### 25.1 Theorem (Bessel's Inequality)

If $f$ be a piecewise continuous function in $[-\pi, \pi]$, then

$$
\frac{a_{0}^{2}}{2}+\sum_{k=1}^{\infty}\left(a_{k}^{2}+b_{k}^{2}\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^{2}(x) \mathrm{d} x
$$

where $a_{0}, a_{1}, \ldots$ and $b_{1}, b_{2}, \ldots$ are Fourier coefficients of $f$.
Proof: Clearly, we have

$$
\int_{-\pi}^{\pi}\left[f(x)-\frac{a_{0}}{2}-\sum_{k=1}^{n}\left[a_{k} \cos (k x)+b_{k} \sin (k x)\right]\right]^{2} \mathrm{~d} x \geq 0
$$

Expanding the integrands we get

$$
\begin{aligned}
& \int_{-\pi}^{\pi} f^{2}(x) \mathrm{d} x+\frac{a_{0}^{2}}{2} \pi+\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}\left[a_{k} \cos (k x)+b_{k} \sin (k x)\right]\right]^{2} \mathrm{~d} x-a_{0} \int_{-\pi}^{\pi} f(x) \mathrm{d} x \\
& -2 \int_{-\pi}^{\pi} f(x)\left[\sum_{k=1}^{n}\left[a_{k} \cos (k x)+b_{k} \sin (k x)\right]\right] \mathrm{d} x+a_{0} \int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}\left[a_{k} \cos (k x)+b_{k} \sin (k x)\right]\right] \mathrm{d} x \geq 0
\end{aligned}
$$

Using the orthogonality of the trigonometric system and definition of Fourier coefficients we get

$$
\int_{-\pi}^{\pi} f^{2}(x) \mathrm{d} x+\frac{a_{0}^{2}}{2} \pi+\pi \sum_{k=1}^{n}\left(a_{k}^{2}+b_{k}^{2}\right)-a_{0}^{2} \pi-2 \pi \sum_{k=1}^{n}\left(a_{k}^{2}+b_{k}^{2}\right)+0 \geq 0
$$

This can be further simplified

$$
\int_{-\pi}^{\pi} f^{2}(x) \mathrm{d} x-\frac{a_{0}^{2}}{2} \pi-\pi \sum_{k=1}^{n}\left(a_{k}^{2}+b_{k}^{2}\right) \geq 0
$$

This implies

$$
\frac{a_{0}^{2}}{2}+\sum_{k=1}^{n}\left(a_{k}^{2}+b_{k}^{2}\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^{2}(x) \mathrm{d} x
$$

Passing the limit $n \rightarrow \infty$, we get the required Bessel's inequality.
Indeed the above Bessel's inequality turns into an equality named Parseval's identity. However, for the sake of simplicity of proof we state the following theorem for more restrictive function but the result holds under less restrictive conditions (only piecewise continuity) same as in Theorem 25.1.

### 25.2 Theorem (Parseval's Identity)

If $f$ is a continuous function in $[-\pi, \pi]$ and one sided derivatives exit then we have the equality

$$
\begin{equation*}
\frac{a_{0}^{2}}{2}+\sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right)=\frac{1}{\pi} \int_{-\pi}^{\pi} f^{2}(x) \mathrm{d} x \tag{25.1}
\end{equation*}
$$

where $a_{0}, a_{1}, \ldots$ and $b_{1}, b_{2}, \ldots$ are Fourier coefficients of $f$.
Proof: From the Dirichlet's convergence theorem for $x \in(-\pi, \pi)$ we have

$$
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left[a_{n} \cos (n x)+b_{n} \sin (n x)\right]
$$

Integrating by $f(x)$ and integrating term by term from $-\pi$ to $\pi$ we obtain

$$
\int_{-\pi}^{\pi} f^{2}(x) \mathrm{d} x=\frac{a_{0}}{2} \int_{-\pi}^{\pi} f(x) \mathrm{d} x+\sum_{n=1}^{\infty}\left(a_{n} \int_{-\pi}^{\pi} f(x) \cos (n x) \mathrm{d} x+b_{n} \int_{-\pi}^{\pi} f(x) \sin (n x) \mathrm{d} x\right)
$$

Using the definition of Fourier coefficients we get

$$
\int_{-\pi}^{\pi} f^{2}(x) \mathrm{d} x=\frac{\pi a_{0}^{2}}{2}+\pi \sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right)
$$

Dividing by $\pi$ we obtain the required identity.

Remark: As stated earlier Parseval's identity can be proved for piecewise continuous functions. Further, for a piecewise continuous function on $[-L, L]$ we can get Parseval's identity just by replacing $\pi$ by $L$ in (25.1).

### 25.3 Example Problems

### 25.3.1 Problem 1

Consider the Fourier cosine series of $f(x)=x$ :

$$
x \sim 1+\sum_{n=1}^{\infty} \frac{4}{\pi^{2} n^{2}}[\cos (n \pi)-1] \cos \frac{n \pi x}{2}
$$

a) Write Parseval's identity corresponding to the above Fourier series
b) Determine from a) the sum of the series

$$
\frac{1}{1^{4}}+\frac{1}{2^{4}}+\frac{1}{3^{4}}+\ldots
$$

Solution: a) We first find the Fourier coefficient and the period of the Fourier series just by comparing the given series with the standard Fourier series

$$
\begin{aligned}
& a_{0}=2, \quad a_{n}=\frac{4}{\pi^{2} n^{2}}[\cos (n \pi)-1], n=1,2 \ldots, \quad b_{n}=0 \\
& \text { period }=2 L=4 \Rightarrow L=2
\end{aligned}
$$

Writing Parseval's identity as

$$
\frac{1}{L} \int_{-L}^{L} f^{2}(x) \mathrm{d} x=\frac{a_{0}^{2}}{2}+\sum_{n=1}^{\infty}\left(\overline{a_{n}^{2}}+b_{n}^{2}\right)
$$

This implies

$$
\frac{1}{2} \int_{-2}^{2} x^{2} \mathrm{~d} x=\frac{4}{2}+\sum_{n=1}^{\infty} \frac{16}{\pi^{4} n^{4}}(\cos (n \pi)-1)^{2}
$$

This can be simplified to give

$$
\frac{8}{3}=2+\frac{64}{\pi^{4}}\left[\frac{1}{1^{4}}+\frac{1}{3^{4}}+\frac{1}{5^{4}}+\ldots\right]
$$

Then we obtain

$$
\frac{1}{1^{4}}+\frac{1}{3^{4}}+\frac{1}{5^{4}}+\ldots=\frac{\pi^{4}}{96}
$$

b) Let

$$
S=\frac{1}{1^{4}}+\frac{1}{2^{4}}+\frac{1}{3^{4}}+\ldots
$$

This series can be rewritten as

$$
\begin{aligned}
S & =\left(\frac{1}{1^{4}}+\frac{1}{3^{4}}+\frac{1}{5^{4}}+\ldots\right)+\left(\frac{1}{2^{4}}+\frac{1}{4^{4}}+\frac{1}{6^{4}}+\ldots\right) \\
& =\frac{\pi^{4}}{96}+\frac{1}{2^{4}} S
\end{aligned}
$$

Then we have the required sum as $S=\frac{\pi^{4}}{90}$.

### 25.3.2 Problem 2

Find the Fourier series of $x^{2},-\pi<x<\pi$ and use it along with Parseval's theorem to show that

$$
\sum_{n=1}^{\infty} \frac{1}{(2 n-1)^{4}}=\frac{\pi^{4}}{96}
$$

Solution: Since $f(x)=x^{2}$ is an even function, so $b_{n}=0$. The Fourier coefficients $a_{n}$ will be given as

$$
a_{n}=\frac{2}{\pi} \int_{0}^{\pi} f(x) \cos (n x) \mathrm{d} x=\frac{2}{\pi} \int_{0} \pi x^{2} \cos (n x) \mathrm{d} x
$$

This can be further simplified for $n \neq 0$ to

$$
a_{n}=\frac{2}{\pi}\left[0-\frac{2}{n} \int_{0}^{\pi} x \sin (n x) \mathrm{d} x\right]=\frac{4}{n^{2}}(-1)^{n}
$$

The coefficient $a_{0}$ can be evaluated separately as

$$
a_{0}=\frac{2}{\pi} \int_{0} \pi x^{2} \mathrm{~d} x=\frac{2 \pi^{2}}{3}
$$

The the Fourier series of $f(x)=x^{2}$ will be given as

$$
x^{2}=\frac{\pi^{2}}{3}+4 \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^{2}} \cos (n x)
$$

Now by parseval's theorem we have

$$
\frac{1}{\pi} \int_{-\pi}^{\pi} f^{2}(x) \mathrm{d} x=\frac{a_{0}^{2}}{2}+\sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right)
$$

Using $\frac{1}{\pi} \int_{-\pi}^{\pi} x^{4} \mathrm{~d} x=\frac{2 \pi^{4}}{5}$ we get

$$
\frac{4 \pi^{4}}{18}+\sum_{n=1}^{\infty} \frac{16}{n^{4}}=\frac{2 \pi^{4}}{5}
$$

This implies

$$
\sum_{n=1}^{\infty} \frac{1}{n^{4}}=\frac{\pi^{4}}{90}
$$

Now using the idea of splitting of the series from the Example 25.3.1 (b), we have

$$
\sum_{n=1}^{\infty} \frac{1}{(2 n-1)^{4}}=\sum_{n=1}^{\infty} \frac{1}{n^{4}}-\frac{1}{16} \sum_{n=1}^{\infty} \frac{1}{n^{4}}=\frac{15}{16} \sum_{n=1}^{\infty} \frac{1}{n^{4}}
$$

Substituting the value of $\sum_{k=1}^{\infty} \frac{1}{n^{4}}$ in the above equation we get the required sum.

### 25.3.3 Problem 3

## Given the Fourier series

$$
\cos \left(\frac{x}{2}\right)=\frac{2}{\pi}+\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{\left(4 n^{2}-1\right)} \cos (n x)
$$

deduce the value of

$$
\sum_{n=1}^{\infty} \frac{1}{\left(4 n^{2}-1\right)^{2}}
$$

Solution: By Parseval's theorem for

$$
a_{0}=\frac{4}{\pi}, a_{n}=\frac{4}{\pi} \frac{(-1)^{n+1}}{\left(4 n^{2}-1\right)}, f(x)=\cos (x / 2)
$$

we have

$$
\frac{1}{2} \frac{16}{\pi^{2}}+\frac{16}{\pi^{2}} \sum_{n=1}^{\infty} \frac{1}{\left(4 n^{2}-1\right)^{2}}=\frac{1}{\pi} \int_{-\pi}^{\pi} \cos ^{2}(x / 2) \mathrm{d} x=1
$$

Then,

$$
\sum_{n=1}^{\infty} \frac{1}{\left(4 n^{2}-1\right)^{2}}=\frac{\pi^{2}-8}{16}
$$

## Suggested Readings

Davis, H.F. (1963). Fourier Series and Orthogonal Functions. Dover Publications, Inc. New York.

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## Lesson 26

## Complex Fourier Series

It is often convenient to work with complex form of Fourier series. In deed, the complex form of Fourier series has applications in the field of signal processing which is of great interest to many electrical engineers.

Given the Fourier series of a function $f(x)$ as

$$
\begin{equation*}
f \sim \frac{1}{2} a_{0}+\sum_{n=1}^{\infty}\left[a_{n} \cos (n x)+b_{n} \sin (n x)\right], \quad-\pi<x<\pi \tag{26.1}
\end{equation*}
$$

with

$$
a_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos (n x) \mathrm{d} x, \quad n=0,1,2 \ldots
$$

and

$$
b_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin (n x) \mathrm{d} x, \quad n=1,2 \ldots
$$

We know from Euler's formula

$$
\cos (n x)=\frac{e^{i n x}+e^{-i n x}}{2} \quad \sin (n x)=\frac{e^{i n x}-e^{-i n x}}{2 i}
$$

Substituting these values of $\cos (n x)$ and $\sin (n x)$ into the equation (26.1) we obtain

$$
\begin{aligned}
f & \sim \frac{1}{2} a_{0}+\sum_{n=1}^{\infty}\left[a_{n} \frac{e^{i n x}+e^{-i n x}}{2}+b_{n} \frac{e^{i n x}-e^{-i n x}}{2 i}\right] \\
& =\frac{1}{2} a_{0}+\sum_{n=1}^{\infty}\left[\frac{1}{2}\left(a_{n}-i b_{n}\right) e^{i n x}+\frac{1}{2}\left(a_{n}+i b_{n}\right) e^{-i n x}\right]
\end{aligned}
$$

Let us define new coefficients as

$$
\begin{equation*}
c_{n}=\frac{1}{2}\left(a_{n}-i b_{n}\right), \quad k_{n}=\frac{1}{2}\left(a_{n}+i b_{n}\right) \tag{26.2}
\end{equation*}
$$

Note that $c_{0}=a_{0} / 2$ because $b_{0}=0$. Then the Fourier series becomes

$$
\begin{equation*}
f \sim c_{0}+\sum_{n=1}^{\infty}\left[c_{n} e^{i n x}+k_{n} e^{-i n x}\right] \tag{26.3}
\end{equation*}
$$

where the coefficients are given as

$$
\begin{aligned}
c_{n} & =\frac{1}{2}\left(a_{n}-i b_{n}\right)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(x)[\cos (n x)-i \sin (n x)] \mathrm{d} x=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(x) e^{-i n x} \mathrm{~d} x \\
k_{n} & =\frac{1}{2}\left(a_{n}+i b_{n}\right)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(x)[\cos (n x)+i \sin (n x)] \mathrm{d} x=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(x) e^{i n x} \mathrm{~d} x
\end{aligned}
$$

From the above calculation we get $k_{n}=c_{-n}$. Substituting the value of $k_{n}$ into the Fourier series (26.3) we have

$$
\begin{equation*}
f \sim \sum_{n=-\infty}^{\infty} c_{n} e^{i n x} \tag{26.4}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{n}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(x) e^{-i n x} \mathrm{~d} x, \quad n=0, \pm 1, \pm 2, \ldots \tag{26.5}
\end{equation*}
$$

The series on the right side of equation (26.4) is called complex form of the Fourier series. For a function of period $2 L$ defined in $[-L, L]$, the complex form of the Fourier series can analogously be derived to have

$$
f \sim \sum_{n=-\infty}^{\infty} c_{n} e^{\frac{i n \pi x}{L}}, \quad c_{n}=\frac{1}{2 L} \int_{-L}^{L} f(x) e^{\frac{-i n \pi x}{L}} \mathrm{~d} x, \quad n=0, \pm 1, \pm 2, \ldots
$$

### 26.1 Example Problems

### 26.1.1 Problem 1

Find the complex Fourier series of

$$
f(x)=e^{x} \text { if }-\pi<x<\pi \text { and } f(x+2 \pi)=f(x)
$$

Solution: We calculate the coefficients $c_{n}$ as

$$
\begin{aligned}
c_{n} & =\frac{1}{2 \pi} \int_{-\pi}^{\pi} e^{x} e^{-i n x} \mathrm{~d} x=\frac{1}{2 \pi} \int_{-\pi}^{\pi} e^{(1-i n) x} \mathrm{~d} x \\
& =\left.\frac{1}{2 \pi} \frac{e^{(1-i n) x}}{1-i n}\right|_{-\pi} ^{\pi}=\frac{1}{2 \pi} \frac{1}{1-i n}\left[e^{\pi} e^{-i n \pi}-e^{-\pi} e^{i n \pi}\right]
\end{aligned}
$$

Substituting $e^{ \pm i n \pi}=\cos n \pi \pm i \sin n \pi=(-1)^{n}$ we get

$$
c_{n}=\frac{1}{\pi} \frac{1+i n}{(1-i n)(1+i n)}(-1)^{n} \sinh \pi=\frac{1}{\pi} \frac{1+i n}{\left(1+n^{2}\right)}(-1)^{n} \sinh \pi
$$

Then, the Fourier is given as

$$
f \sim \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty}(-1)^{n} \frac{1+i n}{\left(1+n^{2}\right)} e^{i n x}
$$

### 26.1.2 Problem 2

Determine the complex Fourier series representation of

$$
f(x)=x \text { if }-l<x<l \text { and } f(x+2 l)=f(x)
$$

Solution: The complex Fourier series representation of a function $f(x)$ is given as

$$
f \sim \sum_{n=-\infty}^{\infty} c_{n} e^{\frac{i n \pi x}{l}}
$$

where

$$
c_{n}=\frac{1}{2 l} \int_{-l}^{l} f(x) e^{\frac{-i n \pi x}{l}} \mathrm{~d} x=\frac{1}{2 l} \int_{-l}^{l} x e^{\frac{-i n \pi x}{l}} \mathrm{~d} x
$$

For $n \neq 0$, integrating by parts we get

$$
c_{n}=\frac{1}{2 l}\left[\left.\left(x e^{\frac{-i n \pi x}{l}} \frac{-l}{i n \pi}\right)\right|_{-l} ^{l}+\frac{l}{i n \pi} \int_{-l}^{l} e^{\frac{-i n \pi x}{l}} \mathrm{~d} x\right],
$$

Further application of integration by parts simplifies to

$$
c_{n}=\frac{1}{2 l}\left(-\frac{l^{2}}{i n \pi} e^{-i n \pi}-\frac{l^{2}}{i n \pi} e^{i n \pi}\right)-\frac{l^{2}}{(i n \pi)^{2}} \underbrace{\left.e^{\frac{-i n \pi x}{l}}\right|_{-l} ^{l}}_{=0},
$$

Finally, it simplifies to

$$
c_{n}=\frac{(-1)^{n} i l}{n \pi}, \quad n= \pm 1, \pm 2, \ldots
$$

Now $c_{0}$ can be calculated as

$$
c_{0}=\frac{1}{2 l} \int_{-l}^{l} x \mathrm{~d} x=0
$$

Therefore, the Fourier series is given as

$$
f \sim \frac{i l}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n}}{n} e^{\frac{i n \pi x}{l}}
$$

### 26.1.3 Problem 3

Show that Parseval's identity for the complex form of Fourier series takes the form

$$
\frac{1}{2 \pi} \int_{-\pi}^{\pi}\{f(x)\}^{2} \mathrm{~d} x=\sum_{n=-\infty}^{\infty}\left|c_{n}\right|^{2}
$$

Solution: For the real form of Fourier series the Parseval's identity is given as

$$
\begin{equation*}
\frac{a_{0}^{2}}{2}+\sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right)=\frac{1}{\pi} \int_{-\pi}^{\pi}\{f(x)\}^{2} \mathrm{~d} x \tag{26.6}
\end{equation*}
$$

We know that

$$
c_{0}=\frac{a_{0}}{2}, \quad c_{n}=\frac{1}{2}\left(a_{n}-i b_{n}\right), \quad c_{-n}=\frac{1}{2}\left(a_{n}+i b_{n}\right)
$$

We can deduce that

$$
\begin{equation*}
\left|c_{n}\right|^{2}=\frac{1}{4}\left(a_{n}^{2}+b_{n}^{2}\right), \quad\left|c_{-n}\right|^{2}=\frac{1}{4}\left(a_{n}^{2}+b_{n}^{2}\right) \tag{26.7}
\end{equation*}
$$

Diving the equation (26.6) by 2 and then splitting the second term as

$$
\frac{a_{0}^{2}}{4}+\frac{1}{4} \sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right)+\frac{1}{4} \sum_{n=1}^{\infty}\left(a_{n}^{2}+b_{n}^{2}\right)=\frac{1}{2 \pi} \int_{-\pi}^{\pi}\{f(x)\}^{2} \mathrm{~d} x
$$

Using the relations (26.7) we obtain

$$
c_{0}^{2}+\sum_{n=1}^{\infty}\left|c_{n}\right|^{2}+\sum_{n=1}^{\infty}\left|c_{-n}\right|^{2}=\frac{1}{2 \pi} \int_{-\pi}^{\pi}\{f(x)\}^{2} \mathrm{~d} x
$$

This can be rewritten as

$$
\sum_{n=-\infty}^{\infty}\left|c_{n}\right|^{2}=\frac{1}{2 \pi} \int_{-\pi}^{\pi}\{f(x)\}^{2} \mathrm{~d} x
$$

### 26.1.4 Problem 4

Given the Fourier series

$$
e^{x} \sim \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty}(-1)^{n} \frac{1+i n}{\left(1+n^{2}\right)} e^{i n x} .
$$

deduce the value of

$$
\sum_{n=-\infty}^{\infty} \frac{1}{n^{2}+1}
$$

Solution: From the given series we clearly have

$$
c_{n}=(-1)^{n} \frac{e^{\pi}-e^{-\pi}}{2 \pi} \frac{1+i n}{\left(1+n^{2}\right)}, \quad n=0, \pm 1, \pm 2, \ldots
$$

These coefficients can be simplified

$$
\left|c_{n}\right|^{2}=\frac{\left(e^{\pi}-e^{-\pi}\right)^{2}}{4 \pi^{2}} \frac{\left(1+n^{2}\right)}{\left(1+n^{2}\right)^{2}}=\frac{\left(e^{\pi}-e^{-\pi}\right)^{2}}{4 \pi^{2}} \frac{1}{\left(1+n^{2}\right)}
$$

A simple calculation gives

$$
\frac{1}{2 \pi} \int_{-\pi}^{\pi}\{f(x)\}^{2} \mathrm{~d} x=\frac{1}{2 \pi} \int_{-\pi}^{\pi} e^{2 x} \mathrm{~d} x=\frac{e^{2 \pi}-e^{-2 \pi}}{4 \pi}
$$

Thus, by Parseval's identity we have

$$
\frac{e^{2 \pi}-e^{-2 \pi}}{4 \pi}=\frac{\left(e^{\pi}-e^{-\pi}\right)^{2}}{4 \pi^{2}} \sum_{n=-\infty}^{\infty} \frac{1}{\left(1+n^{2}\right)}
$$

Therefore, we obtain

$$
\sum_{n=-\infty}^{\infty} \frac{1}{\left(1+n^{2}\right)}=\frac{\pi\left(e^{\pi}+e^{-\pi}\right)}{\left(e^{\pi}-e^{-\pi}\right)}=\pi \cot h \pi .
$$

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## Lesson 27

## Fourier Integral

If $f(x)$ is defined on a finite interval $[-l, l]$ and is piecewise continuous then we can construct a Fourier series corresponding to the function $f$ and this series will represent the function on this interval if the function satisfies some additional conditions discussed before. Furthermore, if $f$ is periodic then we may be able to represent the function by its Fourier series on the entire real line. Now suppose the function is not periodic and is defined on the entire real line. Then we do not have any possibility to represent the function by the Fourier series. However, we may still be able to represent the function in terms of sine and cosines using an integral, called Fourier integral, instead of a summation. In this lesson we discuss a representation of a non-periodic function by letting $l \rightarrow \infty$ in the Fourier series of a function defined on $[-l, l]$.

### 27.1 Fourier Series Representation of a Function

Consider any function $f(x)$ defined on $[-l, l]$ that can be represented by a Fourier series as

$$
\begin{equation*}
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left(a_{n} \cos \frac{n \pi x}{l}+b_{n} \sin \frac{n \pi x}{l}\right) . \tag{27.1}
\end{equation*}
$$

For a more general case we can replace left hand side of the above equation by the average value $(f(x+)+f(x-)) / 2$. We now see what will happen if we let $l \rightarrow \infty$. It should be mentioned that as $l$ approaches to $\infty$ the function $f(x)$ becomes non-periodic defined on the real axis. Substituting $a_{n}$ and $b_{n}$ in the equation (27.1) we get
$f(x)=\frac{1}{2 l} \int_{-l}^{l} f(u) \mathrm{d} u+\frac{1}{l} \sum_{n=1}^{\infty}\left(\int_{-l}^{l} f(u) \cos \frac{n \pi u}{l} \mathrm{~d} u \cos \frac{n \pi x}{l}+\int_{-l}^{l} f(u) \sin \frac{n \pi u}{l} \mathrm{~d} u \sin \frac{n \pi x}{l}\right)$
Using the identity $\cos x \cos y+\sin x \sin y=\cos (x-y)$, we get

$$
\begin{equation*}
f(x)=\frac{1}{2 l} \int_{-l}^{l} f(u) \mathrm{d} u+\frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^{l} f(u) \cos \frac{n \pi}{l}(u-x) \mathrm{d} u \tag{27.2}
\end{equation*}
$$

If we assume that $\int_{-\infty}^{\infty}|f(u)| \mathrm{d} u$ converges, the first term on the right hand side approaches to 0 as $l \rightarrow \infty$ since $\left|\frac{1}{2 l} \int_{-l}^{l} f(u) \mathrm{d} u\right| \leq \frac{1}{2 l} \int_{-\infty}^{\infty}|f(u)| \mathrm{d} u$.


Figure 27.1: Sum of area of trapezoid as area under curve in the limiting case

Letting $l \rightarrow \infty$ in equation (27.2), we get

$$
f(x)=\lim _{l \rightarrow \infty} \frac{1}{l} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(u) \cos \frac{n \pi}{l}(u-x) \mathrm{d} u=\lim _{l \rightarrow \infty} \frac{\pi}{l} \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \frac{n \pi}{l}(u-x) \mathrm{d} u
$$

For simplifications, we define

$$
\Delta \alpha=\frac{\pi}{l} \quad \text { and } \quad F(\alpha)=\frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \alpha(u-x) \mathrm{d} u
$$

With these definitions and noting $\Delta \alpha \rightarrow 0$ as $l \rightarrow \infty$, we have

$$
f(x)=\lim _{\Delta \alpha \rightarrow 0} \sum_{n=1}^{\infty} \Delta \alpha F(n \Delta \alpha)
$$

Refereing Figure 27.1, we can write this limit of the sum in the form of improper integral as

$$
f(x)=\int_{0}^{\infty} F(\alpha) \mathrm{d} \alpha=\frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(u) \cos \alpha(u-x) \mathrm{d} u \mathrm{~d} \alpha
$$

This is called Fourier Integral Representation of $f$ on the real line. Equivalently, this can be rewritten as

$$
f(x)=\frac{1}{\pi} \int_{0}^{\infty}\left[\left(\int_{-\infty}^{\infty} f(u) \cos \alpha u \mathrm{~d} u\right) \cos \alpha x+\left(\int_{-\infty}^{\infty} f(u) \sin \alpha u \mathrm{~d} u\right) \sin \alpha x\right] \mathrm{d} \alpha
$$

It is often convenient to write

$$
f(x)=\int_{0}^{\infty}[A(\alpha) \cos \alpha x+B(\alpha) \sin \alpha x] \mathrm{d} \alpha
$$

where the Fourier Integral Coefficients are

$$
A(\alpha)=\frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \alpha u \mathrm{~d} u \quad \text { and } \quad B(\alpha)=\frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin \alpha u \mathrm{~d} u
$$

Remark It should be mentioned that above derivation is not rigorous proof of convergence of the Fourier Integral to the function. This is just to give some idea of transition form Fourier series to Fourier Integral. Nevertheless we summarize the convergence result, without proof, in the next theorem. In addition to all conditions required for the convergence of Fourier series we need one more condition, namely, absolute integrability of $f$. Further, note that Fourier integral representation of $f(x)$ is entirely analogous to $a$ Fourier series representation of a function on finite interval $\left(\sum_{n=1}^{\infty} \cdots\right.$, is replaced with $\left.\int_{0}^{\infty} \cdots d u\right)$.

### 27.1.1 Theorem

Assume that $f$ is piecewise smooth on every finite interval on the $x$ axis (or piecewise continuous and one sided derivatives exist) and let $f$ be absolutely integrable over entire real axis. Then for each $x$ on the entire axis we have

$$
\frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(u) \cos \alpha(u-x) d u=\frac{f(x+)+f(x-)}{2}
$$

As in the convergence of Fourier series if $f$ is continuous and all other conditions are satisfied then the Fourier integral converges

### 27.2 Example Problems

### 27.2.1 Problem 1

Let a be a real constant and the function $f$ is defined as

$$
f(x)= \begin{cases}0, & x<0 \\ x, & 0<x<a \\ 0, & x>a\end{cases}
$$

i) Find the Fourier integral representation of $f$. ii) Determine the convergence of the integral at $x=a$. iii) Find the value of the integral $\int_{0}^{\infty} \frac{1-\cos \alpha}{\alpha^{2}} d \alpha$.
Solution: $i$ ) The integral representation of $f$ is

$$
\begin{equation*}
f(x) \sim \int_{0}^{\infty}[A(\alpha) \cos \alpha x+B(\alpha) \sin \alpha x] \mathrm{d} \alpha \tag{27.3}
\end{equation*}
$$

where

$$
\begin{aligned}
A(\alpha) & =\frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \alpha u \mathrm{~d} u=\frac{1}{\pi} \int_{0}^{a} u \cos \alpha u \mathrm{~d} u=\frac{1}{\pi}\left[\left.\left(\frac{u \sin \alpha u}{\alpha}\right)\right|_{0} ^{a}-\int_{0}^{a} \frac{\sin \alpha u}{\alpha} \mathrm{~d} u\right] \\
& =\frac{1}{\pi}\left[\frac{a \sin \alpha a}{\alpha}+\frac{(\cos \alpha a-1)}{\alpha^{2}}\right]=\frac{1}{\pi}\left[\frac{\cos \alpha a+\alpha a \sin \alpha a-1}{\alpha^{2}}\right] \\
B(\alpha) & =\frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin \alpha u \mathrm{~d} u=\frac{1}{\pi} \int_{0}^{a} u \sin \alpha u \mathrm{~d} u=\frac{1}{\pi}\left[\left.\left(\frac{-u \cos \alpha u}{\alpha}\right)\right|_{0} ^{a}+\int_{0}^{a} \frac{\cos \alpha u}{\alpha} \mathrm{~d} u\right] \\
& =\frac{1}{\pi}\left[\frac{-a \cos \alpha a}{\alpha}+\frac{\sin \alpha a}{\alpha^{2}}\right]=\frac{1}{\pi}\left[\frac{\sin \alpha a-\alpha a \cos \alpha a}{\alpha^{2}}\right]
\end{aligned}
$$

Replacing $A(\alpha)$ and $B(\alpha)$ in equation (27.3), we have

$$
\begin{aligned}
f(x) & \sim \frac{1}{\pi} \int_{0}^{\infty}\left[\left(\frac{\cos \alpha a+\alpha a \sin \alpha a-1}{\alpha^{2}}\right) \cos \alpha x+\left(\frac{\sin \alpha a-\alpha a \cos \alpha a}{\alpha^{2}}\right) \sin \alpha x\right] \mathrm{d} \alpha \\
& =\frac{1}{\pi} \int_{0}^{\infty} \frac{\cos \alpha(a-x)+\alpha a \sin \alpha(a-x)-\cos \alpha x}{\alpha^{2}} \mathrm{~d} \alpha
\end{aligned}
$$

ii) The function is not defined at $x=a$. The value of the Fourier integral at $x=a$ is given as

$$
\frac{1}{\pi} \int_{0}^{\infty} \frac{1-\cos \alpha a}{\alpha^{2}} \mathrm{~d} \alpha=\frac{f(a+)+f(a-)}{2}=\frac{0+a}{2}=\frac{a}{2}
$$

iii) Substituting $a=1$ in the above integral we get

$$
\frac{1}{\pi} \int_{0}^{\infty} \frac{1-\cos \alpha}{\alpha^{2}} \mathrm{~d} \alpha=\frac{1}{2} \Longrightarrow \int_{0}^{\infty} \frac{1-\cos \alpha}{\alpha^{2}} \mathrm{~d} \alpha=\frac{\pi}{2}
$$

### 27.2.2 Problem 2

Determine the Fourier integral representing

$$
f(x)= \begin{cases}1, & 0<x<2 \\ 0, & x<0 \text { and } x>2 .\end{cases}
$$

Further, find the value of the integral $\int_{0}^{\infty} \frac{\sin \alpha}{\alpha} d \alpha$.
Solution: The Fourier integral representation of $f$ is

$$
\begin{equation*}
f(x) \sim \int_{0}^{\infty}[A(\alpha) \cos \alpha x+B(\alpha) \sin \alpha x] \mathrm{d} \alpha \tag{27.4}
\end{equation*}
$$

where

$$
\begin{gathered}
A(\alpha)=\frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \alpha u \mathrm{~d} u=\frac{1}{\pi} \int_{0}^{2} \cos \alpha u \mathrm{~d} u=\left.\frac{1}{\pi} \frac{\sin \alpha u}{\alpha}\right|_{0} ^{2}=\frac{1}{\pi} \frac{\sin 2 \alpha}{\alpha} \\
B(\alpha)=\frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin \alpha u \mathrm{~d} u=\frac{1}{\pi} \int_{0}^{2} \sin \alpha u \mathrm{~d} u=\left.\frac{1}{\pi} \frac{-\cos \alpha u}{\alpha}\right|_{0} ^{2}=\frac{1}{\pi} \frac{(1-\cos 2 \alpha)}{\alpha}
\end{gathered}
$$

Then, substituting calculated values of $A(\alpha)$ and $B(\alpha)$ in equation (27.4), we obtain

$$
\begin{aligned}
f(x) & \sim \frac{1}{\pi} \int_{0}^{\infty}\left[\frac{\sin 2 \alpha}{\alpha} \cos \alpha x+\frac{(1-\cos 2 \alpha)}{\alpha} \sin \alpha x\right] \mathrm{d} \alpha \\
& =\frac{1}{\pi} \int_{0}^{\infty} \frac{\sin \alpha(2-x)+\sin \alpha x}{\alpha} \mathrm{~d} \alpha
\end{aligned}
$$

To find the value of the given integral we substitute $x=1$ in the above Fourier integral and use convergence theorem to get

$$
\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \alpha}{\alpha} \mathrm{d} \alpha=f(1)=1
$$

This gives the value of the desired integral as

$$
\int_{0}^{\infty} \frac{\sin \alpha}{\alpha} \mathrm{d} \alpha=\frac{\pi}{2}
$$

## Suggested Readings

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## Lesson 28

## Fourier Integrals (Cont.)

In this lesson we shall first present complex form of Fourier integral. We then introduce Fourier sine and cosine integral. The convergence of these integrals with its application to evaluate integrals will be discussed. In this lesson will be very useful to introduce Fourier transforms.

### 28.1 The Exponential Fourier Integral

It is often convenient to introduce complex form of Fourier integral. In fact, using complex form of Fourier integral we shall introduce Fourier transform, sometimes referred as Fourier exponential transform, in the next lesson. We start with the following Fourier integral

$$
\begin{equation*}
f(x)=\frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(u) \cos \alpha(u-x) \mathrm{d} u \mathrm{~d} \alpha \tag{28.1}
\end{equation*}
$$

Note that the integral

$$
\int_{-\infty}^{\infty} f(u) \cos \alpha(u-x) \mathrm{d} u
$$

is an even function of $\alpha$ and therefore the integral (28.1) can be written as

$$
\begin{equation*}
f(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \cos \alpha(u-x) \mathrm{d} u \mathrm{~d} \alpha \tag{28.2}
\end{equation*}
$$

Also, note that the integral

$$
\int_{-\infty}^{\infty} f(u) \sin \alpha(u-x) \mathrm{d} u
$$

is an odd function of $\alpha$ and therefore we have the following result

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \sin \alpha(u-x) \mathrm{d} u \mathrm{~d} \alpha=0 \tag{28.3}
\end{equation*}
$$

Multiplying the equation (28.3) by $i$ and adding into the equation (28.2) we obtain

$$
\begin{equation*}
f(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)[\cos \alpha(u-x)+i \sin \alpha(u-x)] \mathrm{d} u \mathrm{~d} \alpha \tag{28.4}
\end{equation*}
$$

This may be rewritten as

$$
\begin{equation*}
f(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i \alpha(u-x)} \mathrm{d} u \mathrm{~d} \alpha \tag{28.5}
\end{equation*}
$$

If we subtract the equation (28.3) after multiplying by $i$ from the equation (28.2) we obtain

$$
\begin{equation*}
f(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-i \alpha(u-x)} \mathrm{d} u \mathrm{~d} \alpha \tag{28.6}
\end{equation*}
$$

Either (28.5) or (28.6) are exponential form of the Fourier integral.

### 28.1.1 Example

Compute the complex Fourier integral representation of $f(x)=e^{-a|x|}$.
Solution: The complex integral representation of $f$ is given as

$$
\begin{equation*}
f(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i \alpha(u-x)} \mathbf{d} u \mathrm{~d} \alpha=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-i \alpha x} \int_{-\infty}^{\infty} f(u) e^{i \alpha u} \mathrm{~d} u \mathrm{~d} \alpha \tag{28.7}
\end{equation*}
$$

We first compute the inner integral

$$
\int_{-\infty}^{\infty} f(u) e^{i \alpha u} \mathrm{~d} u=\int_{-\infty}^{0} e^{a u} e^{i \alpha u} \mathrm{~d} u+\int_{0}^{\infty} e^{-a u} e^{i \alpha u} \mathrm{~d} u=\left[\frac{e^{(a+i \alpha) u}}{a+i \alpha}\right]_{-\infty}^{0}+\left[-\frac{e^{-(a-i \alpha) u}}{a-i \alpha}\right]_{0}^{\infty}
$$

This can be further simplified

$$
\int_{-\infty}^{\infty} f(u) e^{-i \alpha u} \mathrm{~d} u=\left(\frac{1}{a+i \alpha}+\frac{1}{a-i \alpha}\right)=\frac{2 a}{a^{2}+\alpha^{2}}
$$

Then the complex Fourier integral representation of $f$ is

$$
\begin{equation*}
f(x)=\frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^{2}+\alpha^{2}} e^{-i \alpha x} \mathrm{~d} \alpha \tag{28.8}
\end{equation*}
$$

### 28.2 Fourier Sine and Cosine Integrals

Consider the Fourier integral representation of a function $f$ as

$$
f(x) \sim \int_{0}^{\infty}[A(\alpha) \cos \alpha x+B(\alpha) \sin \alpha x] \mathrm{d} \alpha
$$

where the Fourier Integral Coefficients are

$$
A(\alpha)=\frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \alpha u \mathrm{~d} u \quad \text { and } \quad B(\alpha)=\frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin \alpha u \mathrm{~d} u
$$

If the function $f$ is an even function, the integral of $A(\alpha)$ has an even integrand. Therefore we can simplify the integral to

$$
A(\alpha)=\frac{2}{\pi} \int_{0}^{\infty} f(u) \cos \alpha u \mathrm{~d} u
$$

Since the integrand of the integral in $B(\alpha)$ is odd and therefore $B(\alpha)=0$. Thus for even function $f$ we have

$$
f(x) \sim \int_{0}^{\infty} A(\alpha) \cos \alpha x \mathrm{~d} \alpha
$$

Similarly, for an odd function $f$ we have

$$
A(\alpha)=0 \quad \text { and } \quad B(\alpha)=\frac{2}{\pi} \int_{0}^{\infty} f(u) \sin \alpha u \mathrm{~d} u
$$

and

$$
f(x) \sim \int_{0}^{\infty} B(\alpha) \sin \alpha x \mathrm{~d} \alpha
$$

Remark: Similar to half range Fourier series, we can represent a function defined for all real $x>0$ by Fourier sine or Fourier cosine integral by extending the function as an odd function or as an even function over the entire real axis, respectively.

We summarize the above results in the following theorem:

### 28.2.1 Theorem

Assume that $f$ is piecewise smooth function on every finite interval on the positive $x$-axis and let $f$ be absolutely integrable over 0 to $\infty$. Then $f$ may be represented by either: a) Fourier cosine integral

$$
f(x) \sim \int_{0}^{\infty} A(\alpha) \cos \alpha x, d \alpha \quad 0<x<\infty
$$

where

$$
A(\alpha)=\frac{2}{\pi} \int_{0}^{\infty} f(u) \cos \alpha u d u
$$

b) Fourier sine integral

$$
f(x) \sim \int_{0}^{\infty} B(\alpha) \sin \alpha x d \alpha \quad 0<x<\infty
$$

where

$$
B(\alpha)=\frac{2}{\pi} \int_{0}^{\infty} f(u) \sin \alpha u d u
$$

Moreover, the above Fourier cosine and sine integrals converge to $[f(x+)+f(x-)] / 2$.

### 28.3 Example Problems

### 28.3.1 Problem 1

For the function

$$
f= \begin{cases}0, & -\infty<x<-\pi \\ -1, & -\pi<x<0 \\ 1, & 0<x<\pi \\ 0, & \pi<x<\infty\end{cases}
$$

determine the Fourier integral. To what value does the integral converge at $x=-\pi$ ?
Solution: Since the given function is an odd function we can directly put $A(\alpha)=0$ and evaluate $B(\alpha)$ as

$$
B(\alpha)=\frac{2}{\pi} \int_{0}^{\infty} f(u) \sin \alpha u \mathrm{~d} u=\frac{2}{\pi} \int_{0}^{\pi} \sin \alpha u \mathrm{~d} u=\frac{2}{\pi \alpha}(1-\cos \alpha \pi)
$$

Therefore, the Fourier integral representation is

$$
f(x) \sim \frac{2}{\pi} \int_{0}^{\infty} \frac{1-\cos \alpha \pi}{\alpha} \sin \alpha x \mathrm{~d} \alpha
$$

The function is not defined at $x=-\pi$ and therefore the Fourier integral at $x=-\pi$ will converge to the average value $\frac{0-1}{2}=-\frac{1}{2}$.

### 28.3.2 Problem 2

Find a Fourier sine and cosine integral representation of

$$
f= \begin{cases}1, & 0<x<\pi \\ 0, & \pi<x<\infty\end{cases}
$$

## Hence evaluate

$$
\int_{0}^{\infty} \frac{\sin \pi \alpha \cos \pi \alpha}{\alpha} d \alpha \quad \text { and } \quad \int_{0}^{\infty} \frac{(1-\cos \pi \alpha) \sin \pi \alpha}{\alpha} d \alpha
$$

Solution: Fourier sine representation is given as

$$
f(x) \sim \int_{0}^{\infty} B(\alpha) \sin \alpha x \mathrm{~d} \alpha
$$

where

$$
B(\alpha)=\frac{2}{\pi} \int_{0}^{\infty} f(u) \sin \alpha u \mathrm{~d} u=\frac{2}{\pi} \int_{0}^{\pi} \sin \alpha u \mathrm{~d} u=\frac{2}{\pi} \frac{(1-\cos \pi \alpha)}{\alpha}
$$

Therefore

$$
f(x) \sim \frac{2}{\pi} \int_{0}^{\infty} \frac{(1-\cos \pi \alpha)}{\alpha} \sin \alpha x \mathrm{~d} \alpha
$$

Using convergence theorem, we have

$$
\frac{2}{\pi} \int_{0}^{\infty} \frac{(1-\cos \pi \alpha)}{\alpha} \sin \alpha x \mathrm{~d} \alpha= \begin{cases}0, & x>\pi \\ 1 / 2, & x=\pi \\ 1, & 0<x<\pi\end{cases}
$$

To get the desired integral we substitute $x=\pi$ in the above integral

$$
\frac{2}{\pi} \int_{0}^{\infty} \frac{(1-\cos \pi \alpha)}{\alpha} \sin \pi \alpha \mathrm{d} \alpha=\frac{1}{2} \Longrightarrow \int_{0}^{\infty} \frac{(1-\cos \pi \alpha)}{\alpha} \sin \pi \alpha \mathrm{d} \alpha=\frac{\pi}{4}
$$

For the Fourier cosine representation we evaluate

$$
A(\alpha)=\frac{2}{\pi} \int_{0}^{\infty} f(u) \cos \alpha u \mathrm{~d} u=\frac{2}{\pi} \int_{0}^{\pi} \cos \alpha u \mathrm{~d} u=\frac{2}{\pi} \frac{\sin \pi \alpha}{\alpha}
$$

Thus, the Fourier cosine integral representation is given as

$$
f(x) \sim \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \pi \alpha \cos \alpha x}{\alpha} \mathrm{~d} \alpha
$$

Applying convergence theorem we have

$$
\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \pi \alpha \cos \alpha x}{\alpha} \mathrm{~d} \alpha= \begin{cases}0, & x>\pi \\ 1 / 2, & x=\pi \\ 1, & 0<x<\pi\end{cases}
$$

To get the required integral we now substitute $x=\pi$ into the above integral

$$
\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \pi \alpha \cos \pi \alpha}{\alpha} \mathrm{d} \alpha=\frac{1}{2} \quad \Longrightarrow \quad \int_{0}^{\infty} \frac{\sin \pi \alpha \cos \pi \alpha}{\alpha} \mathrm{d} \alpha=\frac{\pi}{4}
$$

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## Lesson 29

## Fourier Sine and Cosine Transform

In this lesson we introduce Fourier cosine and sine transforms. Evaluation and properties of Fourier cosine and sine transform will be discussed. The parseval's identities for Fourier cosine and sine transform will be given.

### 29.1 Fourier Cosine and Sine Transform

Consider the Fourier cosine integral representation of a function $f$ as $f(x)=\frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \cos \alpha u \mathrm{~d} u \cos \alpha x \mathrm{~d} \alpha=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty}\left(\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) \cos \alpha u \mathrm{~d} u\right) \cos \alpha x \mathrm{~d} \alpha$ In this integration representation, we set

$$
\begin{equation*}
\hat{f}_{c}(\alpha)=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) \cos \alpha u \mathrm{~d} u \tag{29.1}
\end{equation*}
$$

and then

$$
\begin{equation*}
f(x)=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_{c}(\alpha) \cos \alpha x \mathrm{~d} \alpha \tag{29.2}
\end{equation*}
$$

The function $\hat{f}_{c}(\alpha)$ as given by (29.1) is known as the Fourier cosine transform of $f(x)$ in $0<x<\infty$. We shall denote Fourier cosine transform by $F_{c}(f)$. The function $f(x)$ as given by (29.2) is called inverse Fourier cosine transform of $\hat{f}_{c}(\alpha)$. It is denoted by $F_{c}^{-1}\left(\hat{f}_{c}\right)$.

Similarly we define Fourier sine and inverse Fourier sine transform by

$$
F_{s}(f)=\hat{f}_{s}(\alpha):=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) \sin \alpha u \mathrm{~d} u \text { and } F_{s}^{-1}(\hat{f})=f(x):=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_{s}(\alpha) \sin \alpha x \mathrm{~d} \alpha
$$

### 29.2 Properties

We mention here some important properties of Fourier cosine and sine transform that will be used in the application to solving differential equations.

1. Linearity: Let $f$ and $g$ are piecewise continuous and absolutely integrable functions. Then for constants $a$ and $b$ we have

$$
F_{c}(a f+b g)=a F_{c}(f)+b F_{c}(g) \quad \text { and } \quad F_{s}(a f+b g)=a F_{s}(f)+b F_{c}(g)
$$

Note that these properties are obvious and can be proved just using linearity of the integrals.
2. Transform of Derivatives: Let $f(x)$ be continuous and absolutely integrable on $x$-axis. Let $f^{\prime}(x)$ be piecewise continuous and on each finite interval on $[0, \infty]$ and $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Then,

$$
F_{c}\left\{f^{\prime}(x)\right\}=\alpha F_{s}\{f(x)\}-\sqrt{\frac{2}{\pi}} f(0) \text { and } F_{s}\left\{f^{\prime}(x)\right\}=-\alpha F_{c}\{f(x)\}
$$

Proof: By the definition of Fourier cosine transform we have

$$
F_{c}\left\{f^{\prime}(x)\right\}=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f^{\prime}(x) \cos \alpha x \mathrm{~d} x
$$

Integrating by parts we get

$$
F_{c}\left\{f^{\prime}(x)\right\}=\sqrt{\frac{2}{\pi}}\left[\left.(f(x) \cos \alpha x)\right|_{0} ^{\infty}+\alpha \int_{0}^{\infty} f(x) \sin \alpha x \mathrm{~d} x\right]
$$

Using the definition of Fourier sine integral we obtain

$$
F_{c}\left\{f^{\prime}(x)\right\}=-\sqrt{\frac{2}{\pi}} f(0)+\alpha F_{s}\{f(x)\}
$$

Similarly the other result for Fourier sine transform can be obtained.

Remark: The above results can easily be extended to the second order derivatives to have

$$
F_{c}\left\{f^{\prime \prime}(x)\right\}=-\alpha^{2} F_{c}\{f(x)\}-\sqrt{\frac{2}{\pi}} f^{\prime}(0) \text { and } F_{s}\left\{f^{\prime \prime}(x)\right\}=\sqrt{\frac{2}{\pi}} \alpha f(0)-\alpha^{2} F_{s}\{f(x)\}
$$

Note that here we have assumed continuity of $f$ and $f^{\prime}$ and piecewise continuity of $f^{\prime \prime}$. Further, we also assumed that $f$ and $f^{\prime}$ both goes to 0 as $x$ approaches to $\infty$.
3. Parseval's Identities: For Fourier sine and cosine transform we have the following identities
i) $\int_{0}^{\infty} \hat{f}_{c}(\alpha) \hat{g}_{c}(\alpha) \mathrm{d} \alpha=\int_{0}^{\infty} f(x) g(x) \mathrm{d} x$
ii) $\int_{0}^{\infty}\left[\hat{f}_{c}(\alpha)\right]^{2} \mathrm{~d} \alpha=\int_{0}^{\infty}[f(x)]^{2}$
iii) $\int_{0}^{\infty} \hat{f}_{s}(\alpha) \hat{g}_{s}(\alpha) \mathrm{d} \alpha=\int_{0}^{\infty} f(x) g(x) \mathrm{d} x$
iv) $\int_{0}^{\infty}\left[\hat{f}_{s}(\alpha)\right]^{2} \mathrm{~d} \alpha=\int_{0}^{\infty}[f(x)]^{2}$

Proof: We prove the first identity and rest can be proved similarly. We take the right hand side of the identity and use the definition of the inverse cosine transform to get

$$
\int_{0}^{\infty} f(x) g(x) \mathrm{d} x=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \int_{0}^{\infty} \hat{g}_{c}(\alpha) \cos (\alpha x) \mathrm{d} \alpha \mathrm{~d} x
$$

Changing the order of integration we obtain

$$
\int_{0}^{\infty} f(x) g(x) \mathrm{d} x=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{g}_{c}(\alpha) \int_{0}^{\infty} f(x) \cos (\alpha x) \mathrm{d} x \mathrm{~d} \alpha=\int_{0}^{\infty} \hat{f}_{c}(\alpha) \hat{g}_{c}(\alpha) \mathrm{d} \alpha
$$

### 29.3 Example Problems

### 29.3.1 Problem 1

Find the Fourier sine transform of $e^{-x}, x>0$. Hence show that

$$
\int_{0}^{\infty} \frac{x \sin m x}{1+x^{2}} d x=\frac{\pi e^{-m}}{2}, m>0
$$

Solution: Using the definition of Fourier sine transform

$$
F_{s}\left\{e^{x}\right\}=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-x} \sin \alpha x \mathrm{~d} x
$$

Let us denote the integral on the right hand side by $i$ and evaluate it by integrating by parts as

$$
I=\int_{0}^{\infty} e^{-x} \sin \alpha x \mathbf{d} x=-\left.e^{-x} \sin \alpha x\right|_{0} ^{\infty}+\alpha \int_{0}^{\infty} e^{-x} \cos \alpha x \mathbf{d} x=\alpha \int_{0}^{\infty} e^{-x} \cos \alpha x \mathbf{d} x
$$

Again integrating by parts

$$
I=\alpha\left[-\left.e^{-x} \cos \alpha x\right|_{0} ^{\infty}-\alpha \int_{0}^{\infty} e^{-x} \sin \alpha x \mathrm{~d} x\right]=\alpha[1-\alpha I]
$$

This implies

$$
I=\frac{\alpha}{1+\alpha^{2}}
$$

Finally substituting the value of $I$ to the expression of Fourier sine transform above we get

$$
F_{s}\left\{e^{x}\right\}=\sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{1+\alpha^{2}}\right)
$$

Taking inverse Fourier transform

$$
e^{-x}=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_{s}(\alpha) \sin \alpha x \mathrm{~d} \alpha=\frac{2}{\pi} \int_{0}^{\infty} \frac{\alpha}{1+\alpha^{2}} \sin \alpha x \mathrm{~d} \alpha
$$

Changing $x$ to $m$ and $\alpha$ to $x$ we obtain

$$
\int_{0}^{\infty} \frac{x}{1+x^{2}} \sin (x m) \mathrm{d} x=\frac{\pi}{2} e^{-m}
$$

### 29.3.2 Problem 2

Find the Fourier cosine transform of $e^{-x^{2}}, x>0$.
Solution: By the definition of Fourier cosine transform we have

$$
F_{c}\left\{e^{-x^{2}}\right\}=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) \cos (\alpha u) \mathrm{d} u=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-u^{2}} \cos (\alpha u) \mathrm{d} u
$$

Let us denote the integral on the right hand side by $I$ and differentiate it with respect to $\alpha$ as

$$
\frac{d I}{d \alpha}=\frac{d}{d \alpha} \int_{0}^{\infty} e^{-u^{2}} \cos (\alpha u) \mathrm{d} u=-\int_{0}^{\infty} e^{-u^{2}} \sin (\alpha u) u \mathrm{~d} u
$$

Integrating by parts we get

$$
\frac{d I}{d \alpha}=\frac{1}{2}\left[e^{-u^{2}} \sin (\alpha u)-\alpha \int_{0}^{\infty} e^{-u^{2}} \cos (\alpha u) \mathrm{d} u\right]=-\frac{\alpha}{2} I
$$

This implies

$$
I=c e^{-\alpha^{2} / 4}
$$

Using $I(0)=\frac{\sqrt{\pi}}{2}$, we evaluate the constant $c=\frac{\sqrt{\pi}}{2}$. Then we have

$$
I=\frac{\sqrt{\pi}}{2} e^{-\alpha^{2} / 4}
$$

Therefore the desired Fourier cosine transform is given as

$$
F_{c}\left\{e^{-x^{2}}\right\}=\sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{2} e^{-\alpha^{2} / 4}=\frac{1}{\sqrt{2}} e^{-\alpha^{2} / 4}
$$

### 29.3.3 Problem 3

Using Parseval's identities, prove that
i) $\int_{0}^{\infty} \frac{d t}{\left(a^{2}+t^{2}\right)\left(b^{2}+t^{2}\right)}=\frac{\pi}{2 a b(a+b)}$
ii) $\int_{0}^{\infty} \frac{t^{2}}{\left(t^{2}+1\right)} d t=\frac{\pi}{4}$

Solution: $i$ ) For the first part let $f(x)=e^{-a x}$ and $g(x)=e^{-b x}$. It can easily be shown that

$$
\begin{aligned}
& F_{c}\{f\}=\hat{f}_{c}(\alpha)=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-a x} \cos \alpha x \mathrm{~d} x=\sqrt{\frac{2}{\pi}} \frac{a}{a^{2}+\alpha^{2}} \\
& F_{c}\{g\}=\hat{f}_{c}(\alpha)=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-b x} \cos \alpha x \mathrm{~d} x=\sqrt{\frac{2}{\pi}} \frac{b}{b^{2}+\alpha^{2}}
\end{aligned}
$$

Using Parseval's identity we get

$$
\int_{0}^{\infty} \hat{f}_{c}(\alpha) \hat{g}_{c}(\alpha) \mathrm{d} \alpha=\int_{0}^{\infty} f(x) g(x) \mathrm{d} x \Rightarrow \frac{2}{\pi} \int_{0}^{\infty} \frac{a}{a^{2}+\alpha^{2}} \frac{b}{b^{2}+\alpha^{2}} \mathrm{~d} \alpha=\int_{0}^{\infty} e^{-(a+b) x} \mathrm{~d} x
$$

This can be further simplified as

$$
\frac{2 a b}{\pi} \int_{0}^{\infty} \frac{\mathrm{d} \alpha}{\left(a^{2}+\alpha^{2}\right)\left(b^{2}+\alpha^{2}\right)} \mathrm{d} \alpha=-\left.\frac{e^{-(a+b) x}}{a+b}\right|_{0} ^{\infty}
$$

Thus we get

$$
\int_{0}^{\infty} \frac{\mathrm{d} \alpha}{\left(a^{2}+\alpha^{2}\right)\left(b^{2}+\alpha^{2}\right)}=\frac{\pi}{a b(a+b)}
$$

ii) For the second part we use Fourier sine transform of $e^{-x}$ as

$$
F_{s}\left\{e^{-x}\right\}=\hat{f}_{s}(\alpha)=\sqrt{\frac{2}{\pi}} \frac{\alpha}{1+\alpha^{2}}
$$

Using Parseval's identity we obtain

$$
\frac{2}{\pi} \int_{0}^{\infty} \frac{\alpha^{2}}{\left(1+\alpha^{2}\right)^{2}} \mathrm{~d} \alpha=\int_{0} \infty\left(e^{-x}\right)^{2} \mathrm{~d} x=\frac{1}{2}
$$

Hence we have the desired results

$$
\int_{0}^{\infty} \frac{\alpha^{2}}{\left(1+\alpha^{2}\right)^{2}} \mathrm{~d} \alpha=\frac{\pi}{2}
$$

## Suggested Readings

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## Lesson 30

## FOURIER TRANSFORM

In this lesson we describe Fourier transform. We shall connect Fourier series with the Fourier transform through Fourier integral. Several interesting properties of the Fourier Transform such as linearity, shifting, scaling etc. will be discussed.

### 30.1 Fourier Transform

Consider the Fourier integral defined in earlier lessons as

$$
f(x)=\frac{1}{\pi} \int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(u) \cos \alpha(u-x) d u\right] d \alpha
$$

Since the inner integral is an even function of $\alpha$ we have

$$
\begin{equation*}
f(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u) \cos \alpha(u-x) d u\right] d \alpha \tag{30.1}
\end{equation*}
$$

Further note that

$$
\begin{equation*}
0=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \sin \alpha(u-x) d u d \alpha \tag{30.2}
\end{equation*}
$$

as the integral

$$
\int_{-\infty}^{\infty} f(u) \sin \alpha(u-x) d u d \alpha
$$

is an odd function of $\alpha$. Multiplying the equation (30.2) by the imaginary unit $i$ and adding to the equation (30.1), we obtain

$$
\begin{equation*}
f(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i \alpha(u-x)} d u d \alpha \tag{30.3}
\end{equation*}
$$

This is the complex Fourier integral representation of $f$ on the real line. Now we split the exponential integrands and the pre-factor $1 /(2 \pi)$ as

$$
\begin{equation*}
f(x)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(u) e^{i \alpha u} d u\right] e^{-i \alpha x} d \alpha \tag{30.4}
\end{equation*}
$$

The term in the parentheses is what we will the Fourier transform of $f$. Thus the Fourier transform of $f$, denoted by $\hat{f}(\alpha)$, is defined as

$$
\hat{f}(\alpha)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(u) e^{i \alpha u} d u
$$

Now the equation (30.4) can be written as

$$
\begin{equation*}
f(x)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i \alpha x} d \alpha \tag{30.5}
\end{equation*}
$$

The function $f(x)$ in equation (30.5) is called the inverse Fourier transform of $\hat{f}(\alpha)$. We shall use $F$ for Fourier transformation and $F^{-1}$ for inverse Fourier transformation in this lesson.

Remark: It should be noted that there are a number of alternative forms for the Fourier transform. Different forms deals with a different pre-factor and power of exponential. For example we can also define Fourier and inverse Fourier transform in the following manner.

$$
f(x)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i \alpha x} d \alpha \text { where } \hat{f}(\alpha)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(u) e^{-i \alpha u} d u
$$

or

$$
f(x)=\int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i \alpha x} d \alpha \text { where } \hat{f}(\alpha)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} f(u) e^{-i \alpha u} d u
$$

or

$$
f(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i \alpha x} d \alpha \text { where } \hat{f}(\alpha)=\int_{-\infty}^{\infty} f(u) e^{-i \alpha u} d u
$$

We shall remain with our original form because it is easy to remember because of the same pre-factor in front of both forward and inverse transforms.

### 30.2 Properties

We now list a number of properties of the Fourier transform that are useful in their manipulation.

1. Linearity: Let $f$ and $g$ are piecewise continuous and absolutely integrable functions. Then for constants $a$ and $b$ we have

$$
F(a f+b g)=a F(f)+b F(g)
$$

Proof: Similar to the Fourier sine and cosine transform this property is obvious and can be proved just using linearity of the Fourier integral.
2. Change of Scale Property: If $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$ then

$$
F[f(a x)]=\frac{1}{|a|} \hat{f}\left(\frac{\alpha}{a}\right), \quad a \neq 0
$$

Proof: By the definition of Fourier transform we get

$$
F[f(a x)]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(a x) e^{i \alpha x} d x
$$

Substituting $a x=t$ so that $a d x=d t$, we have

$$
F[f(a x)]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(t) e^{i \alpha \frac{t}{a}} \frac{d t}{a}=\frac{1}{|a|} \hat{f}\left(\frac{\alpha}{a}\right)
$$

3. Shifting Property: If $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$ then

$$
F[f(x-a)]=e^{i \alpha a} F[f(x)]
$$

Proof: By definition, we have

$$
\begin{aligned}
F[f(x-a)] & =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(x-a) e^{i \alpha x} d x \\
& =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(t) e^{i \alpha(t+a)} d t=e^{i \alpha a} \hat{f}(\alpha)
\end{aligned}
$$

3. Duality Property: If $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$ then

$$
F[\hat{f}(x)]=f(-\alpha)
$$

Proof: By definition of the inverse Fourier transform, we have

$$
f(x)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i \alpha x} d \alpha
$$

Renaming $x$ to $\alpha$ and $\alpha$ to $x$, we have

$$
f(\alpha)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{-i \alpha x} d x
$$

Replacing $\alpha$ to $-\alpha$, we obtain

$$
f(-\alpha)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{i \alpha x} d x=F[\hat{f}(x)]
$$

Now we evaluate Fourier transform of some simple functions.

### 30.3 Example Problems

### 30.3.1 Problem 1

Find the Fourier transform of the following function

$$
X_{[-a, a]}(x)= \begin{cases}1, & |x|<a  \tag{30.6}\\ 0, & |x|>a\end{cases}
$$

Solution: By the definition of Fourier transform, we have

$$
F\left[X_{[-a, a]}(x)\right]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} X_{[-a, a]}(x) e^{i \alpha x} d x
$$

Using the given value of given function we get

$$
\begin{aligned}
F\left[X_{[-a, a]}(x)\right] & =\frac{1}{\sqrt{2 \pi}} \int_{-a}^{a} e^{i \alpha x} d x=\frac{1}{\sqrt{2 \pi}} \frac{1}{i \alpha}\left(e^{i \alpha a}-e^{-i \alpha a}\right) \\
& =\frac{2}{\sqrt{2 \pi}}\left(\frac{e^{i \alpha a}-e^{-i \alpha a}}{2 i \alpha}\right)=\frac{2}{\sqrt{2 \pi}}\left(\frac{\sin (\alpha a)}{\alpha}\right) .
\end{aligned}
$$

### 30.3.2 Problem 2

Find the Fourier transform of $e^{-a x^{2}}$.
Solution: Using the definition of the Fourier Transform

$$
F\left(e^{-a x^{2}}\right)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-a x^{2}} e^{i \alpha x} d x
$$

Further simplifications leads to

$$
\begin{aligned}
F\left[e^{-a x^{2}}\right] & =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{\left[-a\left(x-\frac{i \alpha}{2 a}\right)^{2}-\frac{\alpha^{2}}{4 a}\right]} d x \\
& =\frac{1}{\sqrt{2 \pi}} e^{-\frac{\alpha^{2}}{4 a}} \int_{-\infty}^{\infty} e^{-a y^{2}} d y=\frac{1}{\sqrt{2 a}} e^{-\frac{\alpha^{2}}{4 a}}
\end{aligned}
$$

If $a=1 / 2$ then $F\left[e^{-\frac{1}{2} x^{2}}\right]=e^{-\frac{\alpha^{2}}{2}}$. This shows $F[f(x)]=f(\alpha)$ such function is said to be self-reciprocal under the Fourier transformation.

### 30.3.3 Problem 3

Find the inverse Fourier transform of $\hat{f}(\alpha)=e^{-|\alpha| y}$, where $y \in(0, \infty)$.
Solution: By the definition of inverse Fourier transform

$$
\begin{aligned}
F^{-1}[\hat{f}(\alpha)] & =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i \alpha x} d \alpha=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-|\alpha| y} e^{-i \alpha x} d \alpha \\
& =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{0} e^{\alpha y} e^{-i \alpha x} d \alpha+\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty} e^{-\alpha y} e^{-i \alpha x} d \alpha
\end{aligned}
$$

Combining the two exponentials in the integrands

$$
F^{-1}[\hat{f}(\alpha)]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{0} e^{(y-i x) \alpha} d \alpha+\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty} e^{-(y+i x) \alpha} d \alpha
$$

Now we can integrate the aboye two integrals to get

$$
F^{-1}[\hat{f}(\alpha)]=\frac{1}{\sqrt{2 \pi}}\left[\frac{e^{(y-i x) \alpha}}{(y-i x)}\right]_{-\infty}^{0}+\frac{1}{\sqrt{2 \pi}}\left[\frac{e^{-(y+i x) \alpha}}{-(y+i x)}\right]_{0}^{\infty}
$$

Noting $\lim _{\alpha \rightarrow-\infty} e^{(y-i x) \alpha}=0$ and $\lim _{\alpha \rightarrow \infty} e^{-(y+i x) \alpha}=0$, we obtain

$$
F^{-1}[\hat{f}(\alpha)]=\frac{1}{\sqrt{2 \pi}} \frac{1}{y-i x}+\frac{1}{\sqrt{2 \pi}} \frac{1}{y+i x}
$$

This can be further simplified to give

$$
F^{-1}[\hat{f}(\alpha)]=\frac{1}{\sqrt{2 \pi}} \frac{y+i x+y-i x}{(y-i x)(y+i x)}
$$

Hence we get

$$
f(x)=\sqrt{\frac{2}{\pi}} \frac{y}{\left(x^{2}+y^{2}\right)}
$$

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## Lesson 31

## Fourier Transform (Cont.)

In this lesson we continues further on Fourier transform. Here we discuss some more properties of the Fourier transform and evolution of Fourier transform of some special functions. Some applications of Parseval's identity and convolution property will be demonstrated.

### 31.1 Fourier Transforms of Derivatives

### 31.1.1 Theorem

If $f(x)$ is continuously differential and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, then

$$
F\left[f^{\prime}(x)\right]=(-i \alpha) F[f(x)]=(-i \alpha) \hat{f}(\alpha)
$$

Proof: By the definition of Fourier transform we have

$$
F\left[f^{\prime}(x)\right]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f^{\prime}(x) e^{i \alpha x} d x
$$

Integrating by parts we obtain

$$
F\left[f^{\prime}(x)\right]=\frac{1}{\sqrt{2 \pi}}\left\{\left[f(x) e^{i \alpha x}\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty} f(x) e^{i \alpha x}(i \alpha) d x\right\} .
$$

Since $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, we get

$$
F\left[f^{\prime}(x)\right]=-i \alpha \hat{f}(\alpha) .
$$

This proves the result.
Note that the above result can be generalized. If $f(x)$ is continuously $n$-times differentiable and $f^{k}(x) \rightarrow 0$ as $|x| \rightarrow \infty$ for $k=1,2, \ldots, n-1$, then the Fourier transform of $n$th derivative is

$$
F\left[f^{n}(x)\right]=(-i \alpha)^{n} \hat{f}(\alpha) .
$$

### 31.2 Convolution for Fourier Transforms

### 31.2.1 Theorem

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is $\sqrt{2 \pi}$ times the product of the Fourier transforms of $f(x)$ and $g(x)$, i.e.,

$$
F[f * g]=\sqrt{2 \pi} F(f) F(g) .
$$

Proof: By definition, we have

$$
F[f * g]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y) g(x-y) \mathrm{d} y\right) e^{i \alpha x} \mathrm{~d} x
$$

Changing the order of integration we obtain

$$
F[f * g]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(x-y) e^{i \alpha x} \mathrm{~d} x \mathrm{~d} y
$$

By substituting $x-y=t \Rightarrow d x=d t$ we get

$$
F[f * g]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(t) e^{i \alpha(y+t)} \mathrm{d} t \mathrm{~d} y
$$

Splitting the integrals we get

$$
F[f * g]=\sqrt{2 \pi}\left(\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(y) e^{i \alpha y} \mathrm{~d} y\right)\left(\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} g(t) e^{i \alpha t} \mathrm{~d} t\right)
$$

Finally we have the following result

$$
F[f * g]=\sqrt{2 \pi} F[f] F[g]=\sqrt{2 \pi} \hat{f}(\alpha) \hat{g}(\alpha)
$$

This proves the result.
The above result is sometimes written by taking the inverse transform on both the sides as

$$
(f * g)(x)=\int_{-\infty}^{\infty} \hat{f}(\alpha) \hat{g}(\alpha) e^{-i \alpha x} \mathrm{~d} \alpha
$$

or

$$
\int_{-\infty}^{\infty} f(y) g(x-y) \mathrm{d} y=\int_{-\infty}^{\infty} \hat{f}(\alpha) \hat{g}(\alpha) e^{-i \alpha x} \mathrm{~d} \alpha
$$

### 31.3 Perseval's Identity for Fourier Transforms

### 31.3.1 Theorem

If $\hat{f}(\alpha)$ and $\hat{g}(\alpha)$ are the Fourier transforms of the $f(x)$ and $g(x)$ respectively, then

$$
\text { (i) } \int_{-\infty}^{\infty} \hat{f}(\alpha) \overline{\hat{g}(\alpha)} \mathrm{d} \alpha=\int_{-\infty}^{\infty} f(x) \overline{g(x)} \mathrm{d} x \quad \text { (ii) } \int_{-\infty}^{\infty}|\hat{f}(\alpha)|^{2} \mathrm{~d} \alpha=\int_{-\infty}^{\infty}|f(\alpha)|^{2} \mathrm{~d} \alpha
$$

Proof: (i) Use of the inversion formula for Fourier transform gives

$$
\int_{-\infty}^{\infty} f(x) \overline{g(x)} \mathrm{d} x=\int_{-\infty}^{\infty} f(x)\left(\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \overline{\hat{g}(\alpha)} e^{i \alpha x} \mathrm{~d} \alpha\right) \mathrm{d} x
$$

Changing the order of integration we have

$$
\int_{-\infty}^{\infty} f(x) \overline{g(x)} \mathrm{d} x=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \overline{\hat{g}(\alpha)} e^{i \alpha x} \mathrm{~d} x \mathrm{~d} \alpha
$$

Using the definition of Fourier transform we get

$$
\int_{-\infty}^{\infty} f(x) \overline{g(x)} \mathrm{d} x=\int_{-\infty}^{\infty} \overline{\hat{g}(\alpha)} \hat{f}(\alpha) \mathrm{d} \alpha
$$

(ii) Taking $f(x)=g(x)$ we get,

$$
\int_{-\infty}^{\infty} \hat{f}(\alpha) \hat{f}(\alpha) \mathrm{d} \alpha=\int_{-\infty}^{\infty} f(x) \overline{f(x)} \mathrm{d} x
$$

This implies

$$
\int_{-\infty}^{\infty}|f(x)|^{2} \mathrm{~d} x=\int_{-\infty}^{\infty}|\hat{f(\alpha)}|^{2} \mathrm{~d} \alpha
$$

### 31.4 Example Problems

### 31.4.1 Problem 1

Find the Fourier transform of $f(x)$ defined by

$$
f(x)=\left\{\begin{array}{l}
1, \text { when }|x|<a \\
0, \text { when }|x|>a
\end{array}\right.
$$

and hence evaluate

$$
\text { (i) } \int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} \mathrm{~d} \alpha, \quad \text { (ii) } \int_{0}^{\infty} \frac{\sin \alpha a}{\alpha} \mathrm{~d} \alpha \text { and (iii) } \int_{0}^{\infty} \frac{\sin ^{2} x}{x^{2}} \mathrm{~d} x
$$

Solution: (i) Let $\hat{f}(\alpha)$ be the Fourier transform of $f(x)$. Then, by the definition of Fourier transform

$$
\begin{aligned}
\hat{f}(\alpha) & =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{i \alpha x} f(x) d x=\frac{1}{\sqrt{2 \pi}} \int_{-a}^{a} e^{i \alpha x} d x \\
& =\frac{1}{\sqrt{2 \pi}} \frac{1}{i \alpha}\left(e^{i \alpha a}-e^{-i \alpha a}\right) d x
\end{aligned}
$$

This gives

$$
\hat{f}(\alpha)=\frac{2}{\sqrt{2 \pi}} \frac{\sin a \alpha}{\alpha}
$$

From the definition of inverse Fourier transform we also know that

$$
f(x)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i \alpha x} \mathrm{~d} \alpha
$$

This implies that

$$
\int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i \alpha x} \mathrm{~d} \alpha=\sqrt{2 \pi} f(x)= \begin{cases}\sqrt{2 \pi}, & \text { when }|x|<a \\ 0, & \text { when }|x|>a\end{cases}
$$

Substituting $\hat{f}(\alpha)$ in the above equation we get

$$
\int_{-\infty}^{\infty} \frac{2}{\sqrt{2 \pi}} \frac{\sin a \alpha}{\alpha}(\cos \alpha x-i \sin \alpha x) \mathrm{d} \alpha= \begin{cases}\sqrt{2 \pi}, & \text { when }|x|<a \\ 0, & \text { when }|x|>a\end{cases}
$$

We now split the left hand side into real and imaginary parts to get

$$
\int_{-\infty}^{\infty} \frac{\sin a \alpha \cos x \alpha}{\alpha} \mathrm{~d} \alpha-i \int_{-\infty}^{\infty} \frac{\sin \alpha a \sin \alpha x}{\alpha} \mathrm{~d} \alpha= \begin{cases}\pi, & \text { when }|x|<a \\ 0, & \text { when }|x|>a\end{cases}
$$

Equating real part on both sides we get the desired result as

$$
\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} \mathrm{~d} \alpha=\left\{\begin{array}{l}
\pi, \text { when }|x|<a \\
0, \text { when }|x|>a
\end{array}\right.
$$

(ii) If we set $x=0$ and $a=1$ in the above results, we get

$$
\int_{-\infty}^{\infty} \frac{\sin \alpha}{\alpha} \mathrm{d} \alpha=\pi, \quad \text { Since }|x|<a
$$

Since the integrand is an even function, we get the the desired results

$$
\int_{0}^{\infty} \frac{\sin \alpha}{\alpha}=\frac{\pi}{2}
$$

(ii) We now apply Parseval's identity for Fourier transform

$$
\int_{-\infty}^{\infty}|\hat{f}(\alpha)|^{2} \mathrm{~d} \alpha=\int_{-\infty}^{\infty}|f(\alpha)|^{2} \mathrm{~d} \alpha
$$

Substituting the function $f(x)$ and its Fourier transform we get

$$
\int_{-\infty}^{\infty} \frac{4}{2 \pi} \frac{\sin ^{2} a \alpha}{\alpha^{2}} \mathrm{~d} \alpha=\int_{-a}^{a} \mathrm{~d} \alpha=2 a
$$

This implies

$$
\int_{-\infty}^{\infty} \frac{\sin ^{2} a \alpha}{\alpha^{2}} \mathrm{~d} \alpha=\pi a
$$

Since the integrand is an even function we have the desired result as

$$
\int_{0}^{\infty} \frac{\sin ^{2} a \alpha}{\alpha^{2}} \mathrm{~d} \alpha=\frac{\pi}{2} a
$$

### 31.4.2 Problem 2

Evaluate the Fourier transform of the rectangular pulse function

$$
\Pi(t)= \begin{cases}1, & \text { if }|t|<1 / 2 \\ 0, & \text { otherwise }\end{cases}
$$

Apply the convolution theorem to evaluate the Fourier transform of the triangular pulse function

$$
\Lambda(t)= \begin{cases}1-|t|, & \text { if }|t|<1 \\ 0, & \text { otherwise }\end{cases}
$$

Solution: It is well known result that $\Lambda=\Pi * \Pi$. It can easily be sheen by observing

$$
(\Pi * \Pi)(t)=\int_{-\infty}^{\infty} \Pi(y) \Pi(t-y) d y= \begin{cases}\int_{-1 / 2}^{t+1 / 2} 1 \cdot 1 d y, & \text { if }-1<t<0 \\ \int_{t-1 / 2}^{1 / 2} 1 \cdot 1 d y, & \text { if } 0<t<1 \\ 0 & \text { otherwise }\end{cases}
$$

Clearly, we have

$$
(\Pi * \Pi)(t)=\int_{-\infty}^{\infty} \Pi(y) \Pi(t-y) d y=\left\{\begin{array}{ll}
1+t, & \text { if }-1<t<0 \\
1-t, & \text { if } 0<t<1 \\
0 & \text { otherwise }
\end{array} \quad=\Lambda(t)\right.
$$

Using $a=1 / 2$ in the previous example we have

$$
F(\Pi)=\frac{2}{\sqrt{2 \pi}} \frac{\sin (\alpha / 2)}{\alpha}
$$

Now using convolution result we get

$$
F[\Lambda(t)]=F[(\Pi * \Pi)(t)]=\sqrt{2 \pi} F(\Pi) F(\Pi)=\frac{4}{\sqrt{2 \pi}} \frac{\sin ^{2}(\alpha / 2)}{\alpha^{2}}
$$

## Suggested Readings

Debnath, L. and Bhatta, D. (2007). Integral Transforms and Their Applications. Second Edition. Chapman and Hall/CRC (Taylor and Francis Group). New York.

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## Lesson 32

## Fourier Transform (Cont.)

In this lesson we provide some miscellaneous examples of Fourier transforms. One of the major applications of Fourier transforms for solving partial differential equations will not be discussed in this module. However, we shall highlights some other applications like evaluating special integrals and the idea of solving ordinary differential equations.

### 32.1 Example Problems

### 32.1.1 Problem 1

Find the Fourier transform of

$$
f(x)= \begin{cases}1-x^{2}, & \text { when }|x|<1 \\ 0, & \text { when }|x|>1\end{cases}
$$

and hence evaluate

$$
\int_{0}^{\infty} \frac{-x \cos x+\sin x}{x^{3}} \cos \frac{x}{2} \mathrm{~d} x .
$$

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Solution: Using the definition of Fourier transform we get

$$
\begin{aligned}
\hat{f}(\alpha)=F[f(x)] & =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{i \alpha x} f(x) \mathrm{d} x \\
& =\frac{1}{\sqrt{2 \pi}} \int_{-1}^{1} e^{i \alpha x}\left(1-x^{2}\right) \mathrm{d} x
\end{aligned}
$$

Integrating by parts we obtain

$$
\hat{f}(\alpha)=\left.\frac{1}{\sqrt{2 \pi}} \frac{e^{i \alpha x}}{i \alpha}\left(1-x^{2}\right)\right|_{-1} ^{1}-\int_{-1}^{1} \frac{e^{i \alpha x}}{i \alpha}(-2 x) \mathrm{d} x
$$

Again, the application of integration by parts gives

$$
\hat{f}(\alpha)=\frac{2}{\sqrt{2 \pi}}\left[\left.\frac{e^{i \alpha x}}{(i \alpha)^{2}} x\right|_{-1} ^{1}-\int_{-1}^{1} \frac{e^{i \alpha x}}{(i \alpha)^{2}} \mathrm{~d} x\right]
$$

## Fourier Transform (Cont.)

Further simplifications leads to

$$
\begin{aligned}
\hat{f}(\alpha) & =\frac{2}{\sqrt{2 \pi}}\left[-\frac{1}{\alpha^{2}}\left(e^{i \alpha}+e^{-i \alpha}-\left.\frac{e^{i \alpha x}}{i \alpha}\right|_{-1} ^{1}\right)\right] \\
& =-\frac{1}{\sqrt{2 \pi}} \frac{2}{\alpha^{2}}\left[e^{i \alpha}+e^{-i \alpha}-\frac{e^{i \alpha}}{i \alpha}+\frac{e^{-i \alpha}}{i \alpha}\right]
\end{aligned}
$$

Using Euler's equality we obtain

$$
\begin{aligned}
\hat{f}(\alpha) & =-\frac{1}{\sqrt{2 \pi}} \frac{4}{\alpha^{2}}\left[\cos \alpha-\frac{\sin \alpha}{\alpha}\right] \\
& =\frac{1}{\sqrt{2 \pi}} \frac{4}{\alpha^{3}}[-\alpha \cos \alpha+\sin \alpha]
\end{aligned}
$$

We know from the Fourier inversion formula that

$$
f(x)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i \alpha x} \mathrm{~d} \alpha
$$

This implies

$$
f(x)=\frac{4}{2 \pi} \int_{-\infty}^{\infty} \frac{-\alpha \cos \alpha+\sin \alpha}{\alpha^{3}} e^{-i \alpha x} \mathrm{~d} \alpha
$$

Equating real parts, on both sides we get

$$
\int_{-\infty}^{\infty} \frac{-\alpha \cos \alpha+\sin \alpha}{\alpha^{3}} \cos \alpha x \mathrm{~d} \alpha=\frac{\pi}{2} f(x)
$$

Substituting the value of the function we obtain

$$
\int_{-\infty}^{\infty} \frac{-\alpha \cos \alpha+\sin \alpha}{\alpha^{3}} \cos \alpha x \mathrm{~d} \alpha=\left\{\begin{array}{l}
\frac{\pi}{2}\left(1-x^{2}\right), \text { when }|x|<1 \\
0, \text { when }|x|>1
\end{array}\right.
$$

Substitution $x=1 / 2$ gives

$$
\int_{-\infty}^{\infty} \frac{-\alpha \cos \alpha+\sin \alpha}{\alpha^{3}} \cos \frac{\alpha}{2} \mathrm{~d} \alpha=\frac{\pi}{2}\left(1-\frac{1}{4}\right)
$$

This implies

$$
2 \int_{0}^{\infty} \frac{-\alpha \cos \alpha+\sin \alpha}{\alpha^{3}} \cos \frac{\alpha}{2} \mathrm{~d} \alpha=\frac{3 \pi}{8}
$$

Hence we get the desired result as

$$
\int_{0}^{\infty} \frac{-\alpha \cos \alpha+\sin \alpha}{\alpha^{3}} \cos \frac{\alpha}{2} \mathrm{~d} \alpha=\frac{3 \pi}{16}
$$

### 32.1.2 Problem 2

Find the Fourier transformation of the function $f(t)=e^{-a t} H(t)$, where

$$
H(t)=\left\{\begin{array}{l}
0, \text { when } t<0 \\
1, \text { when } t \geq 0
\end{array}\right.
$$

Solution: Using the definition of Fourier transform

$$
\begin{aligned}
F[f(t)] & =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(t) e^{i \alpha t} \mathrm{~d} t \\
& =\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty} e^{-a t} e^{i \alpha t} \mathrm{~d} t
\end{aligned}
$$

Solving integral leads to

$$
F[f(t)]=\left.\frac{1}{\sqrt{2 \pi}} \frac{e^{(-a+i \alpha) t}}{(-a+i \alpha)}\right|_{0} ^{\infty}
$$

Since we know that

$$
\lim _{t \rightarrow \infty} e^{-a t} e^{i \alpha t}=\lim _{t \rightarrow \infty} e^{-a t}(\cos \alpha t+i \sin \alpha t)=0
$$

We get the required transform as

$$
F[f(t)]=-\frac{1}{\sqrt{2 \pi}} \frac{1}{(-a+i \alpha)}=\frac{1}{\sqrt{2 \pi}}\left(\frac{1}{a-i \alpha}\right) .
$$

### 32.1.3 Problem 3

Find the Fourier transform of Dirac-Delta function $\delta(t-a)$.
Solution: Recall that the Dirac-Delta function can be thought as

$$
\delta(t-a)=\lim _{\epsilon \rightarrow 0} \delta_{\epsilon}(t-a)=\left\{\begin{array}{l}
0, \text { when } t<a, \quad a>0 \\
\frac{1}{\epsilon}, \text { when } a \leq t \leq a+\epsilon \\
0, \text { when } t>a+\epsilon
\end{array}\right.
$$

Applying the definition of Fourier transform we get

$$
\begin{aligned}
F[\delta(t-a)] & =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \delta(t-a) e^{i \alpha t} \mathrm{~d} t \\
& =\frac{1}{\sqrt{2 \pi}} \int_{a}^{a+\epsilon} \lim _{\epsilon \rightarrow 0} \frac{1}{\epsilon} e^{i \alpha t} \mathrm{~d} t
\end{aligned}
$$

On integrating we obtain

$$
\begin{aligned}
F[\delta(t-a)] & =\left.\lim _{\epsilon \rightarrow 0} \frac{1}{\sqrt{2 \pi}} \frac{1}{\epsilon} \frac{e^{i \alpha t}}{i \alpha}\right|_{a} ^{a+\epsilon} \\
& =\lim _{\epsilon \rightarrow 0} \frac{1}{\sqrt{2 \pi}} \frac{1}{\epsilon} \frac{1}{i \alpha}\left(e^{i \alpha(a+\epsilon)}-e^{i \alpha a}\right) \\
& =\frac{1}{\sqrt{2 \pi}} e^{i \alpha a} \lim _{\epsilon \rightarrow 0} \frac{e^{i \alpha \epsilon}-1}{i \alpha \epsilon}=\frac{1}{\sqrt{2 \pi}} e^{i \alpha a}
\end{aligned}
$$

With this results we deduce that $F^{-1}(1)=\sqrt{2 \pi} \delta(t)$.

### 32.1.4 Problem 4

Find the Fourier transform of

$$
f(t)=e^{-a|t|}, \quad-\infty<t<\infty, a>0
$$

Solution: Using the definition of Fourier transform we have

$$
\begin{aligned}
F\left[e^{-a|t|}\right] & =\frac{1}{\sqrt{2 \pi}}\left[\int_{-\infty}^{0} e^{a t} e^{i \alpha t} \mathrm{~d} t+\int_{0}^{\infty} e^{-a t} e^{i \alpha t} \mathrm{~d} t\right] \\
& =\frac{1}{\sqrt{2 \pi}}\left[\left.\frac{e^{(a+i \alpha) t}}{a+i \alpha}\right|_{-\infty} ^{0}+\left.\frac{e^{(-a+i \alpha) t}}{-a+i \alpha}\right|_{0} ^{\infty}\right] \\
& =\frac{1}{\sqrt{2 \pi}}\left[\frac{1}{a+i \alpha}+(-1) \frac{1}{-a+i \alpha}\right] \\
& =\frac{1}{\sqrt{2 \pi}}\left[\frac{1}{a+i \alpha}+\frac{1}{a-i \alpha}\right]=\frac{1}{\sqrt{2 \pi}} \frac{2 a}{a^{2}+\alpha^{2}}
\end{aligned}
$$

### 32.1.5 Problem 5

Find the inverse Fourier transform of $\hat{f}(\alpha)=\frac{1}{2 \pi(a-i \alpha)^{2}}$.
Solution: Writing the given function as a product of two functions as

$$
F^{-1}[\hat{f}(\alpha)]=F^{-1}\left[\frac{1}{\sqrt{2 \pi}(a-i \alpha)} \frac{1}{\sqrt{2 \pi}(a-i \alpha)}\right]
$$

Application of convolution theorem gives

$$
f(t)=\frac{1}{\sqrt{2 \pi}} F^{-1}\left[\frac{1}{\sqrt{2 \pi}(a-i \alpha)}\right] * F^{-1}\left[\frac{1}{\sqrt{2 \pi}(a-i \alpha)}\right]=\frac{1}{\sqrt{2 \pi}}\left[e^{-a t} H(t) * e^{-a t} H(t)\right]
$$

Evaluating the convolution

$$
f(t)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-a x} H(x) e^{-a(t-x)} H(t-x) d x=\frac{e^{-a t}}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} H(x) H(t-x) d x
$$

Note that $H(x) H(t-x)=0$ when $x<0$ or when $t-x<0$, i.e.,

$$
H(x) H(t-x)= \begin{cases}1, & \text { if } 0<x<t \\ 0, & \text { otherwise }\end{cases}
$$

Hence we have

$$
f(t)=\frac{e^{a t}}{\sqrt{2 \pi}} \int_{0}^{t} d x= \begin{cases}\frac{t e^{-a t}}{\sqrt{2 \pi}}, & \text { if } t>0 \\ 0, & \text { if } t<0\end{cases}
$$

Thus we get

$$
f(t)=\frac{t e^{-a t}}{\sqrt{2 \pi}} H(t)
$$

### 32.1.6 Problem 6

Using Fourier transform, find the solution of the differential equation

$$
y^{\prime}-2 y=H(t) e^{-2 t}, \quad-\infty<t<\infty
$$

Solution: Taking Fourier transform on both sides we get

$$
F\left[y^{\prime}\right]-2 F[y]=\frac{1}{\sqrt{2 \pi}}\left(\frac{1}{-2+i \alpha}\right)
$$

Aplying the property of Fourier transform of derivatives we get

$$
-i \alpha \hat{y}-2 \hat{y}=-\frac{1}{\sqrt{2 \pi}}\left(\frac{1}{-2+i \alpha}\right)
$$

Simple algebraic calculation gives the value of transformed variable as

$$
\hat{y}=-\frac{1}{\sqrt{2 \pi}} \frac{1}{4+\alpha^{2}}
$$

Taking inverse Fourier transform we get the desired solution as $y=-\frac{1}{4} e^{-2|t|}$.

## Suggested Readings

Debnath, L. and Bhatta, D. (2007). Integral Transforms and Their Applications. Second Edition. Chapman and Hall/CRC (Taylor and Francis Group). New York.

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## Lesson 33

## Finite Fourier Transform

The Fourier transform, cosine transform and sine transform are all motivated by the respective integral representations of a function. Applying the same line of reasoning, but using Fourier cosine and sine series instead of integrals, we obtain the so called finite transforms. It has applications in solving partial differential equations in finite domain.

### 32.1 Finite Fourier Transformations

Let $f(x)$ be defined in $(0, L)$ and satisfies Dirichlet's conditions in that finite domain. We begin with the cosine Fourier series

$$
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty} a_{n} \cos \frac{n \pi x}{L},
$$

where the Fourier coefficients are given by

$$
a_{n}=\frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n \pi x}{L} \mathrm{~d} x, n=1,2, \ldots
$$

The sine Fourier series is given as

$$
f(x)=\sum_{n=1}^{\infty} b_{n} \sin \frac{n \pi x}{L}
$$

where

$$
b_{n}=\frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n \pi x}{L} \mathrm{~d} x, \quad n=1,2, \ldots
$$

We now define the finite Fourier cosine transform as

$$
F_{c}[f]=\hat{f}_{c}(n)=\int_{0}^{L} f(x) \cos \frac{n \pi x}{L} \mathrm{~d} x
$$

The function $f(x)$ is called inverse finite Fourier cosine transform and is given by

$$
F_{c}^{-1}\left[\hat{f}_{c}(n)\right]=f(x)=\frac{1}{L} \hat{f}_{c}(0)+\frac{2}{L} \sum_{n=1}^{\infty} \hat{f}_{c}(n) \cos \frac{n \pi x}{L} \mathrm{~d} x
$$

The finite Fourier sine transform is

$$
F_{s}(f)=\hat{f}_{s}(n)=\int_{0}^{L} f(x) \sin \frac{n \pi x}{L} \mathrm{~d} x
$$

The inverse finite Fourier sine transform is given by

$$
f(x)=\frac{2}{L} \sum_{n=1}^{\infty} \hat{f}_{s}(n) \sin \frac{n \pi x}{L} \mathrm{~d} x
$$

Remark: $\quad$ The factor $\frac{2}{L}$ may be associated with either the transformation or the inverse of the transformation or the factor $\sqrt{\frac{2}{L}}$ may be associated with both the transform and the inverse.

### 32.2 Finite Fourier Transform (Complex form)

Similar to the finite Fourier sine and cosine transform we can also define finite Fourier transform from complex form of Fourier series as

$$
F[f]=\int_{-L}^{L} f(x) e^{\frac{-i n \pi x}{L}} \mathrm{~d} x=\hat{f}(n)
$$

The inverse finite Fourier transform is defined as

$$
f(x)=\frac{1}{2 L} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{\frac{i n \pi x}{L}}
$$

### 32.3 Derivatives of Finite Fourier Sine and Cosine Transforms

### 32.3.1 Theorem

Let $f(x)$ and $f^{\prime}(x)$ be continuous and $f^{\prime \prime}(x)$ be piecewise continuous on the interval $[0, l]$, then
(i) $F_{s}\left[f^{\prime}(x)\right]=-\left(\frac{n \pi}{L}\right) \hat{f}_{c}(n)$
(ii) $F_{s}\left[f^{\prime \prime}(x)\right]=-\left(\frac{n \pi}{L}\right)^{2} \hat{f}_{s}(n)+\left(\frac{n \pi}{L}\right)\left[f(0)+(-1)^{n+1} f(L)\right]$
(iii) $F_{c}\left[f^{\prime}(x)\right]=\left(\frac{n \pi}{L}\right) \hat{f}_{c}(n)+(-1)^{n} f(L)-f(0)$
(iv) $F_{c}\left[f^{\prime \prime}(x)\right]=-\left(\frac{n \pi}{L}\right)^{2} \hat{f}_{c}(n)+(-1)^{n} f^{\prime}(L)-f^{\prime}(0)$

Proof: (i) Using the definition of Fourier sine transform

$$
F_{s}\left[f^{\prime}(x)\right]=\int_{0}^{L} f^{\prime}(x) \sin \frac{n \pi x}{L} \mathrm{~d} x
$$

Integrating by parts, we get

$$
F_{s}\left[f^{\prime}(x)\right]=\left[\left.f(x) \sin \frac{n \pi x}{L}\right|_{0} ^{L}-\int_{0}^{L} f(x) \cos \frac{n \pi x}{L} \frac{n \pi}{L} \mathrm{~d} x\right]
$$

This implies

$$
F_{s}\left[f^{\prime}(x)\right]=-\left(\frac{n \pi}{L}\right) \hat{f}_{c}(n)
$$

(ii) By the definition of finite Fourier transform, we get

$$
F_{c}\left[f^{\prime}(x)\right]=\int_{0}^{L} f^{\prime}(x) \cos \frac{n \pi x}{L} \mathrm{~d} x
$$

Integrating by parts gives

$$
F_{c}\left[f^{\prime}(x)\right]=\left[\left.f(x) \cos \frac{n \pi x}{L}\right|_{0} ^{L}-\int_{0}^{L} f(x) \sin \frac{n \pi x}{L} \frac{n \pi}{L} \mathrm{~d} x\right]
$$

Thus we get

$$
F_{c}\left[f^{\prime}(x)\right]=(-1)^{n} f(L)-f(0)+\left(\frac{n \pi}{L}\right) \hat{f}_{c}(n)
$$

Repeated applications of these above two will give (ii) and (iv).

### 32.4 Example Problems

### 32.4.1 Problem 1

Find the finite Fourier sine and cosine transform of $f(x)=x^{2}$, if $0<x<4$.

Solution: Using the definition of Fourier sine transform

$$
F_{s}[f(x)]=\int_{0}^{4} f(x) \sin \frac{n \pi x}{4} d x=\int_{0}^{4} x^{2} \sin \frac{n \pi x}{4} d x \text { if } n=1,2,3 \ldots
$$

Integration by parts leads to

$$
\begin{aligned}
F_{s}[f(x)] & =\left[x^{2}\left\{-\frac{\cos (n \pi x) / 4}{n \pi / 4}\right\}\right]_{0}^{4}-\int_{0}^{4} 2 x \frac{\cos (n \pi x) / 4}{n \pi / 4} d x \\
& =-\frac{64 \cos n \pi}{n \pi}+\frac{8}{n \pi} \int_{0}^{4} x \cos \frac{n \pi x}{4} d x
\end{aligned}
$$

Evaluating the integral we get

$$
F_{s}[f(x)]=\frac{64(-1)^{n+1}}{n \pi}-\frac{32}{n^{2} \pi^{2}}\left[-\frac{\cos (n \pi x) / 4}{n \pi / 4}\right]_{0}^{4}=\frac{64(-1)^{n+1}}{n \pi}+\frac{128}{n^{3} \pi^{3}}\left[(-1)^{n}-1\right]
$$

We have used the fact that

$$
\cos (n \pi)=(-1)^{n}= \begin{cases}1 & \text { if } n \text { even } \\ -1 & \text { if } n \text { odd }\end{cases}
$$

Now, by the definition of Fourier cosine transform, we get

$$
F_{c}[f(x)]=\int_{0}^{4} f(x) \cos \frac{n \pi x}{4} d x=\int_{0}^{4} x^{2} \sin \frac{n \pi x}{4} d x \text { if } n=1,2,3 \ldots
$$

Proceeding as before we get

$$
\begin{aligned}
F_{s}[f(x)] & =\left[x^{2} \frac{\sin (n \pi x) / 4}{n \pi / 4}\right]_{0}^{4}-\int_{0}^{4} 2 x \frac{\sin (n \pi x) / 4}{n \pi / 4} d x \\
& =-\frac{8}{n \pi} \int_{0}^{4} x \sin \frac{n \pi x}{4} d x=\frac{128(-1)^{n}}{n^{2} \pi^{2}}
\end{aligned}
$$

If $n=0$, the $F_{s}[f(x)]=\int_{0}^{4} f(x) d x=\int_{0}^{4} x^{2} d x=\frac{64}{3}$.

### 32.4.2 Problem 2

Find the finite Fourier sine and cosine transform of the function

$$
f(t)=|t| \quad \text { for } \quad-1<t \leq 1,
$$

Solution: For $n \geq 1$, we note that $|t| \cos (n \pi t)$ is even and hence by the finite Fourier cosine transform we have

$$
\begin{aligned}
\hat{f}_{c}(n) & =\int_{-1}^{1} f(t) \cos (n \pi t) d t \\
& =2 \int_{0}^{1} t \cos (n \pi t) d t \\
& =2\left[\frac{t}{n \pi} \sin (n \pi t)\right]_{t=0}^{1}-2 \int_{0}^{1} \frac{1}{n \pi} \sin (n \pi t) d t \\
& =0+\frac{1}{n^{2} \pi^{2}}[\cos (n \pi t)]_{t=0}^{1}=\frac{2\left((-1)^{n}-1\right)}{n^{2} \pi^{2}}
\end{aligned}
$$

For $n=0$, we find

$$
\hat{f}_{c}(n)=\int_{-1}^{1}|t| d t=1
$$

Now, we notice that $|t| \sin (n \pi t)$ is odd and, therefore, finite Fourier sine transform is calculated as

$$
\hat{f}_{s}(n)=\int_{-1}^{1} f(t) \sin (n \pi t) d t=0
$$

### 32.4.3 Problem 3

Find the finite Fourier sine transform of the function

$$
f(x)= \begin{cases}x, \quad \text { if } 0 \leq x \leq \pi / 2 ; \\ \pi-x, & \text { if } \pi / 2 \leq x \leq \pi .\end{cases}
$$

Solution: By the definition of finite Fourier transform, we have

$$
\hat{f}_{s}(n)=\int_{0}^{\pi} f(x) \sin (n x) d x=\int_{0}^{\pi / 2} f(x) \sin (n x) d x+\int_{\pi / 2}^{\pi} f(x) \sin (n x) d x
$$

Using the given values of $f(x)$ we get

$$
\hat{f}_{s}(n)=\int_{0}^{\pi / 2} x \sin (n x) d x+\int_{\pi / 2}^{\pi}(\pi-x) \sin (n x) d x
$$

Integrating by parts leads to

$$
\hat{f}_{s}(n)=\left[x\left(-\frac{\cos n x}{n}\right)-\left(-\frac{\sin n x}{n^{2}}\right)\right]_{0}^{\pi / 2}+\left[(\pi-x)\left(-\frac{\cos n x}{n}\right)-\left(-\frac{\sin n x}{n^{2}}\right)\right]_{\pi / 2}^{\pi}
$$

Finally, we get the finite Fourier sine transform as

$$
\hat{f}_{s}(n)=\frac{2}{n^{2}} \sin \frac{n \pi}{2} .
$$

## Suggested Readings

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## Lesson 34

## Partial Differential Equations

### 34.1 Introduction of Partial Differential Equations

Many physical processes in real world are modelled by partial differential equations. Any equation that involves one or more terms with partial derivatives of the dependent variable is called a partial differential equation (p.d.e). For a function $z$ depending on two independent variables $x$ and $y$, i.e., $z(x, y)$, a partial differential equation may be written as:

$$
2 \frac{\partial^{2} z}{\partial x^{2}}+3 \frac{\partial z}{\partial x}+\frac{\partial^{2} z}{\partial y^{2}}+5 \frac{\partial z}{\partial y}=\sin (x+y)
$$

In general, a p.d.e may be written as:

$$
\begin{equation*}
f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^{2} z}{\partial x^{2}}, \frac{\partial^{2} z}{\partial y^{2}}, \frac{\partial^{2} z}{\partial x y}, \ldots . .\right)=0 \tag{34.1}
\end{equation*}
$$

The domain of the function $\mathrm{z}(x, y)$ is a subset of $\mathbb{R}^{2}$. It is to be noted that if the dependent function $z$ is depending on $n$ independent variables, say $z=z\left(x_{1}, x_{2}, x_{3}, \ldots, x_{n}\right)$, then the domain of $z$ will be a sub set of $\mathbb{R}^{n}$.

Evidently, for each point $(x, y)$ in $\Omega$ Subset of $\mathbb{R}^{2}$, there exists a value for $z(x, y)$, and this set of points $\{(x, y, z)\}$ generates a surface in $\mathbb{R}^{3} . z=z(x, y)$, is the solution of a p.d.e. In the same manner, one has to visualize higher dimensional surfaces as solutions of p.d.es involving 3 or more independent variables.

### 34.2 Basic Concepts

Order of the p.d.e: The highest order Partial derivative in the equation decides the order of the p.d.e. For example, $\left(\frac{\partial z}{\partial x}\right)^{3}+\frac{\partial z}{\partial y}=0$ is a first order p.d.e, while $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial z}{\partial y}=0$ is a second order p.d.e.

A set $\Omega$ in the $n$-dimensional Enclidean space $\mathbb{R}^{n}$ is called a domain if it is an open and connected set. A region is a set consisting of a domain plus some or all of its boundary points. For example, the interior of a circle $x^{2}+y^{2}<a^{2}$ with radius $a$ is called the domain while the circle with its boundary $x^{2}+y^{2} \leq a^{2}$ is called the region.

In general, the partial differential equation is assumed to take values from the interior of the region. The boundary and initial conditions are specified on its boundary.

By a solution of the partial differential equation (1) we mean a continuously differential function $\mathrm{z}=\mathrm{z}(\mathrm{x}, \mathrm{y})$ with respect to the independent variables $x$ and $y$ at all points of the domain and it satisfies the differential equation.

Observe that (i) $z(x, y)=(x+y)^{3}$ satisfies the p.d.e $\frac{\partial^{2} z}{\partial x^{2}}-\frac{\partial^{2} z}{\partial y^{2}}=0$.

The partial derivatives are $\frac{\partial z}{\partial x}=3(x+y)^{2}, \frac{\partial^{2} z}{\partial x^{2}}=6(x+y), \frac{\partial z}{\partial y}=3(x+y)^{2}$ and $\frac{\partial^{2} z}{\partial y^{2}}=6(x+y)$. Also observe (ii) $z(x, y)=\sin (x-y)$ is a solution of the same p.d.e, as: $\frac{\partial z}{\partial x}=\cos (x-y), \frac{\partial^{2} z}{\partial x^{2}}=-\sin (x-y)$ and $\frac{\partial z}{\partial y}=-\cos (x-y), \frac{\partial^{2} z}{\partial y^{2}}=-\sin (x-y)$ satisfies the same equation.

This illustrates that a partial differential equation can have more than one solution i.e., uniqueness of solution is not seen.

## Classification of First Order Partial Differential Equations (p.d.es)

The general representation of a first order partial differential equation as given in equation (34.1) represents a non-linear p.d.e as the function f is a general function of the dependent variable and all its partial derivatives of various orders.

When we restrict to the first order partial derivatives of $z(x, y)$ in equation (1), we get a first order p.d.e. The most general form of a non- linear $1^{\text {st }}$ order p.d.e may be written as $f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)=0$

## Classification of the first order p.d.e

The first order p.d.e given by (1) is said to be a linear equation if it is linear $z$, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. It is of the form

$$
\begin{equation*}
A(x, y) \frac{\partial z}{\partial x}+B(x, y) \frac{\partial z}{\partial y}+C(x, y) z=S(x, y) \tag{34.3}
\end{equation*}
$$

where the coefficients $A, B, C$ and $S$ are continuous functions of $x \& y$ in $\Omega$. $S(x, y)$ is called the non-homogeneous function. If $S(x, y) \equiv 0 . \forall(x, y) \in \Omega$, then the equation is called a homogeneous p.d.e.

Examples are: $x \frac{\partial z}{\partial x}+y \frac{\partial z}{\partial y}=x y z+x y, \quad \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$.

Equation (34.2) is said to be a semi-linear p.d.e if it is linear in $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and the coefficient of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are functions of $x$ and $y$ only. The semi linear p.d.e may be written as

$$
\begin{equation*}
A(x, y) \frac{\partial z}{\partial x}+B(x, y) \frac{\partial z}{\partial y}=C(x, y, z) \tag{34.4}
\end{equation*}
$$

Examples are : $\quad x \frac{\partial z}{\partial x}+y \frac{\partial z}{\partial y}=x y z^{3}, \quad x \frac{\partial z}{\partial x}-y \frac{\partial z}{\partial y}=\sin z$.

Equation (2) is called a quasi-linear p.d.e if it is linear in $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and it is written in the form

$$
\begin{equation*}
A(x, y, z) \frac{\partial z}{\partial x}+B(x, y, z) \frac{\partial z}{\partial y}=C(x, y, z) \tag{34.5}
\end{equation*}
$$

An example is $\quad\left(x^{2}-z^{2}\right) \frac{\partial z}{\partial x}+x y \frac{\partial z}{\partial y}=y z^{2}+x^{2}$
Equation (34.2) represents a general first order non-linear equation, a simple example for it may be written as $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}=z^{2}$.

We use these notations for the first order partial derivatives of $z=z(x, y)$ as:

$$
p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y} .
$$

### 34.3 Formation of Partial Differential Equations

Given a one parameter family of plane curves, we can find an ordinary differential equation for which the given one parameter family is a solution. This is done by eliminating the arbitrary constant in the family of curves. In the same way, given an arbitrary surface in $\mathbb{R}^{3}$ or in higher dimensional spaces, elimination of the arbitrary function leads to the partial differential equation for which the given surface is a solution. The following examples give more insight into formation of partial differential equations associated with the given surface.

Example 1: Eliminate the arbitrary function $F$ from the below given surfaces:
(i) $z=x+y+F(x y)$
(ii) $F(x-z, y-z)=0$.

## Solution:

(i) we eliminate the arbitrary function by finding the partial derivatives $p$ and $q$.

$$
\begin{aligned}
& p=\frac{\partial z}{\partial x}=1+\frac{d F(x y)}{d(x y)} \cdot \frac{\partial(x y)}{\partial x}=1+\frac{d F(x y)}{d(x y)} \cdot y \\
& q=\frac{\partial z}{\partial y}=1+\frac{d F(x y)}{d(x y)} \cdot \frac{\partial(x y)}{\partial y}=1+\frac{d F(x y)}{d(x y)} \cdot x
\end{aligned}
$$

Eliminating $\frac{d F(x y)}{d(x y)}$ from the above, we obtain $x p-y q=x-y$ which is the required p.d.e. Further, note that it is a linear p.d.e
(ii) Given $F(u, v)=0$ where $u=x-z, v=y-z$. Using chain rule of differentiation, we get

$$
p=\frac{\partial F}{\partial u} \frac{\partial u}{\partial x}+\frac{\partial F}{\partial v} \frac{\partial v}{\partial y}=\frac{\partial F}{\partial u}(1-p)+\frac{\partial F}{\partial u}(-p)=0
$$

Similarly $\quad q=\frac{\partial F}{\partial u} \frac{\partial u}{\partial y}+\frac{\partial F}{\partial v} \frac{\partial v}{\partial y}=\frac{\partial F}{\partial u}(-q)+\frac{\partial F}{\partial u}(1-q)=0$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from the above two equation we get $-p q+(1-q)(1-p)=0 \Rightarrow p+q=1$ which is the required linear p.d.e.

Thus eliminating an arbitrary function resulted in a linear partial differential equation.

Example 2: Eliminate the arbitrary parameters from the following functions.
(i) $2 z=(a x+y)^{2}+b$
(ii) $z=a x+b y$

## Solution:

(i) Note that in the surface $2 z=(a x+y)^{2}+b$, the arbitrary constant $a$ is nonlinearly involved. Differentiating partially we obtain,

$$
\begin{aligned}
& 2 p=2(a x+y) \cdot a \Rightarrow p x=a^{2} x^{2}+a x y \\
& 2 q=2(a x+y) \cdot 1 \text { or } q=(a x+y) \Rightarrow q y=y^{2}+a x y
\end{aligned}
$$

$$
\text { Also } 2 z=q^{2}+b \Rightarrow b=2 z-q^{2}
$$

$$
q^{2}=2 z-b=(a x+y)^{2}=p x+q y
$$

$\therefore p x+q y=q^{2}$ is the required non-linear p.d.e
(ii) $z=a x+b y$; the arbitrary constants $a \& b$ are linearly involved in the function. Differentiating partially we obtain $p=a$ and $q=b$. So the required p.d.e is $x p+y q=z$ which is a linear p.d.e.

The following are some Standard Partial Differential Equations which occur in physics:

1. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$
(Transport equation)
2. $\frac{\partial u^{2}}{\partial x}+\frac{\partial u^{2}}{\partial y}=1$
(Eikonal equation)
3. $\frac{\partial^{2} u}{\partial t^{2}}-\frac{\partial^{2} u}{\partial x^{2}}=0$
(Wave equation)
4. $\frac{\partial u}{\partial t}-\frac{\partial^{2} u}{\partial x^{2}}=0$
(Heat or Diffusion equation)
$\begin{array}{ll}\text { 5. } \frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}=0 & \text { (Laplace equation) } \\ \text { 6. } \frac{\partial^{2} u^{2}}{\partial x \partial y}-\frac{\partial^{2} u}{\partial x^{2}} \cdot \frac{\partial^{2} u}{\partial y^{2}}=f(x, y) & \text { (Monge-Ampere equation) }\end{array}$

In the above, equations $34.1,34.3,34.4,34.5$ are linear and homogeneous equations while equations $34.2,34.6$ are non-homogeneous equations. In equation 6, if $f(x, y) \equiv 0 \forall(x, y) \in \Omega$, the equation is a non-linear and homogeneous equation.

### 34.4 Checking Linearity of the given Partial Differential Equation

The Linear equation (34.3) can be written in the operator form as:

$$
\begin{equation*}
L z=S \tag{34.6}
\end{equation*}
$$

where $L$ is the linear operator defined as:

$$
L: A \frac{\partial}{\partial x}+B \frac{\partial}{\partial y}+C
$$

The homogeneous equation corresponding to (34.6) is $L z=0$.

Let us check the linearity of equation $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0$.

Definition: An operator $L$ is said to be linear if and only if, for two functions $Z_{1}(x, y)$ and $Z_{2}(x, y)$ with arbitrary constants $c_{1}$ and $c_{2} \in \mathbb{R}$, the following property holds. :

$$
L\left(c_{1} z_{1}+c_{2} z_{2}\right)=c_{1} L z_{1}+c_{2} L z_{2} .
$$

The operator L is non-linear if the above property is not satisfied.
In equation (34.7), the operator is $L=\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}$

Now consider

$$
\begin{aligned}
L\left[c_{1} z_{1}(x, y)+c_{2} z_{2}(x, y)\right] & =\frac{\partial^{2}}{\partial x^{2}}\left(c_{1} z_{1}+c_{2} z_{2}\right)+\frac{\partial^{2}}{\partial y^{2}}\left(c_{1} z_{1}+c_{2} z_{2}\right) \\
& =\left(c_{1} \frac{\partial^{2} z_{1}}{\partial x^{2}}+c_{2} \frac{\partial^{2} z_{2}}{\partial x^{2}}\right)+\left(c_{1} \frac{\partial^{2} z_{1}}{\partial y^{2}}+c_{2} \frac{\partial^{2} z_{1}}{\partial y^{2}}\right) \\
& =c_{1}\left[\frac{\partial^{2} z_{1}}{\partial x^{2}}+\frac{\partial^{2} z_{2}}{\partial x^{2}}\right]+c_{2}\left[\frac{\partial^{2} z_{1}}{\partial y^{2}}+\frac{\partial^{2} z_{2}}{\partial y^{2}}\right] \\
& =c_{1} L z_{1}+c_{2} L z_{2}
\end{aligned}
$$

$\therefore \frac{\partial^{2} z_{1}}{\partial x^{2}}+\frac{\partial^{2} z_{2}}{\partial x^{2}}=0$ is a linear equation.

Let us test the equation

$$
\begin{equation*}
\frac{\partial z}{\partial x}+z \frac{\partial z}{\partial y}=0 \tag{34.8}
\end{equation*}
$$

for linearity. Consider $\frac{\partial}{\partial x}\left(c_{1} z_{1}+c_{2} z_{2}\right)+\left(c_{1} z_{1}+c_{2} z_{2}\right) \frac{\partial}{\partial y}\left(c_{1} z_{1}+c_{2} z_{2}\right)$

$$
\begin{aligned}
& =c_{1} \frac{\partial z_{1}}{\partial x}+c_{2} \frac{\partial z_{2}}{\partial x}+\left(c_{1} z_{1}+c_{2} z_{2}\right)\left(c_{1} \frac{\partial z_{1}}{\partial y}+c_{2} \frac{\partial z_{2}}{\partial y}\right) \\
& \neq\left(c_{1} \frac{\partial z_{1}}{\partial x}+c_{2} \frac{\partial z_{2}}{\partial x}\right)+\left(c_{1} z_{1} \frac{\partial z_{1}}{\partial y}+c_{2} z_{2} \frac{\partial z_{2}}{\partial y}\right)
\end{aligned}
$$

Hence equation (34.8) is a non-linear equation.

Keywords: Order, linear equation, semi-linear, quasi-linear

## Exercise 1

Eliminate the arbitrary function / constants from the following surfaces to form an appropriate partial differential equation.
(i) $z=(x+a)(y+b)$ (ii) $z^{2}=8(x+a y+b)^{3}$
(iii) $z=F\left[\left(x^{2}+y^{2}\right)^{\frac{1}{2}}\right]$ (iv) $z=x y+F\left(x^{2}+y^{2}\right)$
(v) $z=F\left(\frac{x y}{z}\right)$

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## Lesson 35

## Linear First Order Equation

### 35.1 Classification of Integrals (Solutions of p.d.es)

A surface $\mathrm{z}=\mathrm{z}(\mathrm{x}, \mathrm{y})$ which is continuously differentiable with respect to both the variables $x$ and $y$ in a domain $\Omega \subseteq \mathbb{R}^{2}$ that satisfies the given p.d.e $f(x, y, z, p, q)=0$ is called an integral surface of it.

Let $z=F(x, y, a, b)$ be a 2-parameter family of surfaces, with $a, b$ as arbitrary parameters. Now $p=\frac{\partial z}{\partial x}=\frac{\partial F}{\partial x}$ and $q=\frac{\partial z}{\partial y}=\frac{\partial F}{\partial y}$.

Using $p$ and $q$, we can eliminate $a$ and $b$ from $z=F(x, y, a, b)$ and form a first order p.d.e $f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)=0$.

Also the surface $z=F(x, y, a, b)$ is a solution of the p.d.e $f(x, y, z, p, q)=0$.
The solution of the partial differential equation is called an integral surface of it. It is classified as (i) complete integral (ii) general integral and (iii) singular integral.
(i) Complete Integral: A two parameter family of surfaces $z=F(x, y, a, b)$ that satisfies $f(x, y, z, p, q)=0$ is called a complete integral if in the domain of definition $\Omega$, the rank of the matrix $A=\left(\begin{array}{ccc}\frac{\partial F}{\partial a} & \frac{\partial^{2} F}{\partial x \partial a} & \frac{\partial^{2} F}{\partial y \partial a} \\ \frac{\partial F}{\partial b} & \frac{\partial^{2} F}{\partial x \partial b} & \frac{\partial^{2} F}{\partial y \partial b}\end{array}\right)$ is 2 .

Let us see some examples:

Example 1: Consider the surface

## Solution:

$(x-a)^{2}+(y-b)^{2}+z^{2}=1$ and the p.d.e $z^{2}\left(1+p^{2}+q^{2}\right)=1$.

$$
\begin{aligned}
& F(x, y, z, a, b)=(x-a)^{2}+(y-b)^{2}+z^{2}-1 \\
& \frac{\partial F}{\partial a}=-2(x-a), \quad \frac{\partial^{2} F}{\partial x \partial a}=-2, \quad \frac{\partial^{2} F}{\partial y \partial a}=0, \\
& \frac{\partial F}{\partial b}=-2(y-b), \quad \frac{\partial^{2} F}{\partial x \partial b}=-2, \quad \frac{\partial^{2} F}{\partial y \partial b}=0 .
\end{aligned}
$$

The matrix $\quad A=\left(\begin{array}{lll}2(a-x) & -2 & 0 \\ 2(b-x) & 0 & -2\end{array}\right)$ is with a non-vanishing $2 \times 2$ minor and hence its rank is 2 . We check whether the given surface is a solution of the given p.d.e. We find $p=\frac{x-a}{-z}$ and $q=\frac{y-b}{-z}$ and using $p$ and $q$ in $z^{2}\left(1+p^{2}+q^{2}\right)=1$, we see $z^{2}\left(1+\left(\frac{x-a}{-z}\right)^{2}+\left(\frac{y-b}{-z}\right)^{2}\right)=1$ or $(x-a)^{2}+(y-b)^{2}+z^{2}=1$ is the given surface that is satisfying the p.d.e.
$\therefore$ This surface is a complete integral of the given p.d.e.

## Exercises

1. Show that $z=a x+\frac{y}{a}+b$ is a complete integral of $p q=1$.
2. Show that the 2-parameter family of surfaces $z=a x+b y+a^{2}+b^{2}$ is a complete integral of the p.d.e $z-p x-q x-p^{2}-q^{2}=0$.
(ii) General Integral: The general integral is also a solution of the partial differential equation that involves an arbitrary function. In the two parameter family of solutions $z=F(x, y, a, b)$, take $a=\varphi(b)$, we get a one parameter family of solutions of $f(x, y, z, p, q)=0$. We obtain $z=F(x, y, \varphi(b), b)$ which is a subfamily of the given two parameter family of complete integral of $f(x, y, z, p, q)=0$. Find the envelope of this one parameter sub-family by eliminating b between $\quad z=F(x, y, \varphi(b), b) \quad$ and $\frac{\partial F(x, y, \varphi(b), b)}{\partial a} \varphi^{\prime}(b)+\frac{\partial F(x, y, \varphi(b), b)}{\partial b}=0$ if exists. This way we will be able to find $b=b(x, y)$ and substituting for $b$ in the one parameter sub family, we obtain $z=F(x, y, \varphi(b(x, y)), b(x, y))$. If the function $\varphi$ which defines this subfamily is arbitrary, then such a solution is called a general integral of $f(x, y, z, p, q)=0$. Different choices of $\varphi$ give different particular integrals of the p.d.e. Let us illustrate this with examples.

Example 2: Find the general solution of the equation $\frac{\partial z}{\partial x}+z=e^{-x}$.

## Solution:

Integrating the homogeneous equation $\frac{\partial z}{\partial x}+z=0$ with respect of $x$, holding $y$ as a constant, we obtain $z(x, y)=e^{-x} \bullet f(y)$ where f is an arbitrary function which is a continuously differentiable function of $y$.

By inspection, we note that $x e^{-x}$ satisfies the given equation. This is a particular solution. Thus the given general solution of this p.d.e is written as $z(x, y)=e^{-x} f(y)+x e^{-x}$.

### 35.2 General Solution of the Linear Equation

Let us now derive the form of the general solution of the linear first order homogeneous equation.

$$
\begin{equation*}
A(x, y) z_{x}+B(x, y) z_{y}+C(x, y) z_{z}=0 \tag{35.1}
\end{equation*}
$$

Where A,B,C are continuously differentiable functions in some domain in $\mathbb{R}^{2}$. Chose the transformation $\xi=\xi(x, y), \eta=\eta(x, y),(x, y) \in \Omega$, with Jacobian $J=\left|\begin{array}{ll}\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y}\end{array}\right| \neq 0$ on $\Omega$.

Clearly,

$$
\begin{equation*}
\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial x}+\frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial x}, \text { and } \frac{\partial z}{\partial y}=\frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial y}+\frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial y} \tag{35.2}
\end{equation*}
$$

Using these in the linear equation (35.1), we obtain

$$
\begin{equation*}
\left(A \frac{\partial \xi}{\partial x}+B \frac{\partial \xi}{\partial y}\right) \frac{\partial z}{\partial \xi}+\left(A \frac{\partial \eta}{\partial x}+B \frac{\partial \eta}{\partial y}\right) \frac{\partial z}{\partial \eta}+C z=0 \tag{35.3}
\end{equation*}
$$

Chose $\eta$ such that

$$
\begin{equation*}
\left(A \frac{\partial \eta}{\partial x}+B \frac{\partial \eta}{\partial y}\right)=0 \tag{35.4}
\end{equation*}
$$

This is a meaningful choice because of the following argument.

Assume that $A(x, y) \neq 0$ and consider the o.d.e

$$
\begin{equation*}
\frac{\partial y}{\partial x}=\frac{B(x, y)}{A(x, y)} \tag{35.5}
\end{equation*}
$$

Write its general solution as $\eta(x, y)=k, k$ is an arbitrary constant and $\frac{\partial \eta}{\partial y} \neq 0$. Then for, $\quad \eta(x, y)=k$
we have $d \eta(x, y)=d k=0 \quad$ or $\frac{\partial \eta}{\partial x} d x+\frac{\partial \eta}{\partial y} d y=0$

In view of this, equation (35.4) is satisfied.

The one parameter family of curves given by (35.6) that are obtained from equation (35.5) are called characteristic curves of the differential equation (35.1).

Now chose $\xi=\xi(x, y)=x$, such that

$$
J=J=\left|\begin{array}{cc}
1 & 0 \\
\eta_{x} & \eta_{y}
\end{array}\right|=\eta_{y} \neq 0 \quad \forall(x, y) \in \Omega
$$

Now the transformation $\xi=x, \eta=\eta(x, y)$ which is an invertible transformation (having one to one correspondence between ( $\xi, \eta$ ) and ( $x, y$ ) transforms the equation (35.3) to the following simple form

$$
\begin{equation*}
A(\xi, \eta) \frac{\partial z}{\partial \xi}+C(\xi, \eta) z=0 \tag{35.7}
\end{equation*}
$$

This equation is called canonical form for the linear partial differential equation (35.1). This can be solved as an o.d.e.

Under the same transformation, the non-homogeneous linear equation

$$
\begin{equation*}
A(x, y) z_{x}+B(x, y) z_{y}+C(x, y) z_{z}=D(x, y) \tag{35.8}
\end{equation*}
$$

gets transformed to

$$
\begin{equation*}
A(\xi, \eta) z_{\xi}+C(\xi, \eta) z=D(\xi, \eta) \tag{35.9}
\end{equation*}
$$

We describe the Lagrange method for finding the general integral of the given quasi-linear p.d.e in the next lesson.

Example 3: Find the general solution of the linear p.d.e

$$
x z-y z_{y}+y^{2} z=y^{2}, \quad(x, y) \neq 0
$$

## Solution:

Given $A(x, y)=x, B(x, y)=-y, C(x, y)=y^{2}, D(x, y)=y^{2}$

Now equation (35.5) gives $\frac{d y}{d x}=-\frac{y}{x}$ which gives its general solution as $x y=k \quad$ where $\quad k$ is an arbitrary constant. Now set $\xi=x, \eta=x y$ as the coordinate transformation; This gives $J=x \neq 0$.

Now $\frac{d z}{d x}=\frac{d z}{d \xi} \frac{d \xi}{d x}+\frac{d z}{d \eta} \frac{d \eta}{d x}=z_{\xi} \cdot 1+z_{\eta} \cdot y$, and $\frac{d z}{d y}=\frac{d z}{d \xi} \frac{d \xi}{d y}+\frac{d z}{d \eta} \frac{d \eta}{d y}=y z_{\eta}$,

$$
A(x, y)=x=\xi, \quad B(x, y)=-y=-\frac{\eta}{\xi}, \quad C(x, y)=\frac{\eta^{2}}{\xi^{2}}, \quad D(x, y)=\frac{\eta^{2}}{\xi^{2}} .
$$

The canonical form of the given equation is:

$$
\xi \cdot z_{\xi}+\frac{\eta^{2}}{\xi^{2}} \cdot z=\frac{\eta^{2}}{\xi^{2}} \quad \text { or } z_{\xi}+\frac{\eta^{2}}{\xi^{3}} z=\frac{\eta^{2}}{\xi^{3}} .
$$

This can be solved as an p.d.e, by fixing $\eta$ as a constant in $z(\xi, \eta)$. Thus we obtain

$$
\begin{aligned}
z(\xi, \eta) & =e^{-\int \frac{\eta^{2}}{\xi^{3}} d \xi}\left[f(\eta)+\int \frac{\eta^{2}}{\xi^{3}} e^{\int \frac{\eta^{2}}{\xi^{2}} d \xi} d \xi\right] \\
& =e^{\frac{\eta^{2}}{2 \xi^{2}}}\left[f(\eta)+e^{-\frac{\eta^{2}}{2 \xi^{2}}}\right]
\end{aligned}
$$

$=f(\eta) e^{\frac{\eta^{2}}{2 \xi^{2}}}+1$, where $f(\eta)$ is an arbitrary function.

Thus $z(x, y)=f(x y) e^{\frac{y^{2}}{2}}+1$ is the general solution of the given p.d.e.

Example 4: Find the general solution of the Euler equation $x z_{x}+y z_{y}=n z,(x, y) \neq 0$.

## Solution:

Given $A(x, y)=x ; B(x, y)=y ; C(x, y)=\eta, D(x, y)=0$, equation (35.5) gives $\frac{d y}{d x}=\frac{y}{x} \Rightarrow \frac{1}{x} d x=\frac{1}{y} d y, \quad$ leading to $\quad \ln x=\ln k+\ln y \quad$ or $\quad \frac{x}{y}=k \quad$ as its characteristic curve. Now set $\xi=x, \eta=\frac{x}{y}$,
$J=-\frac{x}{y^{2}} \neq 0$. Also, note that $A(x, y)=\xi, C(x, y)=\eta$, and the canonical form for the given p.d.e is $\quad \xi_{Z_{x}}+\eta Z=0$, or $\quad Z_{\xi}+\frac{\eta}{\xi} Z=0$. Its general solution is $z(\xi, \eta)=\xi^{-n} f(\eta)$ or $z(x, y)=x^{-n} f\left(\frac{x}{y}\right)$ where $f$ is an arbitrary function.

Exercises 3: Find the general solutions of
(i) $x z_{x}+y z_{y}=x^{n}$
(ii) $a z_{x}+b z_{y}+c z_{z}=d$ where $a, b, c, d$ are constants such that $a^{2}+b^{2} \neq 0$.
(iii) Singular integral: Find the envelope of the two parameter family of solutions $z=F(x, y, a, b)$, if exists. This is obtained by eliminating $a$ and $b$ from the equations $z=F(x, y, a, b), \frac{\partial F(x, y, a, b)}{\partial a}=0, \frac{\partial F(x, y, a, b)}{\partial b}=0$. This is called the singular integral of the given p.d.e.

Example 5: Obtain the singular integral for $z-p x-q y-p^{2}-q^{2}=0$.

## Solution:

The given equation has the two parameter family of curves $z=a x+b y+a^{2}+b^{2}$ as its complete integral. Now

$$
\begin{aligned}
& \frac{\partial F(x, y, a, b)}{\partial a}=0 \Rightarrow x+2 a=0 \\
& \frac{\partial F(x, y, a, b)}{\partial b}=0 \Rightarrow y+2 b=0
\end{aligned}
$$

Eliminating $a$ and $b$ from the equations $z=a x+b y+a^{2}+b^{2}$, $x+2 a=0, y+2 b=0$, we obtain the singular integral as $4 z=-\left(x^{2}+y^{2}\right)$.

Keywords: Complete Integral, General Integral, Singular Integral, Characteristic Curves

## References

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## Lesson 36

## Geometric Interpretation of a First Order Equation

### 36.1 Geometric Interpretation of a First Order Equation

Consider the general quasi linear partial differential equation

$$
\begin{equation*}
P(x, y, u) u_{x}+Q(x, y, u) u_{y}=R(x, y, u) \tag{36.1}
\end{equation*}
$$

A possible solution written in implicit form as

$$
\begin{equation*}
f(x, y, u)=u(x, y)-u=0 \tag{36.2}
\end{equation*}
$$

which is a surface in $(x, y, u)$ space. At any point $(x, y, u)$ on this surface, $\nabla f=\left(u_{x}, u_{y},-1\right)$ gives the normal to the surface. Equation (1) can be re-written as:

$$
\begin{equation*}
(P, Q, R) \cdot\left(u_{x}, u_{y},-1\right)=0 \tag{36.3}
\end{equation*}
$$

This shows that the vector ( $P, Q, R$ ) must be a tangent vector of the surface given by (36.2) at $(x, y, u)$ and this determines a direction filed called the Characteristic Direction for the integral surface for of the given p.d.e. In brief; $f(x, y, u)=u(x, y)-u=0$ is a solution of (1) if and only if the direction vector field ( $\mathrm{P}, \mathrm{Q}, \mathrm{R}$ ) lies in the tangent plane of this integral surface at each point $(x, y, u)$.


If $\Gamma$ is a curve with the parametric rep $x=x(t), y=y(t), u=u(t)$. If this space curve lies on the surface $u=u(x, y)$, then at $(x, y, u)$, the tangent to the curve $\Gamma$ will have the direction cosines as $(P, Q, R)$ where $(P, Q, R) \cdot\left(u_{x}, u_{y},-1\right)=0$ is the partial differential equation for which $u=u(x, y)$ is the solution.

Definition: A curve in $(x, y, u)$ - space, whose tangent at every point coincides with the characteristic direction field $(P, Q, R)$ is called a characteristic curve. If parametric representation of this characteristic curve is

$$
\begin{equation*}
x=x(t), y=y(t), u=u(t) \tag{36.4}
\end{equation*}
$$

then the tangent vector to this curve is $\left(\frac{d x}{d t}, \frac{d y}{d t}, \frac{d u}{d t}\right)$ which must coincide with ( $P, Q, R$ ).

The system of ordinary differential equations representing these characteristic curve is given by

$$
\begin{equation*}
\frac{d x}{d t}=P(x, y, u) ; \frac{d y}{d t}=Q(x, y, u) ; \frac{d u}{d t}=R(x, y, u) \tag{36.5}
\end{equation*}
$$

These are called the characteristic equations of the Quasi linear equation (36.1). Its solution consist of a 2-p family of curves in ( $\mathrm{x}, \mathrm{y}, \mathrm{u}$ ) - space.

The characteristic equations in non-parametric form are written as:

$$
\begin{equation*}
\frac{d x}{P}=\frac{d y}{Q}=\frac{d u}{R} \tag{35.6}
\end{equation*}
$$

### 36.2 Method of Characteristics to obtain the general integral: (Lagrange Method)

The general solution of the quasi-linear partial differential equation (also known as the Lagrange's equation) $P(x, y, u) u_{x}+Q(x, y, u) u_{y}=R(x, y, u)$ is written as $F(\phi, \psi)=0$ where $F$ is an arbitrary function of $\phi$ and $\psi$ and $\phi(x, y, u)=C_{1}$ and $\psi(x, y, u)=C_{2}$ are two functionally independent solutions of the characteristic system $\frac{d x}{P}=\frac{d y}{Q}=\frac{d u}{R}$. This general solution can also be written as: $\phi=G(\psi)$.

Example 1: Find the general integral of the quasi linear p.d.e $y z z_{x}+x z z_{y}=x y$.

## Solution:

The characteristic system is: $\frac{d x}{y z}=\frac{d y}{x z}=\frac{d z}{x y}$

Taking 2 equations at a time and integrating, we get
(i) $\frac{d x}{y}=\frac{d y}{x} \Rightarrow x^{2}-y^{2}=C_{1}$
(ii) $\frac{d y}{z}=\frac{d z}{y} \Rightarrow z^{2}-y^{2}=C_{2}$

The general solution is $F\left(x^{2}-y^{2}, z^{2}-y^{2}\right)=0$, where $F$ is and arbitrary function, this general solution can also be written in the form $z^{2}=y^{2}+G\left(x^{2}-y^{2}\right), G$ is any arbitrary function.

Example 2: Find the general integral of $z_{x}+z z_{y}=0$

## Solution:

The characteristic system is $\quad \frac{d x}{1}=\frac{d y}{z}=\frac{d z}{0}$

Which admits solutions as: (i) $z=C_{1}$ and (ii) $y-z x=C_{2}$.

So the general solution is written as $F(z, y-z x)=0$, where $F$ is an arbitrary function, or is also written as $z=G(y-z x)$, where $G$ is arbitrary function.

Exercises 1: Find the General Integral of

1) $z\left(x z_{x}-y z_{y}\right)=y^{2}-x^{2}$
2) $y z Z_{x}+x z z_{y}=x+y$
3) $x(y-z) z_{x}+y(z-x) z_{y}=z(x-y)$

### 36.3 Linear Equation to 3-Variables

Now, let us consider the extension of the linear equation in 3-variable for the function $u(x, y, z)$ as

$$
A(x, y, z) u_{x}+B(x, y, z) u_{y}+C(x, y, z) u_{z}=0 .
$$

For this equation, the characteristic system is given by

$$
\frac{d x}{A(x, y, z)}=\frac{d y}{B(x, y, z)}=\frac{d z}{C(x, y, z)}
$$

and this gives the family of characteristic curves as (say) $g(x, y, z)=C_{1}$ and $h(x, y, z)=C_{2}$ which are two functionally independent solution of the above system, then the general solution is written as $u=F(g, h)$.

The functions $g(x, y, z), h(x, y, z)$ are called functionally independent if rank $\left(\begin{array}{lll}\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z}\end{array}\right)$

$$
\text { is } 2 .
$$

Example 3: Find the general solution of the linear equation in 3 independent variables

$$
(y-z) u_{x}+(z-x) u_{y}+(x-y) u_{z}=0
$$

## Solution:

The characteristic curves are obtained from the characteristic system

$$
\frac{d x}{(y-z)}=\frac{d y}{(z-x)}=\frac{d z}{(z-y)} .
$$

Note that $d x+d y+d z=(y-z+z-x+x-y)=0$, and $x d x+y d y+z d z=x(y-z)+y(z-x)+z(x-y)=0$.

Integrating, these equations give
$g(x, y, z)=x+y+z=C_{1}$
and $h(x, y, z)=x^{2}+y^{2}+z^{2}=C_{2}$.

Then the general solution is written as $u=F(g, h) \quad$ i.e., $u(x, y, z)=F\left(x+y+z, x^{2}+y^{2}+z^{2}\right):$ where $F$ is an arbitrary function.

Exercises 2: Find the general solution of the equations

1) $x(y-z) u_{x}+y(z-x) u_{y}+z(x-y) u_{z}=0$
2) $x u_{x}+y u_{y}+z u_{z}=u$.

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## Lesson 37

## Integral Surface Through a Given Curve - The Cauchy Problem

### 37.1 Integral Surface through a given Curve - The Cauchy Problem

For the quasi linear p.d.e. $P(x, y, z) z_{x}+Q(x, y, z) z_{y}=R(x, y, z)$, with its general integral $F(\phi, \psi)=0$ where $\phi(x, y, z)=C_{1}$ and $\psi(x, y, z)=C_{2}$ are two functionally independent solutions of the characteristic system $\frac{d x}{P}=\frac{d y}{Q}=\frac{d z}{R}$, can we find a particular integral containing the given curve $C$ whose parametric equations are given by $\quad x=x_{0}(s), y=y_{0}(s)$ and $z=z_{0}(s)$ where $s$ is the parameter. This is similar to finding the arbitrary constants in the general solution of an ordinary differential equation using the initial conditions.

Thus fixing the arbitrary function in the general solution of the given p.d.e by making it to pass through the given initial data is called the Cauchy Problem.

Suppose $z=z(x, y)$ is the integral surface passing through the initial data curve $C$ then we require that the equations $\phi\left(x_{0}(s), y_{0}(s), z_{0}(s)\right)=C_{1}$ and $\psi\left(x_{0}(s), y_{0}(s), z_{0}(s)\right)=C_{2}$ be satisfied. Now eliminating $s$ from these two equations we obtain $F\left(C_{1}, C_{2}\right)=0$ or $C_{1}=G\left(C_{2}\right)$. This fixes the arbitrary function $F$ (or $G$ ) and produces the required surface.

Let us illustrate this by considering few examples:

Example 1: For the p.d.e $z(x+y) z_{x}+z(x-y) z_{y}=x^{2}+y^{2}$

Find the integral surface that satisfies the Cauchy data $z=0$ on the curve $y=2 x$.

## Solution:

Step 1: Find the general solution:

The Characteristic system is:

$$
\frac{d x}{z(x+y)}=\frac{d y}{z(x-y)}=\frac{d z}{x^{2}+y^{2}}
$$

Note that $-x d x+y d y+z d z=0$ and $\quad y d x+x d y-z d y=0$.

On integrating we get $z^{2}-x^{2}+y^{2}=C_{1}$ and $x y-\frac{1}{2} z^{2}=C_{2}$

Thus the two characteristic curves are $\phi=z^{2}-x^{2}+y^{2}=C_{1}$ and $\psi=2 x y-z^{2}=C_{2}$.

The general solution is written as: $F\left(z^{2}-x^{2}+y^{2}, 2 x y-z^{2}\right)=0$ or $\quad z^{2}=\left(x^{2}-y^{2}\right)+G\left(2 x y-z^{2}\right)$, here $G$ is any arbitrary function

Step 2: Fixing the arbitrary function:
We are given the Cauchy data as $z=0$ on $y=2 x$. Its parametric representation is, $x=s, y=2 s, z=0$.

Using this in the integrals $\phi=C_{1}$ and $\psi=C_{2} ; 0 \cdot-s^{2}+4 s^{2}=C_{1} ; 2 . s .2 s-0=C_{2}$ or $3 s^{2}=C_{1} ; 4 s^{2}=C_{2}$

$$
\Rightarrow \frac{3}{4}=\frac{C_{1}}{C_{2}} \Rightarrow 4 C_{1}=3 C_{2}
$$

Thus the solution is written as:

$$
4\left(z^{2}-x^{2}+y^{2}\right)=3\left(2 x y-z^{2}\right) \text { or } 7 z^{2}=6 x y+4 x^{2}-4 y^{2}
$$

Alternative to Step 2: We have $z^{2}=\left(x^{2}-y^{2}\right)+G\left(2 x y-z^{2}\right)$

Using the Cauchy data, we get $0=s^{2}-4 s^{2}+G(2 . s .2 s)$ or $3 s^{2}=G\left(4 s^{2}\right)$
Put $4 s^{2}=t \Rightarrow s^{2}=\frac{t}{4}$ leads to $3 s^{2}=\frac{3}{4} t$

$$
\therefore G(t)=\frac{3}{4} t
$$

This gives the integral surface as $z^{2}=\left(x^{2}-y^{2}\right)+\frac{3}{4}\left(2 x y-z^{2}\right)$
or $7 z^{2}=6 x y+4\left(x^{2}-y^{2}\right)$.

Example 2: Find the integral surface of the equation
$(2 x y-1) p+\left(z-2 x^{2}\right) q=2(x-y z)$ Passing through the Cauchy data $x_{0}(s)=1, y_{0}(s)=0, z_{0}(s)=s$.

## Solution:

Step 1: The characteristic system is $\frac{d x}{2 x y-1}=\frac{d y}{z-2 x^{2}}=\frac{d z}{2(x-y z)}$

Note that

$$
\text { (i) } z d x+d y+x d z=0 \Rightarrow u=x z+y=C_{1}
$$

(ii) $x d x+y d y+d z=0 \quad \Rightarrow v=x^{2}+y^{2}+z=C_{2}$
$\therefore$ Integral surface is $F\left(x z+y, x^{2}+y^{2}+z\right)=0$

Step 2: using the data: $x_{0}(s)=1, y_{0}(s)=0, z_{0}(s)=s$ In the integrals; we obtain
$1 \cdot s+0=C_{1} \& 1+0+s=C_{2}$ leads to $C_{1}=s ; 1+s=C_{2}$ or $1+C_{1}=C_{2}$.
$\therefore$ The required integral surface is $1+x z+y=x^{2}+y^{2}+z$

$$
\text { or } \quad x^{2}+y^{2}-x z-y+z-1=0
$$

### 37.2 Existence and Uniqueness of solution for the Cauchy problem:

The following result ensures the existence and uniqueness of an integral surface for the Cauchy problem.

Statement: Consider the first order quasi linear p.d.e $P(x, y, z) z_{x}+Q(x, y, z) z_{y}=R(x, y, z)$ in the domain $\Omega$ where $P, Q, R$ are continuously differentiable functions in $\Omega$. Let $x=x_{0}(s), y=y_{0}(s)$ and $z=z_{0}(s), 0 \leq s \leq 1$ is the initial smooth curve in $\Omega$ and

$$
\begin{equation*}
\frac{d x_{0}}{d s} B\left(x_{0}(s), y_{0}(s), z_{0}(s)\right)-\frac{d y_{0}}{d s} A\left(x_{0}(s), y_{0}(s), z_{0}(s)\right) \neq 0 \tag{37.1}
\end{equation*}
$$

$0 \leq s \leq 1$. Then there exists one and only one solution $z=z(x, y)$ defined in a neighbourhood of this initial curve, which satisfies the equation (37.1) and the initial condition $z_{0}(s)=z\left(x_{0}(s), y_{0}(s)\right), 0 \leq s \leq 1$.

Note: The condition given in (37.1) excludes the possibility that the initial curve $x=x_{0}(s), y=y_{0}(s)$ could be a characteristic. Let us now illustrate an example where the given p.d.e has a unique, no, infinitely many solutions with the Cauchy data.

Example 3: Consider the p.d.e $y p-x q=0$ whose general solution is $z=F\left(x^{2}+y^{2}\right)$ where $F$ is an arbitrary function.

Case 1: Consider the initial curve

$$
x=x_{0}(s)=s, y=y_{0}(s)=0, z=z_{0}(s)=s^{2}
$$

which is the parabola in ( $x, z$ ) plane. The condition (37.1) becomes

$$
1 \cdot-s-0 \cdot 0=-s \neq 0
$$

This ensures that the Cauchy Problem has unique solution.

Eliminating $F: s^{2}=F\left(s^{2}\right) \Rightarrow F(t)=t \Rightarrow z=x^{2}+y^{2}$ is the circular paraboloid contains the initial curve (Parabola).

Thus $z=x^{2}+y^{2}$ is the required integral surface.

Case 2: Consider the initial curve $x_{0}(s)=\cos s, y_{0}(s)=\sin s, z_{0}(s)=\sin s$.

This is the parametric representation for the ellipse $x^{2}+y^{2}=1, z=y$.

The condition (37.1) becomes: $(-\sin s)(-\cos s)-(\sin s)(\cos s)=0$

This the condition fails, so either there is no solution or there are infinitely many solutions. (i.e., either existence of the solution is lost or the uniqueness of the solution is lost). The integral surface $z=F\left(x^{2}+y^{2}\right)$ becomes
$y=F(1)$ which is inconsistency that a constant $F(1)$ is equal to a variable $y$. This implies there is no solution to the Cauchy Problem (Existence of solution is lost). Note: The tangent vector $(-\sin s, \cos s, \cos s)$ to the given curve is nowhere parallel to the characteristic vector $(\sin s,-\cos s, 0)$.

Case 3: Consider the initial curve $x_{0}(s)=\cos s, y_{0}(s)=\sin s, z_{0}(s)=1$, which is the circle $x^{2}+y^{2}=1, z=1$. The condition (37.1) means $-\sin s \cdot-\cos s-\cos s \cdot \sin s=0$. The integral surface is containing the curve results in $1=F(1)$. This is possible for any function $F$ such that $F(1)=1$ (i.e., $\left.F(w)=w^{n}\right)$. There are infinitely many representations of this function $F$, in this case with each of $F, z=F\left(x^{2}+y^{2}\right)$ is an integral surface that contains the curve. In this case, it is to be noted that the initial data curve is a characteristic curve.

Example 4: Solve the p.d.e $z z_{x}+z_{y}=\frac{1}{2}$ with the initial condition $z(s, s)=\frac{s}{4}, 0 \leq s \leq 1$.

Solution: The initial curve satisfies the condition given in (37.1) for $s \neq 4$. The characteristic system can also be written as:

$$
\frac{d x}{d t}=z, \frac{d y}{d t}=1, \frac{d z}{d t}=\frac{1}{2} \quad \text { with the initial conditions }
$$

$$
x(s, 0)=s, y(s, 0)=s, z(s, 0)=\frac{s}{4} .
$$

Solving the above system of ordinary differential equations using the initial conditions, we get

$$
\begin{aligned}
z & =\frac{1}{2} t+C_{1}(s, 0)=\frac{1}{2} t+\frac{s}{4}, \quad y=t+C_{2}(s, 0)=t+s \quad \text { and } \\
\frac{d x}{d t} & =z=\frac{1}{2} t+\frac{s}{4} \quad \text { or } x=\frac{t^{2}}{4}+\frac{t s}{4}+C_{3}(s, 0) \quad \text { or } x=\frac{t^{2}}{4}+\frac{t s}{4}+s .
\end{aligned}
$$

Now eliminating $s$ and $t$ from the above we obtain

$$
s=\frac{4 x-y^{2}}{4-y} \text { and } t=\frac{4(y-x)}{4-y}
$$

Hence the integral surface having the given Cauchy data is:

$$
z=\frac{s}{4}+\frac{t}{2}=\frac{1}{4}\left(\frac{4 x-y^{2}}{4-y}\right)+\frac{1}{2}\left(\frac{4(y-x)}{4-y}\right) \text { or } z=\frac{8 y-4 x-y^{2}}{4(4-y)} \text { for } y=s \neq 4 \text {. }
$$

## Exercises:

1. Solve the Cauchy Problem for the p.d.e $2 z_{x}+y z_{y}=z$ containing the initial data curve $x=x_{0}(s)=s, y=y_{0}(s)=s^{2}, z=z_{0}(s)=s, 1 \leq s \leq 2$.
2. Find the solution of $p-z q+z=0$ for all $y$ and $x>0$, for the initial data $x_{0}=0, y_{0}=s, z_{0}=-2 s,-\infty<s<\infty$.
3. Show that the integral surface for the p.d.e $p+q=z^{2}$ with the initial condition $z(x, 0)=f(x)$ is $z(x, y)=\frac{f(x-y)}{1-y f(x-y)}$.

## References

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## Lesson 38

## Non-Linear First order p.d.e - Compatible system

### 38.1 Non-linear first order p.d.e - compatible system

Two first order partial differential equations

$$
\begin{equation*}
f(x, y, z, p, q)=0 \tag{38.1}
\end{equation*}
$$

and

$$
\begin{equation*}
g(x, y, z, p, q)=0 \tag{38.2}
\end{equation*}
$$

are said to be compatible if they have common solutions. In fact these two equations admit a one parameter family of common solutions under some conditions.

Definition: The equation (38.1) and (38.2) are compatible on a domain $\Omega$ if
(i)

$$
\begin{equation*}
J=\frac{\partial(f, g)}{\partial(p, q)} \neq 0 \text { on } \Omega \tag{38.3}
\end{equation*}
$$

and
(ii)

$$
\begin{equation*}
p=\phi(x, y, z), q=\psi(x, y, z) \tag{38.4}
\end{equation*}
$$

obtained by solving (1) and (2) render the equation

$$
\begin{equation*}
d z=\phi d x+\psi d y \tag{38.5}
\end{equation*}
$$

integrable. Below we state a necessary and sufficient condition for the integrability of the equation (38.5).

Result: A necessary and sufficient condition for the integrability of the equation (38.5) is:

$$
\frac{\partial(f, g)}{\partial(x, p)}+p \frac{\partial(f, g)}{\partial(z, p)}+\frac{\partial(f, g)}{\partial(y, q)}+q \frac{\partial(f, g)}{\partial(z, q)}=0 .
$$

We now consider some examples to check compatibility condition for the given equations.

Example 1: Find the domain in which the equation $f=x p-y q-x=0$ and $g=x^{2} p+q-x z=0$ are compatible.

## Solution:

Condition in equation (38.3) means

$$
J=\frac{\partial(f, g)}{\partial(p, q)}=\left|\begin{array}{ll}
f_{p} & g_{p} \\
f_{q} & g_{q}
\end{array}\right|=\left|\begin{array}{cc}
x & x^{2} \\
-y & 1
\end{array}\right|=x(1+x y) \neq 0
$$

So the domain $\Omega$ should not contain points $(x, y)$ such that $x=0$ or $1+x y=0$. In such a domain $\Omega$, these two equations admit common solutions.

Example 2: Find the one parameter family of common solutions to the p.d.es $f=p^{2}+q^{2}-1=0$ and $g=\left(p^{2}+q^{2}\right) x-p z=0$.

## Solution:

Step 1: Let us find the domain in which these equations admit common solutions:
$J=\left|\begin{array}{cc}2 p & 2 q \\ 2 p x-z & 2 q x\end{array}\right|=2 q z, \quad J \neq 0 \Rightarrow z \neq 0$ in $\Omega$

Step 2: Solve for $p$ and $q$ from $f=0$ and $g=0$.
This gives $p=\frac{x}{z}=\phi(x, y, z)$ and $q^{2}=1-p^{2} \Rightarrow q=\frac{\sqrt{z^{2}-x^{2}}}{z}=\psi(x, y, z)$ (say)

Step 3: Integrability of $d z=\phi d x+\psi d y$ leads to $d z=\frac{x}{z} d x+\frac{\sqrt{z^{2}-x^{2}}}{z} d y$
Or $z d z=x d x+\sqrt{z^{2}-x^{2}} d y$, this admits the solution $z^{2}=x^{2}+(y+c)^{2}$.
which is the 1-parameter family of common integrals to $f=0$ and $g=0$.

Example 3: Show that the equations $x p-y q=0$ and $z(x p+y p)=2 x y$ are compatible and solve them.

## Solution:

Step 1: $J \neq 0 \Rightarrow x \neq 0$ (we always assume that both $p$ and $q$ are non zero).
Step 2: Solving $f=0$ and $g=0$ for finding $p$ and $q$ :

$$
\text { we obtain } p=\frac{y}{z}=\phi(x, y, z) \text { and } q=\frac{x}{z}=\psi(x, y, z)
$$

Step 3: Integrability of $d z=\phi d x+\varphi d y \quad \Rightarrow z d z=y d x+x d y$
$\Rightarrow z^{2}=2 x y+c$, is the 1-parameter family of common solutions to $f=0$ and $g=0$.

## Exercise:

1. Show that $f=x p-y q-x=0$ and $g=x^{2} p+q-x z=0$ are compatible. Show also that $z=x(y+1)$ is a solution of $f=0$ but not of $g=0$. Hence conclude that "not all solutions of $f=0$ are solutions of $g=0$ ".
2. Show that $z=\frac{(x+y)}{\sqrt{2}}$ is a solution of $f=p^{2}+q^{2}-1=0$ and not of $g=\left(p^{2}+q^{2}\right) x-p z=0$ though $f=0$ and $g=0$ are compatible. Also find the $1-$ parameter family of common solutions.

Keywords: Common Solutions, Integrability , Non-Linear First Order P.D.E Compatible System

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## Lesson 39

Non - linear p.d.e of $1^{\text {st }}$ order complete integral - Charpit's method

### 39.1 Non - linear p.d.e of $1^{\text {st }}$ order complete integral - Charpit's method.

Given a first order p.d.e

$$
\begin{equation*}
f(x, y, z, p, q)=0 \tag{39.1}
\end{equation*}
$$

its complete integral can be obtained by considering a one parameter family p.d.e.

$$
\begin{equation*}
g(x, y, z, p, q, a)=0 \tag{39.2}
\end{equation*}
$$

which is compatible with $f=0$ for each value of $a$. We know if $f=0$ and $g=0$ are compatible, they admit common solutions.

Choose equation (38.2) such that (a) equations (38.1) and (38.2) on solving for $p$ and $q$ give

$$
\begin{equation*}
p=\phi(x, y, z, a) \text { and } q=\psi(x, y, z, a) \tag{39.3}
\end{equation*}
$$

and (b) the equation

$$
\begin{equation*}
d z=\phi d x+\psi d y \tag{39.4}
\end{equation*}
$$

is integrable. When such a p.d.e $g(x, y, z, p, q, a)=0$ is found, the solution of equation (39.4), which can be written as:

$$
\begin{equation*}
F(x, y, z, a, b)=0 \tag{39.5}
\end{equation*}
$$

containing two arbitrary constants $a$ and $b$ will form the complete integral of (39.1).

Now we see the Construction of such $g(x, y, z, p, q, a)=0$.

As $f=0$ and $g=0$ are compatible, we have

$$
\begin{equation*}
[f, g]=f_{p} \frac{\partial g}{\partial x}+f_{q} \frac{\partial g}{\partial y}+\left(p f_{p}+q f_{q}\right) \frac{\partial g}{\partial z}-\left(f_{x}+p f_{z}\right) \frac{\partial g}{\partial p}-\left(f_{y}+q f_{z}\right) \frac{\partial g}{\partial q}=0 . \tag{39.6}
\end{equation*}
$$

Note that equation (38.6) is obtained by expanding equation

$$
[f, g]=\frac{\partial(f, g)}{\partial(x, p)}+p \frac{\partial(f, g)}{\partial(z, p)}+\frac{\partial(f, g)}{\partial(y, q)}+q \frac{\partial(f, g)}{\partial(z, q)}=0
$$

Equation (39.6) is a quasi linear first order p.d.e for $g$ with $x, y, z, p$ and $q$ as the independent variables, and the corresponding characteristic system is

$$
\begin{equation*}
\frac{d x}{f_{p}}=\frac{d y}{f_{q}}=\frac{d z}{p f_{p}+q f_{q}}=-\frac{d p}{f_{x}+p f_{z}}=-\frac{d q}{f_{y}+q f_{z}} . \tag{39.7}
\end{equation*}
$$

Now we consider any solution of this system which involves $p$ or $q$ or both, which contains an arbitrary constant. This choice gives us a $g(x, y, z, p, q, a)=0$.

Example 1: Find a complete integral of $f=x p q+y q^{2}-1=0$

## Solution:

Equation (39.7) becomes

$$
\frac{d x}{x q}=\frac{d y}{2 y q+x p}=\frac{d z}{2 x p q+2 y q^{2}}=-\frac{d p}{p q}=-\frac{d q}{q^{2}}
$$

For finding $g=0$ chose $\frac{d p}{p q}=\frac{d q}{q^{2}} \Rightarrow p=a q$.

Now we write $g(x, y, z, p, q, a)=p-a q=0$. Since $f=0$ and $g=0$ are compatible, we find $\quad p=\phi(x, y, z, a)=\frac{a}{\sqrt{a x+y}}$ and $q=\psi(x, y, z, a)=\frac{1}{\sqrt{a x+y}}$
such that $d z=\frac{a}{\sqrt{a x+y}} d x+\frac{1}{\sqrt{a x+y}} d y \quad$ Is $\quad$ integrable, i.e. $d z=\frac{a d x+d y}{\sqrt{a x+y}}$ $\Rightarrow(z+b)=2 \sqrt{a x+y}$ or $(z+b)^{2}=4(a x+y)$ is the complete integral which may be written as $F(x, y, z, a, b)=0$. We also note that the matrix $\left(\begin{array}{lll}F_{a} & F_{a x} & F_{a y} \\ F_{b} & F_{b x} & F_{b y}\end{array}\right)$ is of rank two (verify!).

Example 2: Solve $f=q+x p-p^{2}$.

Solution: Equation (36.7) gives the characteristic system as

$$
\frac{d x}{x-2 p}=\frac{d y}{1}=\frac{d z}{-2 p^{2}+x p+q}=\frac{d p}{-p}=\frac{d q}{0} .
$$

Chose $g(x, y, z, p, q, a)=0$ as $p=a e^{-y}$ or $g=p-a e^{-y}=0$.

Solving $f=0, g=0$ for $q$, we get $q=-a x e^{-y}+a^{2} e^{-2 y}$.
Then $d z=p d x+q d y$ becomes $\quad d z=a e^{-y} d x+\left(a^{2} e^{-2 y}-a x e^{-y}\right) d y$

On integrating we get $z=a x e^{-y}-\frac{1}{2} a^{2} e^{-2 y}+b$ as the complete integral of $f=0$ where $a$ and $b$ are arbitrary constants.

## Exercises:

1. Find a complete integral of $f=z^{2}-p q x y=0$ by Charpit's method.
2. Find a complete integral of the non-linear p.d.e

$$
f=\left(p^{2}+q^{2}\right) y-q z=0 .
$$

3. Use Charpit's method to solve the non-linear $1^{\text {st }}$ order p.d.e

$$
f=x^{2} p^{2}+y^{2} q^{2}-4=0 .
$$

4. Solve $16 p^{2} z^{2}+9 q^{2} z^{2}+4 z^{2}-4=0$.
5. Solve $p=(z+q y)^{2}$.
6. Find the complete integral of $2(y+z q)=q(x p+y q)$.

Keywords: Characteristic System, Complete Integral

## References

Amaranath, T. (2003). An Elementary Course in Partial Differential Equations. Narosa Publishing House, New Delhi

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## Lesson 40

## Special Types of First Order Non-Linear p.d.e

### 40.1 Special Types of First Order Non-Linear p.d.e

We now consider 4 special types of first order non-linear p.d.es for which the complete integral can be obtained easily. The underlying principle in the first three types is that of the Charpit's method.

Consider the general p.d.e is $f(x, y, z, p, q)=0$.

Type I: The equation is free from $x, y, z$, i.e., $f(p, q)=0$

Here $f_{x}=0, f_{y}=0, f_{z}=0$.

The auxiliary system equation (38.7) simplifies to $\frac{d x}{f_{p}}=\frac{d y}{f_{q}}=\frac{d z}{p f_{p}+q f_{q}}=\frac{d p}{0}=\frac{d q}{0}$.

On solving, we get either $p=a$ (or $q=a$ ). Without loss of generality, take $g=p-a=0$ Using this, find $q$ from $f=0$, denote it by $q=Q(a)$.

Then $d z=a d x+Q(a) d y$, on integration we get $z=a x+Q(a) y=b$ as the complete integral of $f(p, q)=0$.

## Example 1

Find a complete integral of $f=p(1-q)+q=0$.

Solution: We have $\frac{d x}{1-q}=\frac{d y}{-p+1}=\frac{d z}{p+q-2 p q}=\frac{d p}{0}=\frac{d q}{0}$.

From the last equation, we have $q=a$, constant.

Now using this in $f=p(1-q)+q=0$ we get
$p(1-a)+a=0 \Rightarrow p=\frac{a}{a-1}$.
$\therefore d z=\frac{a}{a-1} d x+a d y \Rightarrow z=\frac{a}{a-1} x+a y=b$ is the complete integral.

Type II: The equation is free from $x, y$, i.e., $f(z, p, q)=0$.

From the characteristic system of equations we consider $\frac{d p}{p}=\frac{d q}{q}$, on solving we get $p=a q$. Using this we find $q$ as $q=Q(a, z)$.
[Note: similarly, one can write $q=a p$ and $p=Q(a, z)$ ]

Now $d z=p d x+q d y=Q(a, z)(a d x+d y)$, on integrating we get
$\int \frac{d z}{Q(a, z)}=a x+y+b$ as the complete integral.

## Example 2

$z=p^{2}+q^{2}$

Solution: Choose $p=a q \Rightarrow q^{2}=z-a^{2} q^{2}$
$q=\frac{\sqrt{z}}{\left(1+a^{2}\right)^{\frac{1}{2}}}$.
$d z=\frac{a \sqrt{z}}{\left(1+a^{2}\right)^{\frac{1}{2}}} d x+\frac{1}{\left(1+a^{2}\right)^{\frac{1}{2}}} \sqrt{z} d y$
or $\int \frac{1}{\sqrt{z}} d z=\int \frac{a d x+d y}{\left(1+a^{2}\right)^{\frac{1}{2}}}=\frac{1}{\left(1+a^{2}\right)^{\frac{1}{2}}}(a x+y)+b$,
or on simplifying we get
$4\left(1+a^{2}\right) z=(a x+y+b)^{2}$ as the complete integral.

Type III: Consider a special form for $f(x, y, z, p, q)=0$ in a separable type such as $g(x, p)=h(y, q)$.

In this case, the auxiliary equations are
$\frac{d x}{g_{p}}=\frac{d y}{-h_{q}}=\frac{d z}{p g_{p}-q h_{q}}=\frac{d p}{-g_{x}}=\frac{d q}{h_{y}}$
Solving the first and fourth together, we get $g_{x} d x+g_{p} d p=0$ or $d g(x, p)=0 \Rightarrow g(x, p)=a$, a constant.

Since $g(x, p)=h(y, q) \Rightarrow h(y, q)=a$,
solving for $p$ and $q$, we get $p=A(a, x)$ and $q=B(a, y)$
and the complete integral becomes $z=\int A(a, x) d x+\int B(a, y) d y+b$.

## Example 3:

Solve $p-x^{2}=q+y^{2}$.

Solution: The auxiliary equations are $\frac{d x}{1}=\frac{d y}{-1}=\frac{d z}{p+q}=\frac{d p}{2 x}=\frac{d q}{2 y}$.

The first and the fourth equations give $2 x d x-d p=0 \Rightarrow p-x^{2}=a \Rightarrow p=a+x^{2}$ also $q+y^{2}=a \Rightarrow q=a-y^{2}$. hence $z=\int\left(a+x^{2}\right) d x+\int\left(a-y^{2}\right) d y+b, \quad$ or $z=a x+\frac{x^{3}}{3}+a y-\frac{y^{3}}{3}+b$ is the complete solution.

Type IV: The p.d.e is in the special form given by $z=p x+q y+h(p, q)$
which is known as the Clairaut equation. Its complete integral is written as $z=a x+b y+h(a, b)$, which clearly satisfies the given p.d.e and also the rank of the $\operatorname{matrix}\left(\begin{array}{lll}x+h_{a} & 1 & 0 \\ y+h_{b} & 0 & 1\end{array}\right)$ is two.

## Example 4

The complete integral of $z=p x+q y+\log p q$ is the surface given by $z=a x+b y+\log a b$.

## Exercises

Find the complete integral of the p.d.es

1. $p^{2}+q^{2}=9$.
2. $p q+p+q=0$.
3. $z=p x+q y+p^{2} q^{2}$.
4. $p\left(1-q^{2}\right)=q(1-z)$.
5. $1+p^{2}=q z$.
6. $q+p x=p^{2}$.
7. $\sqrt{p}-\sqrt{q}+3 x=0$.
8. $x y p+q y+p q=y z$.
9. $z\left(p^{2}+q^{2}\right)+p x+q y=0$.

Keywords: Charpit's method, Clairaut equation.

## References

Ian Sneddon, (1957). Elements of Partial Differential Equations. McGraw-Hill, Singapore

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## Lesson 41

## Classification of Semi-linear $\mathbf{2 d}^{\text {nd }}$ order Partial Differential Equations

### 41.1 Classification of $2^{\text {nd }}$ order Partial Differential Equations: Parabolic Hyperbolic - Elliptic Equations

$\mathrm{A} 2^{\text {nd }}$ order semi linear partial differential equation can be put in the form $L u+g\left(x, y, u, u_{x}, u_{y}\right)=0(41.1)$
where the linear operator $L \equiv R(x, y) \frac{\partial^{2}}{\partial x^{2}}+S(x, y) \frac{\partial}{\partial x \partial y}+T(x, y) \frac{\partial^{2}}{\partial y^{2}}$ is such that the coefficient functions $R, S$ and $T$ are continuous function of x , y and $R^{2}+S^{2}+T^{2} \neq 0$.

We change the independent variables $(x, y)$ to $(\xi, \eta)$ as $\xi=\xi(x, y)$ and $\eta=\eta(x, y)$, to enforce the one - to - one correspondence of this transformation, we assume $\xi_{x} \eta_{y}-\eta_{x} \xi_{y} \neq 0$.

The coefficients and the partial derivatives in the given equation are written in terms of the transformed variables. The first and second order partial derivatives become:

$$
\begin{aligned}
& u_{x}=u_{\xi} \xi_{x}+u_{\eta} \eta_{x} ; \quad u_{y}=u_{\xi} \xi_{y}+u_{\eta} \eta_{y} \\
& u_{x y}=u_{\xi \xi} \xi_{y} \xi_{x}+u_{\xi \eta} \eta_{y} \xi_{x}+u_{\xi} \xi_{x y}+u_{\xi \eta} \xi_{y} \eta_{x}+u_{\eta \eta} \eta_{y} \eta_{x}+u_{\eta} \eta_{x y} \\
& u_{x x}=u_{\xi \xi} \xi_{x}^{2}+u_{\xi \eta} \eta_{x} \xi_{x}+u_{\xi} \xi_{x x}+u_{\xi \eta} \xi_{x} \eta_{x}+u_{\eta \eta} \eta_{x}^{2}+u_{\eta} \eta_{x x}
\end{aligned}
$$

$$
\begin{aligned}
& u_{y y}=u_{\xi \xi} \xi_{y}^{2}+u_{\xi \eta} \eta_{y} \xi_{y}+u_{\xi} \xi_{y y}+u_{\xi \eta} \xi_{y} \eta_{y}+u_{\eta \eta} \eta_{y}^{2}+u_{\eta} \eta_{y y} \\
& \therefore R u_{x x}+S u_{x y}+T u_{y y}=u_{\xi \xi}\left(R \xi_{x}^{2}+S \xi_{x} \xi_{y}+T \xi_{y}^{2}\right) \\
& +u_{\xi \eta}\left(2 R \eta_{x} \xi_{x}+S\left(\eta_{y} \xi_{x}+\xi_{y} \eta_{x}\right)+2 T \xi_{y} \eta_{y}\right) \\
& +u_{\eta \eta}\left(R \eta_{x}^{2}+S \eta_{x} \eta_{y}+T \xi_{y}^{2}\right)+F\left(\xi, \eta, u_{\xi}, u_{\eta}, u\right)
\end{aligned}
$$

Equation (41.1) becomes

$$
\begin{equation*}
A\left(\xi_{x}, \xi_{y}\right) u_{\xi \xi}+2 B\left(\xi_{x}, \xi_{y} ; \eta_{x}, \eta_{y}\right) u_{\xi \eta}+A\left(\eta_{x}, \eta_{y}\right) u_{\eta \eta}=G\left(\xi, \eta, u, u_{\xi}, u_{\eta}\right)( \tag{41.2}
\end{equation*}
$$

where $A(u, v)=R u^{2}+S u v+T v^{2}(41.3)$

$$
B\left(u_{1}, v_{1} ; u_{2}, v_{2}\right)=R u_{1} u_{2}+\frac{1}{2} S\left(u_{1} v_{2}+u_{2} v_{1}\right)+T v_{1} v_{2} \text { (41.4) }
$$

Now, the problem is to determine $\xi \& \eta$ so that the equation (41.2) takes the simplest possible (Canonical) form.

When the sign of the determinant $S^{2}-4 R T$ of the quadratic form (41.3) is everywhere positive, negative or zero it is easy to make the classification.

Case A: When $S^{2}-4 R T>0$ everywhere in the domain.
The new independent variable $\xi \& \eta$ can be so chosen that the coefficients of $u_{\xi \xi}$ and $u_{\eta \eta}$ in (41.2) vanish.

The roots $\lambda_{1} \& \lambda_{2}$ of the equation $R \alpha^{2}+S \alpha+T=0$ are real and distinct. The coefficient of $u_{\xi \xi} \& u_{\eta \eta}$ in (41.2) will vanish if we close $\xi \& \eta$ such that
$\frac{\partial \xi}{\partial x}+\lambda_{1} \frac{\partial \xi}{\partial y} ; \frac{\partial y}{\partial x}=\lambda_{2} \frac{\partial y}{\partial y}$

A suitable choice will be $\xi=f_{1}(x, y), \eta=f_{2}(x, y)$ where
$f_{1}(x, y)=c_{1}, \quad f_{2}(x, y)=c_{2}$ are the solution of the ordinary differential equations
$\frac{d y}{d x}+\lambda_{1}(x, y)=0 ; \frac{d y}{d x}+\lambda_{2}(x, y)=0$ respectively.

It can be verified that
$A\left(\xi_{x}, \xi_{y}\right) A\left(\eta_{x}, \eta_{y}\right)-B^{2}\left(\xi_{x}, \xi_{y} ; \eta_{x}, \eta_{y}\right)=\left(4 R T-S^{2}\right)\left(\xi_{x} \eta_{y}-\xi_{y} \eta_{x}\right)^{2} /$

Now when the $A$ 's are zero;
$B^{2}=\left(S^{2}-4 R T\right)\left(\xi_{x} \eta_{y}-\xi_{y} \eta_{x}\right)^{2}$
Since $S^{2}-4 R T>0 \Rightarrow B^{2}>0$, hence equation (41.2) reduces to $\frac{\partial^{2} u}{\partial \xi \partial \eta}=\phi\left(\xi, \eta, u_{\xi}, u_{\eta}, u\right)(41.6)$

The curves $\xi(x, y)=$ constant, $\eta(x, y)=$ constant are called the characteristic curves of equation (41.1).

Equation (41.6) is called the canonical form of equation

## Example 1

Reduce the equation $u_{x x}-x^{2} u_{y y}=0$ to a canonical form.

## Solution

comparing with the standard form, we note that $R=1, S=0, T=x^{2}$

Then $S^{2}-R T=4 x^{2}>0$.

So $R \alpha^{2}+S \alpha+T=0$ becomes $\alpha^{2}-x^{2}=0 \Rightarrow \alpha= \pm x$.
$\Rightarrow \lambda_{1}=x ; \lambda_{2}=-x$

Now $\frac{d y}{d x}+x=0 \Rightarrow y+\frac{1}{2} x^{2}=c_{1}$
$\frac{d y}{d x}-x=0 \Rightarrow y-\frac{1}{2} x^{2}=c_{2}$

Taking $\xi=y+\frac{1}{2} x^{2} ; \eta=y-\frac{1}{2} x^{2}$
$u_{x}=u_{\xi} \xi_{x}+u_{\eta} \eta_{x}=u_{\xi} x+u_{\eta}(-x)=u_{\xi} x-u_{\eta} x$
$u_{y}=u_{\xi}+u_{\eta}, u_{x x}=x^{2} u_{\xi \xi}-2 x^{2} u_{\xi \eta}+x^{2} u_{\eta \eta}+u_{\xi}-u_{\eta}, u_{y y}=u_{\xi \xi}+2 u_{\xi \eta}+u_{\eta \eta}$
$\therefore u_{x x}-x^{2} u_{y y}=0$ becomes $u_{\xi \eta}=\frac{1}{4 x^{2}}\left(u_{\xi}-u_{\eta}\right)=\frac{1}{4(\xi-\eta)}\left(u_{\xi}-u_{\eta}\right)$.

Case B: If $S^{2}-4 R T=0$
Roots of the equation $R \alpha^{2}+S \alpha+T=0$ are real and equal. We define $\xi$ as in case A and take $\eta$ to be any function of $x, y$ which is independent of $\xi$. In this case we have $A\left(\xi_{x}, \xi_{y}\right)=0$ as before and hence from equation (41.5), $B=0$.

But $A\left(\eta_{x}, \eta_{y}\right) \neq 0 \because \xi \& \eta$ are independent functions.

Hence the canonical form in this case is $\frac{\partial^{2} u}{\partial \eta^{2}}=\phi\left(\xi, \eta, u, u_{x}, u_{y}\right)$

## Example 2

$u_{x x}+2 u_{x y}+u_{y y}=0$ canonical form.

## Solution

comparing with the standard form, we note that
$R=1, S=2, T=1$, and $S^{2}-4 R T=0$.
$R \alpha^{2}+S \alpha+T=\alpha^{2}+2 \alpha+1=(\alpha+1)^{2}=0 \Rightarrow \alpha=-1,-1$.
$\therefore \frac{d y}{d x}-1=0 \Rightarrow x-y=c_{1}, \quad$ take $\xi=x-y$
Then chose $\eta=x+y$.

Using these $\xi \& \eta$ : we have the canonical form as $\frac{\partial^{2} \xi}{\partial \eta^{2}}=0$
$\Rightarrow \xi=\eta f_{1}(\xi)+f_{2}(\xi)$ where $f_{1} \& f_{2}$ are arbitrary functions.
Hence the solution of the given equation is:
$z=(x+y) f_{1}(x-y)+f_{2}(x-y)$.

Case C: $S^{2}-4 R T<0$.
In this case, the roots of the equation $R \alpha^{2}+S \alpha+T=0$ are complexconjugates.
Proceeding as in case A; the canonical form $\frac{\partial^{2} u}{\partial \eta^{2}}=\phi\left(\xi, \eta, u, u_{x}, u_{y}\right)$.

But $\xi \& \eta$ are complex conjugates.To get the real canonical form, we use the transformation $\alpha=\frac{1}{2}(\xi+\eta), \beta=\frac{1}{2} i(\eta-\xi) \Rightarrow \frac{\partial^{2} u}{\partial_{\xi} \partial \eta^{2}}=\frac{1}{4}\left(\frac{\partial^{2} u}{\partial \alpha^{2}}+\frac{\partial^{2} u}{\partial \beta^{2}}\right)$.

So the canonical form in this case is $\frac{\partial^{2} u}{\partial \alpha^{2}}+\frac{\partial^{2} u}{\partial \beta^{2}}=\varphi\left(\alpha, \beta, u, u_{\alpha}, u_{\beta}\right)$.

## Example 3

Reduce the equation $u_{x x}+x^{2} u_{y y}=0$ to canonical form.

## Solution

Clearly, $R=1, S=0, T=x^{2}$, and $S^{2}-4 R T<0$.
$\alpha^{2}+x^{2}=0 \Rightarrow \alpha= \pm i x$, hence $\lambda_{1}=i x ; \lambda_{2}=-i x$,
$\xi=i y+\frac{1}{2} x^{2} ; \eta_{2}=-i y+\frac{1}{2} x^{2}, \alpha=\frac{1}{2} x^{2} ; \beta=y$
$\Rightarrow u_{\alpha \alpha}+u_{\beta \beta}=-\frac{1}{2 \alpha} u_{\alpha}$ is the canonical form

No we classify second order equation of the type (41.1) by their canonical form as:
A) Hyperbolic if $S^{2}-4 R T>0$, B) Parabolic if $S^{2}-4 R T=0$,
C) Elliptic if $S^{2}-4 R T<0$.

Clearly the one dimensional wave equation given by $u_{t t}=c^{2} u_{x x}$ is an example for the Hyperbolic equation,
the one dimensional heat conduction equation given by $u_{t}=\alpha u_{x x}$ is an example for the parabolic equation and the Laplace equation $u_{x x}+u_{y y}=0$ is an example for the elliptic equation.

## Example 4

Discuss the nature of the equation

## Solution

Clearly $S^{2}-R T=\left(x^{2}+y^{2}-2\right)$.
Hence the given equation is Hyperbolic at all points $(x, y)$ such that $x^{2}+y^{2}>2$, Parabolic if $x^{2}+y^{2}=2$ and Elliptic if $x^{2}+y^{2}<2$.

## Exercises

1. Reduce the equation to its canonical form and classify

$$
\mathrm{it}: u_{t t}+4 u_{t x}+4 u_{x x}+2 u_{x}-u_{t}=0
$$

2. Classify the partial differential equation:

$$
u_{t t}+\left(5+2 x^{2}\right) u_{t x}+\left(1+x^{2}\right)\left(4+x^{2}\right) u_{x x}=0
$$

Keywords: Elliptic ,Hyperbolic, Parabolic,

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## Lessons 42

## Solution of Homogeneous and Non-Homogeneous Linear Partial Differential Equations

### 42.1 Introduction

Consider the homogeneous linear equations with constant coefficients $k_{i}$ 's as

$$
\left(D^{n}+k_{1} D^{n-1} D^{\prime}+\ldots+k_{n} D^{\prime n}\right) z=f(x, y) \quad \text { or } F\left(D, D^{\prime}\right) z=f(x, y) \quad \text { where }
$$ $F\left(D, D^{\prime}\right)=\sum_{z} \sum_{s} C_{r s} D^{r} D^{\prime s}, C_{r s}$ are constants $\& D=\frac{\partial}{\partial x} ; D^{\prime}=\frac{\partial}{\partial y}$.

Let us find the Complementary function for this equation.

Result 1: If $u$ is the complementary function and $z_{1}$ a particular integral of a linear differential equation, then $u+z_{1}$ is a general solution of the equation.

We have $F\left(D, D^{\prime}\right) u=0$
and $F\left(D, D^{\prime}\right) z_{1}=f(x, y)$
$\therefore F\left(D, D^{\prime}\right)\left(u+z_{1}\right)=f(x, y)$.

Result 2: If $u_{1}, u_{1}, \ldots, u_{n}$ are solutions of the homogeneous linear partial differential equation $F\left(D, D^{\prime}\right) z=0$, then $c_{1} u_{1}+c_{2} u_{2}+\ldots \ldots . c_{n} u_{n}$ is also a solution; $C_{r}$ 's are arbitrary constants.

Let $F\left(D, D^{\prime}\right)$ be a Linear partial differential operator.

This operator is said to be reducible if it can be written as the product of linear function of the form $\left(D+a r^{\prime}+b\right)$ with $\mathrm{a}, \mathrm{b}$ are constants. For example:

$$
\left(D^{2}-D^{\prime 2}\right)=\left(D+D^{\prime}\right)\left(D-D^{\prime}\right)
$$

It is said to be irreducible if it cannot be so written. For example $\left(D^{r}-D^{\prime}\right)$ is irreducible.

### 42.2 Reducible Equations

Result 3: If the operator $F\left(D, D^{\prime}\right)$ is reducible, the order in which the linear factors occur is unimportant. Any reducible operator can be written in the form.

We have $\left(\alpha_{r} D+\beta_{r} D^{\prime}+r_{r}\right)\left(\alpha_{s} D+\beta_{s} D^{\prime}+r_{s}\right)$
$=\alpha_{r} \alpha_{s} D^{2}+\left(\alpha_{s} \beta_{r}+\alpha_{r} \beta_{s}\right) D D^{\prime}+\beta_{r} \beta_{s} D^{\prime 2}+\left(r_{s} \alpha_{r}+r_{r} \alpha_{s}\right) D+\left(r_{s} \beta_{r}+r_{r} \beta_{s}\right) D^{\prime}+r_{r} r_{s}$
$=\left(\alpha_{s} D+\beta_{s} D^{\prime}+r_{s}\right)\left(\alpha_{r} D+\beta_{r} D^{\prime}+r_{r}\right)$

Similarly this is true for any product of finite number of factors.

Result 4: If $\left(\alpha_{r} D+\beta_{r} D^{\prime}+\gamma_{r}\right)$ is a factor of $F\left(D, D^{\prime}\right)$ and $\phi_{r}(\xi)$ is an arbitrary function of the simple variable $\xi$, then if $\alpha_{r} \neq 0$.

$$
u_{r}=\exp \left(-\frac{\gamma_{r} x}{\alpha_{r}}\right) \phi_{r}\left(\beta_{r} x-\alpha_{r} y\right)
$$

is a solution of the equation $F\left(D, D^{\prime}\right) z=0$.

Proof: $D u_{r}=-\frac{\gamma_{r}}{\alpha_{r}} u_{r}+\beta_{r} \exp \left(\frac{\gamma_{r} x}{\alpha_{r}}\right) \phi^{\prime}\left(\beta_{r} x-\alpha_{r} y\right)$
$D^{\prime} u_{r}=-\alpha_{r} \exp \left(\frac{\gamma_{r} x}{\alpha_{r}}\right) \phi^{\prime}\left(\beta_{r} x-\alpha_{r} y\right)$
so that $\left(\alpha_{r} D+\beta_{r} D^{\prime}+\gamma_{r}\right) u_{r}=0$

Now $F\left(D, D^{\prime}\right)=\left\{\prod_{s=1}^{n}\left(\alpha_{s} D+\beta_{s} D^{\prime}+\gamma_{s}\right)\right\}\left(\alpha_{r} D+\beta_{r} D^{\prime}+\gamma_{r}\right) u_{r}$

The prime after the product denotes that the factor corresponding to $s=r$ is omitted. Combining (42.1) \& (42.2) we get $F\left(D, D^{\prime}\right) u_{r}=0$.

Result 5: If $\left(\beta_{r} D^{\prime}+\gamma_{r}\right)$ is a factor of $F\left(D, D^{\prime}\right)$ and $\phi_{r}(\xi)$ is an arbitrary function of the simple variable $\xi$, then if $\beta_{r} \neq 0 ; u_{r}=\exp \left(\frac{\gamma_{r} x}{\alpha_{r}}\right) \phi_{r}\left(\beta_{r} x\right)$ is a solution of the equation $F\left(D, D^{\prime}\right) z=0$.

Proof: Similar lines to that of result 5 .

If $F\left(D, D^{\prime}\right)$ is decomposed into linear factors such that $\left(\alpha_{r} D+\beta_{r} D^{\prime}+\gamma_{r}\right)$ is a multiple factor; (say $\mathrm{n}=2$ ) then the solution of $F\left(D, D^{\prime}\right) z=0$ is obtained as given below:
$\left(\alpha_{r} D+\beta_{r} D^{\prime}+\gamma_{r}\right)^{2} z=0$
Let $Z=\left(\alpha_{r} D+\beta_{r} D^{\prime}+\gamma_{r}\right) z$.
Then $\left(\alpha_{r} D+\beta_{r} D^{\prime}+\gamma_{r}\right)^{2} Z=0$.

Then by result (4), it has solution

$$
Z=\exp \left(-\frac{\gamma_{r} x}{\alpha_{r}}\right) \phi_{r}\left(\beta_{r} x-\alpha_{r} y\right) \text { if } \alpha_{r} \neq 0
$$

To find $Z$; we have to solve

$$
\left(\alpha_{r} \frac{\partial z}{\partial x}+\beta_{r} \frac{\partial y}{\partial x}+\gamma_{r} z\right)=e^{-\frac{r_{r} x}{\alpha_{r}}} \phi_{r}\left(\beta_{r} x-\alpha_{r} y\right)
$$

Solution: $\frac{d x}{\alpha_{r}}=\frac{d y}{\beta_{r}}=\frac{d z}{-r_{r} z+e^{-\frac{r \alpha_{r} x}{\alpha_{r}}} \phi_{r}\left(\beta_{r} x-\alpha_{r} y\right)}$
With solution:
$\frac{d x}{\alpha_{r}}=\frac{d y}{\beta_{r}} \Rightarrow \beta_{r} x-\alpha_{r} y=C_{1}$
and $\frac{d x}{\alpha_{r}}=\frac{d z}{r_{r} Z+e^{-\frac{r_{r} x}{\alpha_{r}}} \phi_{r} C_{1}}$
$\Rightarrow z=\frac{1}{\alpha_{r}}\left\{\phi_{r}\left(C_{1}\right) x+C_{2}\right\} e^{-\frac{r_{r} x}{\alpha_{r}}}$
$\therefore z=x \phi_{r}\left(\beta_{r} x-\alpha_{r} y\right)+\varphi_{r}\left(\beta_{r} x-\alpha_{r} y\right) e^{-\frac{r_{r} x}{\alpha_{r}}}$
is the solution. $\phi_{r} \& \varphi_{r}$ are arbitrary.

Result 6: (This is generalization of result 5) If $\left(\alpha_{r} D+\beta_{r} D^{\prime}+\gamma_{r}\right)^{n}\left(\alpha_{r} \neq 0\right)$ is a factor of $F\left(D, D^{\prime}\right)$ and if the functions $\phi_{r_{1}}, \phi_{r_{2}}, \ldots, \phi_{r_{n}}$ are arbitrary, then $\exp \left(-\frac{\gamma_{r} X}{\alpha_{r}} \sum_{s=1}^{n} X^{s-1} \phi_{r s}\left(\beta_{r} x-\alpha_{r} y\right)\right.$ is a solution of $F\left(D, D^{\prime}\right)=0$.

Result 7: If $\left(\beta_{r} D^{\prime}+\gamma_{r}\right)^{m}$ is a factor of $F\left(D, D^{\prime}\right)$ and if the functions $\phi_{r_{1}}, \phi_{r_{2}}, \ldots, \phi_{r_{n}} \quad$ are arbitrary, then $\exp \left(-\frac{\gamma_{r} y}{\beta_{r}}\right) \sum_{s=1}^{m} x^{s-1} \phi_{r s}\left(\beta_{r} x\right)$ is a solution of $F\left(D, D^{\prime}\right) z=0$.

Complementary function of $F\left(D, D^{\prime}\right) z=f(x, y)$ when $F\left(D, D^{\prime}\right)$ is reducible. We have $F\left(D, D^{\prime}\right)=\sum_{s=1}^{n}\left(\alpha_{r} D+\beta_{r} D^{\prime}+\gamma_{r}\right)^{m_{r}}$ and if none of $\alpha_{r}^{\prime} s$ is zero, then the corresponding complementary function is:
$u=\sum_{r=1}^{n} \exp \left(-\frac{\gamma_{r} x}{\alpha_{r}}\right) \sum_{s=1}^{n} x^{s-1} \phi_{r s}\left(\beta_{r} x-\alpha_{r} y\right)$ where $\phi_{r s}\left(s=1,2, \ldots, n_{r} ; r=1,2, \ldots, n\right)$ are arbitrary.

Consider the second order equation $\frac{\partial^{2} z}{\partial x^{2}}+k_{1} \frac{\partial^{2} z}{\partial x \partial y}+k_{2} \frac{\partial^{2} z}{\partial y^{2}}=0$
which is written in the operator form as $\left(D^{2}+k_{1} D D+k_{2} D^{\prime 2}\right) z=0$.
Let its roots be denoted by $\frac{D}{D^{\prime}}=m_{1}, m_{2}$.

Case 1: These roots are real and distinct :
Say $\left(D-m_{1} D^{\prime}\right)\left(D-m_{2} D^{\prime}\right) z=0$
$\left(D-m_{2} D^{\prime}\right) z=0 \Rightarrow \frac{d x}{1}=\frac{d y}{-m_{2}}=\frac{d z}{0} \Rightarrow y+m_{2} x=c_{1}, z=c_{2}$
hence $z=\phi\left(y+m_{2} x\right)$, where $\phi$ is an arbitrary function.
Similarly $\left(D-m_{1} D^{\prime}\right) z=0 \Rightarrow z=f\left(y+m_{1} x\right)$, where $f$ is an arbitrary function. Hence the complete solution is $z=f\left(y+m_{1} x\right)+\phi\left(y+m_{2} x\right)$.

Case 2: Let these roots be repeated, say $m_{1}=m_{2}$, then $\left(D-m_{1} D^{\prime}\right)^{2} z=0$;let $\left(D-m_{1} D^{\prime}\right) z=u$, then $\left(D-m_{1} D^{\prime}\right) u=0 \Rightarrow u=\phi\left(y+m_{1} x\right)$ $\left(D-m_{1} D^{\prime}\right) z=\phi\left(y+m_{1} x\right) \Rightarrow \frac{d x}{1}=\frac{d y}{-m_{1}}=\frac{d z}{\phi\left(y+m_{1} x\right)}$ or $y+m_{1} x=c_{1} ; d z=\phi(u) d x$
or $Z=x \phi\left(y+m_{1} x\right)+c_{2}$ or $z=x \phi\left(y+m_{1} x\right)+f\left(y+m_{2} x\right)$ is the complementary function.

## Example 1

$$
\begin{aligned}
& 2 D^{2}+5 D D^{\prime}+2 D^{\prime 2}=0 \\
& 2 m^{2}+5 m+2=0 \Rightarrow m_{1}=-2, m_{2}=-\frac{1}{2} \\
& z=f_{1}(y-2 x)+f_{2}\left(y-\frac{1}{2} x\right) .
\end{aligned}
$$

## Example 2

$$
\begin{aligned}
& r+b s+q t=0 \\
& m^{2}+b m+q=0 \Rightarrow m=-3,-3 \\
& z=f_{1}(y-3 x)+x f_{2}(y-3 x)
\end{aligned}
$$

## Example 3

$\left(D^{2}-D^{\prime 2}\right) z=0 \cdot m^{2}-1=0 \Rightarrow m= \pm 1$
$z=\phi_{1}(x+y)+\phi_{2}(x-y)$.

## Example 4

Find the complementary function of $\left(D^{4}+D^{\prime 4}\right) z-2 D^{2} D^{\prime 2} z=0$.

## Solution

$$
\begin{aligned}
& \left(D+D^{\prime}\right)^{2}\left(D-D^{\prime}\right)^{2} z=0 \\
& \alpha_{1}=\alpha_{2}=1, \gamma_{1}=0
\end{aligned}
$$

So the solution is: $\beta_{1}=\beta_{2}=1 ; \gamma_{2}=0$ $z=x \phi_{1}(x-y)+\phi_{2}(x-y)+x \varphi_{1}(x+y)+\varphi_{2}(x+y)$ where $\varphi$ arbitrary function.

### 42.3 Particular Integral

Result 8: We have $F\left(D, D^{\prime}\right) e^{a \times b y}=F(a, b) e^{a \times b y}$.
$F\left(D, D^{\prime}\right)$ is made up of term of the type $C_{r s} D^{r} D^{\prime s} ; F\left(D, D^{\prime}\right)=\sum_{r} \sum_{s} C_{r s} D^{r} D^{\prime s}$
and $D^{r} e^{a x+b y}=a^{r} e^{a x+b y}$ and $D^{\prime s} e^{a x+b y}=b^{s} e^{a x+b y}$,
so $C_{r s} D^{r} D^{\prime s}=C_{r s} a^{r} b^{s} e^{a x+b y}$
and $F\left(D, D^{\prime}\right) e^{a x+b y}=F(a, b) e^{a x+b y}$.

Result 9: $F\left(D, D^{\prime}\right)\left\{e^{a x+b y} \phi(x, y)\right\}=e^{a x+b y} F\left(D+a, D^{\prime}+b\right) \phi(x, y)$

Solution: $D^{r} e^{a x} \phi=\sum_{\rho=0}^{r}{ }^{r} C_{p}\left(D^{\rho} e^{a x}\right)\left(D^{r-\rho} \phi\right)=e^{a x} \sum_{\rho=0}^{r}\left({ }^{r} C_{p} a^{\rho} D^{r-\rho}\right) \phi(x, y)$
$=e^{a x}(D+a)^{r} \phi$.
Similarly, $D^{\prime s} e^{b x} \phi=e^{a x}\left(D^{\prime}+a\right)^{s} \phi$.
Hence $F\left(D, D^{\prime}\right) e^{a x+b y} \phi=e^{a x+b y} f\left(D+a, D^{\prime}+a\right) \phi(x, y)$.
$f\left(D, D^{\prime}\right) z=F(x, y) \Rightarrow z=\frac{1}{f\left(D, D^{\prime}\right)} F(x, y)$

Case 1: $\frac{1}{f\left(D, D^{\prime}\right)} e^{a x+b y}=\frac{1}{f(a, b)} e^{a x+b y}$, provided $f(a, b) \neq 0$.

## Case 2:

$$
\begin{aligned}
& f\left(D^{2}, D D^{\prime}, D^{\prime 2}\right) \sin (m x+n y)=f\left(-m^{2},-m n,-n^{2}\right) \sin (m x+n y) \cos (m x+n y) \\
& \therefore z=\frac{1}{f\left(-m^{2},-m n,-n^{2}\right)} \sin (m x+n y) \text { or } \cos (m x+n y) .
\end{aligned}
$$

Case 3: $F(x, y)=x^{m} y^{n}, m, n$ constants. P.I. $=\left[f\left(D, D^{\prime}\right)\right]^{-1} x^{m} y^{n}$.
Case 4: $F(x, y)$ is any function of $x$ and $y$, resolve $\frac{1}{f\left(D, D^{\prime}\right)}$, into partial fractions, treating $f\left(D, D^{\prime}\right)$ as a function of $D$ alone and operate each partial function of $F(x, y)$, remembering that

$$
\frac{1}{\left(D-m D^{\prime}\right)} F(x, y)=\int F(x, c-m x) d x
$$

where $c$ is replaced by $y+m x$ after integration.

## Example 5

Find the solution of $\left(D^{2}-D^{\prime 2}\right) z=x-y$
The complementary function is : $\phi_{1}(x+y)+\phi_{2}(x-y)$.

The particular integral is obtained as:
Let $z_{1}=\left(D+D^{\prime}\right) z$
Then $\left(D-D^{\prime}\right) z_{1}=x-y$
$\frac{\partial z_{1}}{\partial x}-\frac{\partial z_{1}}{\partial y}=x-y \Rightarrow z_{1}=\frac{1}{u}(x-y)^{2}+f(x+y), \quad f$ is arbitrary.

## Exercises

Find the solution of the linear p.d.e with constant coefficients:

1. $D^{2}+4 D D^{\prime}-5 D^{\prime 2} z=\sin (2 x+3 y)$
2. $\left(D^{2}-D D^{\prime}\right) z=\cos x \cos 2 y$
3. $D^{3}-2 D^{2} D^{\prime}=2 e^{2 x}+3 x^{2} y$
4. $4 D^{2}-4 D D^{\prime}+D^{\prime 2}=16 \log (x+2 y)$.

### 42.4 The complementary function of irreducible equations

$F\left(D, D^{\prime}\right) z=f(x, y)$
Irreducible factors are treated as follows:
Case 1: The particular integral $z=\frac{1}{F\left(D, D^{\prime}\right)} f(x, y)$ is obtained by Expanding the operator $F^{-1}$ by the binomial theorem and then interpret the operator $D^{-1}, D^{\prime-1}$ as integration.

## Example 6

Find a Particular Integral of the equation $\left(D^{2}-D^{\prime}\right) z=2 y-x^{2}$.

## Solution

$$
\begin{aligned}
& z=\frac{1}{\left(D^{2}-D^{\prime}\right)}\left(2 y-x^{2}\right)=-\left(1-\frac{D^{2}}{D^{\prime}}\right)^{-1} \frac{1}{D^{\prime}}\left(2 y-x^{2}\right) \\
& \text { or } z=\left(1-\frac{1}{D^{\prime}}-\frac{D^{2}}{D^{\prime 2}}-\frac{D^{4}}{D^{\prime 3}}-\ldots \ldots\right)\left(2 y-x^{2}\right) \\
& =\left(-y^{2}+x^{2} y\right)-\frac{1}{D^{\prime 2}}(-2)-\ldots . \\
& =-y^{2}+x^{2} y+y^{2}=x^{2} y .
\end{aligned}
$$

Case 2: If $f(x, y)$ is made of term of the form $\exp (a x+b y)$
then P.I is: $\frac{1}{F(a, b)} e^{(a x+b y)}$ if $F(a, b) \neq 0$.
$\left(D^{2}-D^{\prime}\right) z=e^{(a x+b y)}, F(a, b)=3 \neq 0$
So $\frac{1}{\left(D^{2}-D^{\prime}\right)} e^{(a x+b y)}=\frac{1}{3} e^{(a x+b y)}$.
If $F(a, b)=0$ then $z=w e^{(a x+b y)}$
and $F\left(D+a, D^{\prime}+b\right) w=c$.
$F\left(D, D^{\prime}\right) z=c e^{(a x+b y)}$.

## Example 7

Find the particular solution of $\left(D^{2}-D^{\prime}\right) z=e^{(a x+b y)}$.
Clearly $F(1,1)=0$.
$F\left(D+1, D^{\prime}+1\right)=(D+1)^{2}-\left(D^{\prime}+1\right)=D^{2}+2 D-D^{\prime}$
or $\left(D^{2}+2 D-D^{\prime}\right) w=1$
$\frac{1}{-D^{\prime}\left(1-\frac{D^{2}+2 D}{D^{\prime}}\right)} \cdot 1=\frac{-1}{D^{\prime}}\left(1-\frac{D^{2}+2 D}{D^{\prime}}\right) \cdot 1=\left\{\begin{array}{c}\frac{1}{2} x \\ -y\end{array}\right\}$
$\therefore$ P.I. are $\frac{1}{2} x e^{(a x+b y)} \&-y e^{x+y}$.
$f(x, y)$ involving trigonometric functions Re. or Img. Write it as $\exp (i \ldots .$. use the above method.

Otherwise: method of Undetermined Coefficients.
Ex: $\left(D^{2}-D^{\prime}\right) z=A \cos (l x+m y), A, l, m$ are constants.
Let a P.I. $\quad z=c_{1} \cos (l x+m y)+c_{2} \sin (l x+m y)$.
Find $D^{2} z, D^{1} z$.

Equating the coefficients of sine \& cosine terms.
We get $\left.\begin{array}{l}m c_{1}-l^{2} c_{2}=0 \\ -l^{2} c_{1}+m c_{2}=A\end{array}\right\} \Rightarrow z=\frac{A}{m^{2}-l^{4}}\left\{m \sin (l x+m y)+l^{2} \cos (l x+m y)\right\}$

Exercises: Denote: $\frac{\partial^{2} z}{\partial x^{2}}=r, \frac{\partial^{2} z}{\partial x \partial y}=s, \frac{\partial^{2} z}{\partial y^{2}}=t$. Find the solution of

1. $r+s-2 t=e^{x+y}$.
2. $r-s+2 q-z=x^{2} y^{2}$.
3. $r+s-2 t-p-2 q=0$.
4. Solve $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=e^{-x} \cos y$.

Keywords: Complementary function, Irreducible, Particular integral.

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Private Ltd. New Delhi

## Lesson43

## Non-Homogeneous Linear Equation

### 43.1 Complementary Function and Particular Solution

Consider the non-homogeneous linear equation

$$
f\left(D, D^{\prime}\right) z=F(x, y)_{\text {where }} f\left(D, D^{\prime}\right)=\prod_{r=1}^{n} D_{r}-m D_{r}^{\prime}-C_{r}
$$

for some fixed $r$, the solution may be written as

$$
\frac{d x}{1}=\frac{d y}{-m}=\frac{d z}{c z} \Rightarrow y+m x=a, z=b e^{c x}
$$

Example1: $\left(D^{2}+2 D D^{\prime}+D^{\prime 2}-2 D-2 D^{\prime}\right) z=\sin (x+2 y)$

$$
\left(D+D^{\prime}\right)\left(D+D^{\prime}-2\right) z=\sin (x+2 y)
$$

Solution corresponding to the factor $\left(D+D^{\prime}-2\right)$ is:

$$
z=e^{2 x} \phi(y-x)
$$

and the complementary function is: $\phi_{1}(y-x)+e^{2 x} \phi(y-x)$.

The Particular Integral is
$\frac{1}{\left(D^{2}+2 D D^{\prime}+D^{\prime 2}-2 D-2 D^{\prime}\right)} \sin (x+2 y)$

$$
\begin{aligned}
& =-\frac{1}{2\left(D+D^{\prime}\right)+9} \sin (x+2 y) \\
& =\frac{-2\left(D+D^{\prime}\right)-9}{4\left(D^{2}+2 D D^{\prime}+D^{\prime 2}\right)-81} \sin (x+2 y) \\
& =\frac{1}{39}[2 \cos (x+2 y)-3 \sin (x+2 y)]
\end{aligned}
$$

Exercises: Solve the following non-homogeneous equations

1. $\left(D^{2}+D D^{\prime}+D^{\prime}-1\right) z=e^{-x}$
2. $\left(D+D^{\prime}-1\right)\left(D+2 D^{\prime}-3\right) z=4+3 x+6 y$
3. $\left(D^{\prime}+D D^{\prime}+D^{\prime}\right) z=x^{2}+y^{2}$
4. $\left(2 D D^{\prime}+D^{\prime 2}-3 D^{\prime}\right) z=3 \cos (3 x-2 y)$

Keywords:Non-Homogeneous,

## References

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## Lesson 44

## Method of Separation of Variables

### 44.1 Introduction

This is the oldest systematic procedure for the solving a class of partial differential equations. The underlying principle in this method is to transform the given partial differential equation to a set of ordinary differential equations. The solution of the p.d.e. is then written as either the product $z(x, y)=X(x) \cdot Y(y) \neq 0$ or as a sum $z(x, y)=X(x)+Y(y)$ where $X(x)$ and $Y(y)$ are functions of $x$ and $y$ respectively.

### 44.2 Method of Separation of Variables

Many practical problems in p.d.e. can be solved by the method of separation of variables. Usually, the first order p.d.e. can be solved by this method without the need for Fourier Series which is described in the latter lessons. Let us illustrate the separation of variables technique by few examples.

## Example 1

Solve the first order p.d.e. $z_{x}+2 z_{y}=0$ subject to the condition $z(x=0, y)=4 e^{-2 y}$.

## Solution

We look for a separable solution for $z(x, y)$ in the form $z(x, y)=X(x) \cdot Y(y) \neq 0$. Substituting this in the given p.d.e we obtain $X^{\prime}(x) \cdot Y(y)+2 X(x) \cdot Y^{\prime}(y)=0$.

This can be separated into 2 o.d.es, one in $x$ and the other in $y$ as: $\frac{X^{\prime}(x)}{2 X(x)}=-\frac{Y^{\prime}(y)}{Y(y)}$.

Note that the left hand side of the equality is a function of $x$ alone and it is equated to a function of $y$ alone which is on the right hand side. This is possible only when both are equal to the same constant (say) $k$ which is called an arbitrary separation constant. Thus we have
$\frac{X^{\prime}(x)}{2 X(x)}=k=\frac{Y^{\prime}(y)}{Y(y)}$

This gives two o.d.es. as: $\quad X^{\prime}(x)-2 k X(x)=0, Y^{\prime}(y)-k Y(y)=0$
having solutions $X(x)=A e^{-2 k x}$ and $Y(y)=B e^{-k y}$ where $A$ and $B$ are arbitrary constants. Hence the general solution is $z(x, y)=X(x) \cdot Y(y)=C e^{k(2 x-y)}, C=A B$.

Eliminating the arbitrary constant $C$ using the given condition $z(0, y)=4 e^{-2 y}$ we get $C=4$ and $k=2$. Hence the particular solution is $z(x, y)=4 e^{4 x-2 y}$.

Let us now demonstrate this method for a non-linear p.d.e.

## Example 2

Solve $y^{2} p^{2}+x^{2} q^{2}=(x y z)^{2}$ subject to the condition $u(x, 0)=3 \exp \left(\frac{x^{2}}{4}\right)$.

## Solution

Note that $p=z_{x}$ and $q=z_{y}$ write $z(x, y)=X(x) \cdot Y(y)$ in the given equation. This will produce separate the two variables as
$\frac{1}{x^{2}}\left\{\frac{X^{\prime}(x)}{X(x)}\right\}^{2}=1-\frac{1}{y^{2}}\left\{\frac{Y^{\prime}(y)}{Y(y)}\right\}^{2}=\lambda^{2}$ (say)
$\Rightarrow \frac{1}{X}\left\{\frac{X^{\prime}(x)}{X(x)}\right\}=\lambda \quad$ and $\quad \frac{1}{y}\left\{\frac{Y^{\prime}(y)}{Y(y)}\right\}=\sqrt{1-\lambda^{2}}$.
Solving these two o.d.es $X^{\prime}(x)-\lambda x X(x)=0$ and $Y^{\prime}(y)-\sqrt{1-\lambda^{2}} y Y(y)=0$, we find $X(x)=A e^{\frac{\lambda}{x^{2}} x^{2}}$ and $Y(y)=B e^{\frac{y}{2} \sqrt{1-\lambda^{2}}}$.

Hence the general solution is $\quad z(x, y)=C \exp \left(\frac{\lambda}{2} x^{2}+\frac{y}{2} \sqrt{1-\lambda^{2}}\right), C=A B$.

The boundary condition $u(x, 0)=3 \exp \left(\frac{x^{2}}{4}\right)$ implies $C=4$ and $\lambda=\frac{1}{2}$.
$\therefore$ The particular solution is $z(x, y)=4 \exp \left(\frac{1}{4} x^{2}+\frac{\sqrt{3}}{4} y^{2}\right)$.

Let us now solve a second order equation using this method.

## Example 3

Solve $\frac{\partial^{2} z}{\partial x^{2}}-2 \frac{\partial z}{\partial x}+\frac{\partial y}{\partial x}=0$.

## Solution

Write $\quad Z(x, y)=X(x) \cdot Y(y)$.
$\frac{\partial z}{\partial x}=X^{\prime}(x) \cdot Y(y), \frac{\partial^{2} z}{\partial x^{2}}=X^{\prime \prime}(x) \cdot Y(y)$ and $\frac{\partial z}{\partial y}=X(x) \cdot Y^{\prime}(y)$.

Using these in the given equation, we get $X^{\prime \prime}-2 X^{\prime}-\lambda X=0$ and $Y^{\prime}+\lambda Y=0$
where $\lambda$ is the arbitrary separation constant. Solving these o.d.es., we obtain

$$
X(x)=C_{1} \exp [(1+\sqrt{1+\lambda}) x]+C_{2} \exp [(1-\sqrt{1+\lambda}) x] \text { and } Y(y)=C_{3} \exp [-\lambda y] .
$$

Hence the required solution is

$$
Z(x, y)=\left\{C_{4} \exp [(1+\sqrt{1+\lambda}) x]+C_{5} \exp [(1-\sqrt{1+\lambda}) x]\right\} \exp [-\lambda y]
$$

where $C_{4}\left(=C_{1} C_{3}\right)$ and $C_{5}\left(=C_{2} C_{3}\right)$ are the arbitrary constants.

Exercise: Solve the following using the method of separation of variables.

1. $\frac{\partial u}{\partial x}=4 \frac{\partial u}{\partial y}$, given that $u(0, y)=8 e^{-3 y}$.
2. $4 \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=3 z$ subjected to $z=3 e^{-y}-e^{-5 y}$ when $x=0$.
3. Find a solution of the equation $\frac{\partial^{2} z}{\partial x^{2}}-\frac{\partial z}{\partial x}-2 z=0$ subject to the conditions:

$$
z(x=0, y)=0 \quad \text { And } \frac{\partial z}{\partial x}(x=0, y)=1+e^{-3 y} .
$$

Keywords: Method of Separation of Variables, Separation of variables.

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## Lesson 45

## One Dimensional Heat Equation

### 45.1 Introduction

The one dimensional heat equation is a parabolic partial differential equation. We wish to estimate the heat transfer in a very thin long (finite or infinite) string at some location on the string at any given time. Let $x$ be the coordinate along the thin rod and let $t$ represent the time. Then the 1-dimensional heat conduction equation is given by

$$
\begin{equation*}
\frac{\partial z}{\partial t}=c^{2} \frac{\partial^{2} z}{\partial x^{2}} \tag{45.1}
\end{equation*}
$$

where $z(t, x)$ representing the heat conducting in the material and $c^{2}=\frac{k}{s \rho}$ is the diffusivity constant with $k$ being the thermal conductivity, $\rho$ being the density and $s$ being the specific heat. The problem is well posed if this differential equation is supplemented with an initial condition and two boundary conditions. Let us attempt to solve this equation with suitable initial and boundary conditions using some standard mathematical techniques such as the method of separation of variables and integral transform techniques. By a solution of heat equation, we mean a physically realistic solution that obeys the 'natural' physical process.

### 45.2 Solution of the Heat Equation - Method of separation of variables

Assume that a solution of (45.1) can be written in the form

$$
z(t, x)=X(x) \cdot T(t) .
$$

Finding $\frac{\partial z}{\partial t}$ and $\frac{\partial^{2} z}{\partial x^{2}}$ and substituting in (45.1), we get the set of ordinary differential equation as
$X^{\prime \prime}(x)-\lambda X(x)=0$
and $T^{\prime}(t)-\lambda c^{2} T(t)=0$
with $\lambda$ as the arbitrary separation constant which takes positive or negative or zero. Solving equation (44.2) and (44.3) for these three cases of $\lambda$ we get the following three cases for the solution $z(t, x)$.

Case 1: Take $\lambda>0$, say $\lambda=p^{2}$.
In this case, $X(x)=c_{1} e^{p x}+c_{2} e^{-p x}$ and $T(t)=c_{3} e^{c^{2} p^{2} t}$
i.e., $z(t, x)=\left(c_{4} e^{p x}+c_{4} e^{-p x}\right) e^{c^{2} p^{2} t}$
with $c_{4}=c_{1} c_{3}$ and $c_{5}=c_{2} c_{3}$ are arbitrary constants.

Case 2: Take $\lambda<0$, say $\lambda=-p^{2}$
In this case, $X(x)=c_{6} \cos p x+c_{7} \sin p x$, and $T(t)=c_{8} e^{-c^{2} p^{2} t}$
and $z(t, x)=\left(c_{9} \cos p x+c_{10} \sin p x\right) e^{-c^{2} p^{2} t}$
where $c_{9}=c_{6} c_{8} ; c_{10}=c_{7} c_{8}$ are arbitrary constants.

Case 3: Take $\lambda=0$.
In this case $z(t, x)=\left(c_{14} x+c_{15}\right)$
as $X(x)=\left(c_{11} x+c_{12}\right), T(t)=c_{13}$ where $c_{14}=c_{11} c_{13} ; c_{15}=c_{12} c_{13}$ are arbitrary constants.

Now, among these three possible solutions, we have to choose the one that is physically realistic. In general, the solution of heat conduction problem is exponentially decaying with time ' $t$ '. This property is clearly seen only when $\lambda<0$.

Thus the suitable solution of the heat equation is $z(t, x)=\left(c_{1} \cos p x+c_{2} \sin p x\right) e^{-c^{2} p^{2} t}$ The values of $c_{1}, c_{2}$ and $p$ are found based on the initial and boundary conditions associated with the equation. Let us see this solution procedure in some special situations.

## Example 44.1

solve the heat conduction problem $\frac{\partial z}{\partial t}=\frac{\partial^{2} z}{\partial x^{2}} 0<x<1, t>0$
with the initial condition $z(t=0, x)=\sin n \pi x$ and the boundary conditions $z(t, x=0)=0$ and $z(t, x=1)=0$ for $t>0$.

Solution: The physically realistic solution of the given equation is

$$
z(x, t)=\left(c_{1} \cos p x+c_{2} \sin p x\right) e^{-p^{2} t}
$$

Determining the constants $c_{1}, c_{2}$ and $p$ :
Using the boundary condition at $x=0$, we have $c_{1} e^{-p^{2} t}=0$

This implies $c_{1}=0$ as $e^{-p^{2} t} \neq 0 ; \forall t>0$.
$\therefore z(x, t)=c_{2} \sin p x \cdot e^{-p^{2} t}$

The other boundary condition at $x=1$ gives $c_{2} \cdot \sin p \cdot e^{-p^{2} t}=0$

Now, if we take $c_{2}=0$, then $z(x, t) \equiv 0$ which should be ruled out as we are seeking a non-trivial solution for the given problem.

So $c_{2} \neq 0$, hence $\sin p=0 \Rightarrow p=n \pi, n=0,1,2, \ldots$.
$\therefore z(t, x)=a_{n} \sin n \pi x \cdot e^{-p^{2} t}$

At this stage, note that with each $n, n=0,1,2, \ldots$, we get $z(x, t)=a_{0} \sin 0 \pi x \cdot e^{-p^{2} t}$, $a_{1} \sin 1 \pi x \cdot e^{-p^{2} t}, a_{2} \sin 2 \pi x \cdot e^{-p^{2} t}$ etc. as the solutions.

Using the principle of superposition (valid only for linear p.d.es.), we can write the general solution as the infinite sum of these solutions as $z(x, t)=\sum_{n=0}^{\infty} a_{n} \sin n \pi x \cdot e^{-p^{2} t}$.

Now using the initial condition $z(0, x)=\sin n \pi x$,
we see $z(t, x)=\sum_{n=0}^{\infty} a_{n} \sin n \pi x=\sin n \pi x$

Comparing the coefficients on both sides, we get $a_{n}=1 \forall n$.

Hence the solution of the heat equation satisfying the given initial and boundary conditions is written as $z(t, x)=\sum_{n=0}^{\infty} \sin n \pi x \cdot e^{-p^{2} t}$.

## Example 44.2

Let us replace the boundary condition $z(t, x=1)=0$ by $z(t, x=1)=20 t \quad$ in the example (44.1) and look for the solution.

The solution $z(t, x)=c_{2} \sin p x \cdot e^{-p^{2} t}$, when evaluated at $x=1$, we have

$$
z(t, 1)=c_{2} \sin p \cdot e^{-p^{2} t}=20 t
$$

This will neither give any information about $p$ nor about $c_{2}$. Thus the separation of variables would then be futile. This example clearly indicates the restricted use of the method of separation of variables.

Let us now consider an example with derivative boundary conditions.

## Example 44.3

Solve the equation $\frac{\partial z}{\partial t}=\frac{\partial^{2} z}{\partial x^{2}}, 0<x<L$, subject to the boundary conditions $\frac{\partial z}{\partial x}(t, x=0)=0, \frac{\partial z}{\partial x}=(t, x=L)=0$ and the initial condition $z(t=0, x)=h(x)$.

Solution: Using the separation of variables method the general solution of the heat conduction equation can be written as $z(t, x)=(A \cos p x+B \sin p x) e^{-p^{2} t}$

The boundary condition $\frac{\partial z}{\partial x}(t, 0)=0 \Rightarrow B=0$

Hence $z(t, x)=A \cos p x \cdot e^{-p^{2} t}$.

The other condition $\frac{\partial z}{\partial x}=(t, L)=0 \Rightarrow \sin p L=0(\because A \neq 0) \Rightarrow p=\frac{n \pi}{2} ; n=0,1,2, \ldots$
Thus we can write $z_{n}(t, x)=a_{n} \cos \frac{n \pi x}{L} \cdot e^{\frac{-n^{2} \pi^{2}}{L^{2}} t}$
and $z(t, x)=\sum_{n=0}^{\infty} z_{n}(t, x)=\sum_{n=0}^{\infty} a_{n} \cos \frac{n \pi x}{L} \cdot e^{\frac{-n^{2} \pi^{2}}{L^{2}} t}$.
Using the initial condition, we get

$$
\sum_{n=0}^{\infty} a_{n} \cos \frac{n \pi x}{L}=h(x)
$$

The unknown coefficients $a_{n}$ are computed using the half range Fourier Cosine Series expansion, which gives

$$
a_{0}=\frac{1}{L} \int_{0}^{L} h(x) d x \quad \text { and } \quad a_{n}=\frac{2}{L} \int_{0}^{L} h(x) \cos \frac{n \pi x}{L} d x
$$

Thus for a given function $h(x)$, we find $a_{n}$ 's and the final solution is written as $z(t, x)=\sum_{n=0}^{\infty} a_{n} \cos \frac{n \pi x}{L} \cdot e^{\frac{-n^{2} \pi^{2}}{L^{2}} t}$

## Example 44.4

Two ends $A$ and $B$ of a thin rod of length 10 cm have the temperature at $30^{\circ} \mathrm{C}$ and $80^{\circ} \mathrm{C}$ until steady state is reached. The temperatures of the ends are changed to $40^{\circ} \mathrm{C}$ and $60^{\circ} \mathrm{C}$ respectively. Find the temperature distribution in the rod at time $t$.

Solution: In the steady state condition, $z$ is a function of $x$ i.e., $z(t, x)=z(x)$
and $\frac{\partial z}{\partial t}=\frac{\partial^{2} z}{\partial x^{2}}$ becomes $\frac{\partial^{2} z}{\partial x^{2}}=0$. The steady state solution is $z_{s}(x)=a x+b \ldots$ (i)
The initial temperature at the ends $A$ and $B$ before the steady state is reached are $z(x=0)=30^{\circ} \mathrm{C}$
and $z(x=10)=80^{\circ} \mathrm{C} \ldots$ (iii).
These conditions imply $z(0, x)=30+5 x \ldots$ (iv)
The boundary conditions are

$$
\begin{equation*}
z(t, 0)=40^{\circ} \mathrm{C} . \tag{v}
\end{equation*}
$$

and $z(t, 10)=60^{\circ} \mathrm{C} \quad \forall t \quad .$. (vi)
i.e., the boundary values are non-zero, we split up the temperature function $z(t, x)$ into the sum of $z_{s}(x)$ and $z_{t}(t, x)$ i.e., $z(t, x)=z_{s}(x)+z_{t}(t, x)$ $\qquad$
where $z_{s}(x)$ is the steady state solution (involving $x$ only) satisfying the boundary conditions (v) and (vi);
and $z_{t}(t, x)$ is $z(t, x)-z_{s}(x)$, which is the transient part of the solution which decays with the increase in time.

Since $\quad z_{s}(0)=40$ and $z_{s}(10)=60$, the steady solution $z_{s}(x)=a x+b$ becomes $z_{s}(x)=2 x+40$

The transient solution $z_{t}(t, x)$ is obtained by solving $\frac{\partial z}{\partial t}=\frac{\partial^{2} z}{\partial x^{2}}$ subject to the initial condition $z(x=0)=30+5 x$ and the boundary conditions $z(t, 0)=40^{\circ} C$ and $z(t, 10)=60^{\circ} \mathrm{C}$.
$\therefore z(t, x)=(40+2 x)+\sum_{n=1}^{\infty}\left(a_{n} \cos p x+b_{n} \sin p x\right) e^{-p^{2} t}$.

Now $z(t, 0)=40=40+\sum a_{n} \cos p x \cdot e^{-p^{2} t} \quad \Rightarrow a_{n}=0 \forall n$.

Hence $z(t, x)=(40+2 x)+\sum_{n=1}^{\infty} b_{n} \sin p x \cdot e^{-p^{2} t}$.

The other boundary condition is $z(t, 10)=60$
$\Rightarrow 60=40+20+\sum_{n=1}^{\infty} b_{n} \sin 10 p \cdot e^{-p^{2} t}$

Since $b_{n} \neq 0, \sin 10 p=0 \Rightarrow p=\frac{n \pi}{10}$
$\therefore z(t, x)=(40+2 x)+\sum_{n=1}^{\infty} b_{n} \sin \frac{n \pi x}{10} \cdot e^{-\frac{n \pi t}{10}}$

Now the unknown $b_{n}$ are obtained by making use of the initial condition $z(0, x)=30+5 x$.
$\Rightarrow 30+5 x=40+2 x+\sum_{n=1}^{\infty} b_{n} \sin \frac{n \pi x}{10}$
or $\sum_{n=1}^{\infty} b_{n} \sin \frac{n \pi x}{10}=3 x-10$

Considering the half-range Fourier sine series expansion for $(3 x-10)$, we determine

$$
b_{n}=\frac{2}{10} \int_{0}^{10}(3 x-10) \sin \frac{n \pi x}{10} d x=-\frac{60}{n \pi} \cos n \pi+\frac{20}{n \pi} \cos n \pi-\frac{20}{n \pi}=-\frac{20}{n \pi}[2 \cos n \pi+1] .
$$

Hence the desired solution is

$$
z(t, x)=(40+2 x)-\frac{20}{\pi} \sum_{n=1}^{\infty} \frac{[2 \cos n \pi+1]}{n} \sin \frac{n \pi x}{10} \cdot e^{-\left(\frac{n \pi}{10}\right)^{2} t}
$$

## Exercises:

1. Solve the heat conduction problem $\frac{\partial z}{\partial x}=\alpha^{2} \frac{\partial^{2} z}{\partial x^{2}}, 0<x<l$;

Subject to the boundary and initial conditions

$$
\frac{\partial z}{\partial x}(t, 0)=0, \frac{\partial z}{\partial x}(t, l)=0 ; z(0, x)=x .
$$

2. The temperatures at one end of a bar 10 cm long with insulated sides is kept at $0^{\circ} \mathrm{C}$ and that the other end is kept at $100^{\circ} \mathrm{C}$ until steady state condition attained. The two end are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution in the bar.
3. Solve $\frac{\partial z}{\partial x}=\alpha^{2} \frac{\partial^{2} z}{\partial x^{2}}$ subject to the conditions:
(i) $z$ is decaying as $t \rightarrow \infty$ in $0<x<l$;
(ii) $\frac{\partial z}{\partial x}(t, 0)=0=\frac{\partial z}{\partial x}(t, l)$.

## Keywords: Heat Conducting, Heat equation, Parabolic Partial Differential Equation,

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## Lesson 46

## One Dimensional Wave Equation

### 46.1 Introduction

In general wave motion occurs in vibrating strings, vibrating membranes etc. Waves travelling through a solid media, acoustic waves, water waves, shock waves etc are normally observed in nature. The standard and the simplest example is the vibration of a stretched flexible string which is modelled as the one dimensional wave equation. It is an example for the hyperbolic equation. Mathematically, it is represented as $\quad \frac{\partial^{2} z}{\partial t^{2}}=c^{2} \frac{\partial^{2} z}{\partial x^{2}} \quad$ where $z(t, x)$ denoting the deflection of the string at any position and at any point of time. The constant $c=\sqrt{\frac{P}{m}}$ denotes the wave speed with $P$ denoting the tension in the string and $m$ is the mass per unit length of the string.

The solution of the wave equation should describe the wave motion and this involves periodic sine and cosine terms. A particular solution of this equation can be obtained by specifying two initial conditions and two boundary conditions. Let us now see the solution of this one dimensional wave equation using the separation of variables technique using various types of boundary conditions. Note that we pose either initial displacement or initial velocity or both for the initial conditions to make the mathematical formulation as a well posed problem.
46.2 The Method of Separation of Variables to the 1-D Wave Equation: A finite string of length $L$ that is fixed at both ends and is released from rest with an initial displacement at some position. The mathematical representation of this problem given by
$\frac{\partial^{2} z}{\partial t^{2}}=c^{2} \frac{\partial^{2} z}{\partial x^{2}}$
satisfying the boundary conditions (i) $z(t, x=0)=0$; and (ii) $z(t, x=L)=0$
and the initial conditions (iii) $u(t=0, x)=f(x)$ (Initial displacement),
(iv) $\frac{\partial u}{\partial t}(t=0, x)=0$ (string is released from rest, the initial velocity is zero).

We now write the solution $z(t, x)=X(x) \cdot T(t)$
In the equation (1) it becomes

$$
\begin{equation*}
X(x) \cdot T^{\prime \prime}(t)=c^{2} X^{\prime \prime}(x) \cdot T(t) \quad \text { or } \quad \frac{T^{\prime \prime}}{c^{2} T}=\frac{X^{\prime \prime}}{X}=\lambda \tag{46.3}
\end{equation*}
$$

where $\lambda$ is the arbitrary separation parameter. This will result in two ordinary differential equations as $T^{\prime \prime}-\lambda c^{2} T=0$
and $X^{\prime \prime}-\lambda X=0$

We consider the three cases for $\lambda>0, \lambda<0, \lambda=0$.
Case 1: Take $\lambda>0$ say $\lambda=\bar{p}^{2}$, and solving equations (46.4) and (46.5)

We get the solution as $z(t, x)=\left(c_{1} e^{p x}+c_{2} e^{-p x}\right)\left(c_{3} e^{c p t}+c_{4} e^{-c p t}\right)$

Case 2: When $\lambda<0$ say $\lambda=-p^{2}$ the solution becomes

$$
\begin{equation*}
z(t, x)=\left(c_{5} \cos p x+c_{6} \sin p x\right)\left(c_{7} \cos c p t+c_{8} \sin c p t\right) \tag{46.7}
\end{equation*}
$$

Case 3: When $\lambda=0$; we get $z(t, x)=\left(c_{9} x+c_{10}\right)\left(c_{11} t+c_{12}\right)$

Among these three possible solutions for the wave equation, the physically realistic solutions that represents the periodic functions of $x$ and $t$ is when $\lambda<0$ i.e., $\quad z(t, x)=\left(c_{1} \cos p x+c_{2} \sin p x\right)\left(c_{3} \cos c p t+c_{4} \sin c p t\right)$.

The arbitrary constants $c_{1}, c_{2}, c_{3}, c_{4}$ and $p$ are determined as shown below:

The first boundary condition $z(t, 0)=0 \Rightarrow c_{1}\left(c_{3} \cos c p t+c_{4} \sin c p t\right)=0$

For this to be true for all $t>0$, we should have $c_{1}=0$.

Hence $\quad z(t, x)=c_{2} \sin p x\left(c_{3} \cos c p t+c_{4} \sin c p t\right)$

One of the initial condition $\frac{\partial z}{\partial t}(0, x)=0$ implies
$c_{2} \sin p x\left(-c_{3} \cdot c p \cos c p t+c_{4} \cdot c p \sin c p t\right)=0$.

We have two possibilities, one is $c_{2}=0 \Rightarrow z(t, x)=0$. This is ruled out in order to have a non-trivial solution. The other possibility is $c_{4}=0($ note $c \neq 0 ; p \neq 0)$
$\therefore z(t, x)=c_{2} c_{3} \sin c p t \cos c p t$. At this stage we use the other boundary condition which is given as $z(t, L)=0 \Rightarrow c_{2} c_{3} \sin p L \cos c p t=0 \quad \forall t$.

As $c_{2} \neq 0 ; c_{3} \neq 0$, we have $\sin p L=0 \Rightarrow p=\frac{n \pi}{L} \quad n=0,1,2, \ldots$

Thus $z(t, x)=A \sin \frac{n \pi x}{L} \cos c p t$ where $A=c_{2} c_{3}$. This constant is determined using the other initial condition as: $z(0, x)=f(x) \Rightarrow A \sin \frac{n \pi x}{L}=f(x)$ where $c_{2} c_{3}=A$.

Choose $f(x)=3 \sin \frac{\pi x}{L} \Rightarrow A=3, n=1$, hence the solution is $z(t, x)=3 \sin \frac{\pi x}{L} \cos \frac{\pi t}{L}$.

## Example 2

A tight string, 2 m long with $c=30 \mathrm{~m} / \mathrm{s}$ is initially at rest but is given an initial velocity $300 \sin 4 \pi x$ from its equilibrium position. Determine the displacement at the position $x=\frac{1}{8} m$ of the string.

## Solution:

Given $\quad \frac{\partial^{2} z}{\partial t^{2}}=900 \frac{\partial^{2} z}{\partial x^{2}} \quad$ Subject $\quad$ to $z(t, 0)=0=z(t, 2), \quad$ and $\quad z(0, x)=0 \quad$ and $\frac{\partial z}{\partial t}(0, x)=300 \sin 4 \pi x . \quad$ The solution may be written as
$z(t, x)=(A \cos 30 p t+B \sin 30 p t) \cdot D \sin p x$.
Now, $z(t, 0)=0 \Rightarrow c=0$
$\therefore z(t, x)=(A \cos 30 p t+B \sin 30 p t) \cdot D \sin p x$

Also zero initial displacement $\Rightarrow A=0$
Hence $z(t, x)=B \cdot D \sin p x \cdot \sin 30 p t$.
As $\frac{\partial z}{\partial t}(0, x)=300 \sin 4 \pi x=30 p \cdot B \cdot D \cdot \sin p x \Rightarrow p=4 \pi$ and $B \cdot D=\frac{300}{30 \cdot 4 \pi}=\frac{5}{2 \pi}$
$\therefore z(t, x)=\frac{5}{2 \pi} \sin 120 \pi t \cdot \sin 4 \pi x$.

We now determine the maximum displacement at $x=\frac{1}{8}$ occurs when $\sin 120 \pi t=1$ and then $z_{\max }=\frac{2.5}{\pi}$.

Note that the condition $z(t, 2)=0$ is not used in determining the arbitrary constants but it is satisfied automatically.

## Example 3

A string of length $L$ which is fixed at both ends is initially in equilibrium position. It is set in vibrating mode by given each point a velocity $\frac{v_{0}}{4}\left[3 \sin \frac{\pi x}{L}-\sin \frac{3 \pi x}{L}\right]$. Find the displacement at any point of the string.

## Solution:

The equation of the vibrating sting is $\frac{\partial^{2} z}{\partial t^{2}}=c^{2} \frac{\partial^{2} z}{\partial x^{2}}$.

The boundary conditions are $z(t, 0)=0, z(t, L)=0$. Also given the initial conditions are $z(0, x)=0$ and $\frac{\partial z}{\partial t}(0, x)=\frac{v_{0}}{4}\left[3 \sin \frac{\pi x}{L}-\sin \frac{3 \pi x}{L}\right]$. As seen in the earlier examples, here also the solution of the vibrating string after applying the boundary conditions reduces to

$$
z(t, x)=c_{2} \sin \frac{n \pi x}{L}\left(c_{3} \cos \frac{c n \pi t}{L}+c_{4} \sin \frac{c n \pi t}{L}\right) .
$$

Now the initial condition $z(0, x)=0 \Rightarrow c_{2} c_{3} \sin \frac{n \pi x}{L}=0 \forall x \Rightarrow c_{2} c_{3}=0 \Rightarrow c_{3}=0$ (for non-trivial solution).
$\therefore z(t, x)=b_{n} \sin \frac{n \pi x}{L} \sin \frac{c n \pi t}{L} \quad$ where $b_{n}=c_{2} c_{4}$.
As the wave equation is linear, by the principle of superimposition, we can write the general solution as $z(t, x)=\sum b_{n} \sin \frac{n \pi x}{L} \sin \frac{c n \pi t}{/ L}$.

Applying the other initial condition we have

$$
\frac{\partial z}{\partial t}(0, x)=\frac{v_{0}}{4}\left[3 \sin \frac{\pi x}{L}-\sin \frac{3 \pi x}{L}\right]=\sum_{n=1}^{\infty} \frac{c n \pi}{L} b_{n} \sin \frac{n \pi x}{L} .
$$

Comparing the coefficients of $\sin \frac{n \pi x}{L}$ on both sides we see

$$
b_{1}=\frac{3 L v_{0}}{4 c \pi}, b_{3}=\frac{-L v_{0}}{12 c \pi}, b_{2}=b_{3}=b_{4} \ldots=0
$$

Hence the solution of the given problems is
$z(t, x)=\frac{L v_{0}}{12 c \pi}\left[9 \sin \frac{\pi x}{L} \sin \frac{c \pi t}{L}-\sin \frac{3 \pi x}{L} \sin \frac{3 c \pi t}{L}\right]$.

### 46.3 The D'Alembert Solution of the Wave Equation:

Consider the wave equation $\frac{\partial^{2} z}{\partial t^{2}}=a^{2} \frac{\partial^{2} z}{\partial x^{2}}$
and introduce the new independent variables $\xi=x-a t ; \eta=x+a t$

These are the two characteristics of the hyperbolic equation. The equatin (46.9) is transformed to its canonical form as $\frac{\partial^{2} z}{\partial \xi \partial \eta}=0$

This is because $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \xi}+\frac{\partial z}{\partial \eta} ; \frac{\partial z}{\partial t}=-a \frac{\partial z}{\partial \xi}+a \frac{\partial z}{\partial \eta}, \frac{\partial^{2} z}{\partial u^{2}}=\frac{\partial^{2} z}{\partial \xi^{2}}+2 \frac{\partial^{2} z}{\partial \xi \partial \eta}+\frac{\partial^{2} z}{\partial \eta^{2}}$ and $\frac{\partial^{2} z}{\partial t^{2}}=a^{2} \frac{\partial^{2} z}{\partial \xi^{2}}-2 a^{2} \frac{\partial^{2} z}{\partial \xi \partial \eta}+a^{2} \frac{\partial^{2} z}{\partial \eta^{2}}$.

Substituting these in equation (4.9) results in equation (46.11). Now integrating (46.11) with respect to $\xi$ gives $\frac{\partial z}{\partial \eta}=h(\eta)$, where $h(\eta)$ is a arbitrary function of $\eta$. Integrating again w.r.t $\eta$, we get $z(\eta, \xi)=\int h(\eta) d \eta+g(\xi)$ which can be also be written as $z(\eta, \xi)=f(\eta)+g(\xi)$ with $g(\xi)$ is an arbitrary function of $\xi$ alone and the integral is a function of $\eta$ alone and is written as $f(\eta)$.

Thus the solution is written as $z(t, x)=f(x+a t)+g(x-a t)$
This is the called the $\mathbf{D}$ ' Alembert's solution of the wave equation.
Case 1: Let the initial conditions be $z(0, x)=\phi(x)$ and $\frac{\partial z}{\partial t}(0, x)=0$.

Now differentiating (4) with respect to ' $t$ ' and putting $t=0$,

We obtain $\left.c f^{\prime}(x+c t)\right|_{t=0}-\left.c g^{\prime}(x-c t)\right|_{t=0}=0, \Rightarrow f^{\prime}(x)=g^{\prime}(x) \Rightarrow f(x)=g(x)+k$, where $k$ is a constant. Also, $z(x, 0)=\phi(x)=f(x)+g(x) \quad \Rightarrow 2 g(x)+k=\phi(x)$, $\Rightarrow g(x)=\frac{1}{2}[\phi(x)-k]$. Hence $f(x)=g(x)+k=\frac{1}{2}[\phi(x)-k]$, or $f(x)=\frac{1}{2}[\phi(x)+k]$.

Hence the general solution (46.12) takes the form

$$
\begin{equation*}
z(t, x)=\frac{1}{2}[(\phi(x)+c t)+(\phi(x)-c t)] . \tag{46.13}
\end{equation*}
$$

Case 2: Suppose now that $z(0, x)=0$ and $\frac{\partial z}{\partial t}(0, x)=\theta(x)$
From equation (4), we have $\frac{\partial z}{\partial t}(0, x)=a f^{\prime}(x)-a g^{\prime}(x)=\theta(x)$
$\Rightarrow f(x)-g(x)=\frac{1}{a} \int_{0}^{s} \theta(s) d s+D$
where $s$ is a dummy variable of integration and $D$ is an arbitrary constant.

Also, $z(0, x)=0 \Rightarrow f(x)+g(x)=0$
$\Rightarrow f(x)=-g(x)$ or $f(0)-g(0)=C=2 f(0)=-2 g(0)$.

This $\Rightarrow f(x)=\frac{1}{2 a} \int_{0}^{s} \theta(s) d s+f(0)$ and $g(x)=-\frac{1}{2 a} \int_{0}^{s} \theta(s) d s+g(0)$,
and finally, the solution of the wave equation becomes $z(t, x)=\frac{1}{2 a}\left[\int_{0}^{x+a t} \theta(s) d s-\int_{0}^{x-a t} \theta(s) d s\right]=\frac{1}{2 a} \int_{x-a t}^{x+a t} \theta(s) d s$.

Thus from these two cases, it is evident that a particular solution is obtained for a given $\phi(x)$ and $\theta(x)$ respectively.

## Example 4:

In case, 1 take $\phi(x)=k(\sin x-\sin 2 x)$ and obtain a particular solution of the wave equation.

## Solution:

We have $z(x, t)=\frac{1}{2}[f(x+c t)+f(x-c t)]$

$$
\begin{aligned}
& =\frac{1}{2}[k\{\sin (x+c t)-\sin 2(x+c t)\}+k\{\sin (x-c t)-\sin 2(x-c t)\}] \\
& =k(\sin x \cos c t-\sin 2 x \cos 2 c t) .
\end{aligned}
$$

Keywords: Separation of Variables, Separation of variables, separation of variables, Wave Equation, D' Alembert's solution

## Exercises:

1. Using D'Alembert Method, find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection:
(i) $f(x)=a\left(x-x^{2}\right)$
(ii) $f(x)=\frac{a}{2}(1+\cos 2 k x)$
2. An infinite string is given the initial velocity
$\theta(x)=\left\{\begin{array}{l}0, x<-1 \\ 10(x+1),-1 \leq x \leq 0 \\ 10(1-x), 0 \leq x \leq 1 \\ 0,1<x\end{array}\right.$
If the string has zero initial displacement find the solution of the wave equation.
3. Solve $\frac{\partial^{2} z}{\partial t^{2}}=c^{2} \frac{\partial^{2} z}{\partial x^{2}}, 0<x<L$ subject to the conditions

$$
z(0, t)=z(L, t)=0 ; z(0, x)=\sin \frac{3 \pi x}{L} ; \frac{\partial z}{\partial x}(0, x)=0 .
$$

4. Solve $\frac{\partial^{2} z}{\partial t^{2}}=\frac{\partial^{2} z}{\partial x^{2}}, 0<x<L$, subject to

$$
z(0, t)=0 ; z(L, t)=0 ; z(x, 0)=\mu x(L-x) ; \frac{\partial z}{\partial t}(x, 0)=0
$$

5. The points of trisection of a sting are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time.
6. Solve $\frac{\partial^{2} z}{\partial t^{2}}=\frac{\partial^{2} z}{\partial x^{2}}, 0<x<L ; z(0, t)=z(L, t)=0 ;$

$$
z(x, 0)=\left\{\begin{array}{l}
x, 0 \leq x \leq \frac{1}{2} \\
1-x, \frac{1}{2} \leq x \leq 1
\end{array} ; \frac{\partial z}{\partial t}(x, 0)=0 .\right.
$$

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## Lesson 47

## Laplace Equation in 2-Dimensions

### 47.1 Introduction

Heat conduction in a two dimensional region is given by $\frac{\partial z}{\partial t}=\alpha\left(\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}\right)$ where $z(t, x, y)$ denoting the temperature in the region. This is clearly a parabolic equation. When we consider steady state conditions, $z=z(x, y)$ i.e, $z$ is independent of time and the equation reduces to $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0$ which will be elliptic in nature. Unlike the hyperbolic and parabolic equations where initial conditions are also specified, in case of elliptic equation only boundary conditions are specified, thus making these problems as pure boundary value problems. Let $\Omega$ be the interior of a simple closed differentiable boundary curve $\Gamma$ and $f$ be a continuous function defined on the boundary $\Gamma$. The problem of finding the solution of the above Laplace equation in $\Omega$ such that it coincides with the function $f$ on the boundary $\Gamma$ is called the Dirichlet Problem.

Finding a function $z(x, y)$ that satisfies the Laplace equation in $\Omega$ and satisfies $\frac{\partial z}{\partial \eta}=f(s)$ on $\Gamma$ where $\frac{\partial}{\partial \eta}$ representing the normal derivative along the outward normal direction to the surface $z(x, y)$ that obeys $\int_{\Gamma} f(s) d s=0$ is known as the Neumann Problem. The third boundary value problem, known as the Robin Problem is one in which the solution of the Laplace equation is obtained in $\Omega$ that satisfies the condition $\frac{\partial z}{\partial \eta}+g(s) z(s)=0$ on $\Gamma$ where $g(s) \geq 0$ and $g(s) \neq 0$. We now describe the method of Separation of variables technique for the Laplace equation.

We have the Laplace equation given by $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0$

Let $z(x, y)=X(x) Y(y)$

Finding $\frac{\partial^{2} z}{\partial x^{2}}$ and $\frac{\partial^{2} z}{\partial y^{2}}$ and substituting these in (1) and separating them into two ordinary differential equations, we get
$X^{\prime \prime}-\lambda X=0 \quad$ and $\quad Y^{\prime \prime}+\lambda Y=0$. where $\lambda$ is the arbitrary separation parameter. Solving these equations, we get three possible solutions for $\lambda=p^{2}, \lambda=-p^{2}$ and $\lambda=0$. These forms are:
a) $z(x, y)=\left(c_{1} e^{p x}+c_{2} e^{-p x}\right)\left(c_{3} \cos p y+c_{4} \sin p y\right) ; \lambda>0$
b) $z(x, y)=\left(c_{5} \cos p x+c_{6} \sin p x\right)\left(c_{7} e^{p y}+c_{2} e^{-p y}\right) ; \lambda<0$
and c) $z(x, y)=\left(c_{9} x+c_{10}\right)\left(c_{11} y+c_{12}\right) ; \lambda=0$.

Of these, we take that solution which is consistent with the given boundary conditions.

### 47.2 Dirichlet Problem in a Rectangular Region:

## Example 1:

Solve the Laplace equation $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0$ in the region with the boundary conditions as shown in the figure


## Solution:

$\frac{X^{\prime \prime}}{X}=-\frac{Y^{\prime \prime}}{Y}=p^{2}$

Note: Here we considered $\lambda=p^{2}$ to allow sinusoidal variation with $y$, to be consistent with the boundary conditions.

Then the general solution is
$z(x, y)=\left(A e^{p x}+B e^{-p x}\right)(C \cos \beta y+D \sin \beta y)$.

The boundary conditions are expressed as
$z(x=0, y)=0^{0} C ; z(x=2, y)=50 \sin \pi y^{0} C$
$z(x, y=0)=0^{0} C ; z(x, y=1)=0^{0} C$.

Now $z(0, y)=0 \forall y \Rightarrow A+B=0 \Rightarrow A=-B$.

Hence $z(x, 0)=0 \forall x \Rightarrow D=0$.
$z(x, 1)=0 \forall x \Rightarrow \sin \beta=0 \Rightarrow \beta=\pi$
$\therefore z(x, y)=A C\left(e^{\pi x}-e^{-\pi x}\right) \sin \pi y$

The non-homogeneous boundary condition at $x=2$
$\Rightarrow A C\left(e^{2 \pi}-e^{-2 \pi}\right) \sin \pi y=50 \sin \pi y$
$\Rightarrow A C=\frac{50}{\left(e^{2 \pi}-e^{-2 \pi}\right)}=0.0934 ;(x, y)$

Thus the temperature at any point $(x, y)$ is written as $z(x, y)=0.0934\left(e^{\pi x}-e^{-\pi x}\right) \sin \pi y$.

### 47.3 Temperature Distribution is Studied in an Infinitely Long Plate.

## Example 2:

An infinitely long plane uniform plate is bounded by two parallel edges and at an end at right angles to them as shown in the adjacent figure. Find the temperature distribution at any point of the plate in the steady state.


## Solution:

The steady state temperature distribution in this infinitely long plate is obtained by solving $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0$.

The boundary conditions are $z(x=0, y)=0^{0} C, z(x=\pi, y)=0^{0} C \forall y>0$, $z(x, y=0)=u_{0}{ }^{0} C$ for $0<x<\pi, \quad z(x, y \rightarrow \infty) \rightarrow 0$ for $0<x<\pi$.

Among the three possibilities for solution i.e., solution forms (a), (b), (c), we chose a solution that is consistent with the given boundary conditions. He solution given in equation (a) cannot satisfy the boundary condition $z(0, y)=0 \forall y$ . The solution given in equation (c) cannot satisfy the condition in $z(x, y \rightarrow \infty) \rightarrow 0$ in $0<x<\pi$.

Thus we have the solution as $z(x, y)=(A \cos p x+B \sin p x)\left(C e^{p y}+D e^{-p y}\right)$.

Now $z(0, y)=A\left(C e^{p y}+D e^{-p y}\right)=0 \Rightarrow A=0$
$\therefore z(x, y)=B \sin p x\left(C e^{p y}+D e^{-p y}\right)$.
$z(\pi, y)=0 \forall y \quad \Rightarrow B \sin p \pi\left(C e^{p y}+D e^{-p y}\right)=0 \Rightarrow p=n$, an integer $(\because B \neq 0)$.
$\therefore z(x, y)=B \sin n x\left(C e^{p y}+D e^{-p y}\right)$.

As $z(x, y \rightarrow \infty) \rightarrow 0 \Rightarrow c=0 \quad \therefore z(x, y)=B D \sin n x e^{-n y}$.
Taking $B D=b_{n}$ and write the general form of the solution as $z(x, y)=\sum_{n=1}^{\infty} b_{n} \sin n x e^{-n y}$.

Using the non-homogeneous boundary condition $u(x, 0)=u_{0}=\sum_{n=1}^{\infty} b_{n} \sin n x$
The unknown coefficients are found using the half range Fourier sine series expansion in $(0, \pi)$ as
$b_{n}=\frac{2}{\pi} \int_{0}^{\pi} u_{0} \sin n x d x=\frac{2}{\pi} u_{0}\left[1-(-1)^{n}\right]=\left\{\begin{array}{l}\frac{4 u_{0}}{n \pi}, n=2 m-1 \\ 0, n=2 m\end{array}, m\right.$ is a positive integer.

Thus $z(x, y)=\frac{4 u_{0}}{\pi}\left[e^{-y} \sin x+\frac{1}{3} e^{-3 y} \sin 3 x+\ldots\right]$.

## Example 3

Solve $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0 \quad$ subject to $\quad z(0, y)=z(a, y)=z(x, b)=0 \quad$ and $z(x, 0)=z(a-x), 0<x<a$.

## Solution

Physically realistic solution here is
$z(x, y)=\left(c_{1} \cos p x+c_{2} \sin p x\right)\left(c_{3} e^{p y}+c_{4} e^{-p y}\right)$.
$z(0, y)=0 \Rightarrow c_{1}=0$,
$z(a, y)=0 \Rightarrow \sin p a=0 \Rightarrow p=\frac{n \pi}{a}, n$ is an integer
$\therefore z(x, y)=c_{2} \sin \frac{n \pi x}{a}\left(c_{3} e^{\frac{n \pi y}{a}}+c_{4} e^{\frac{-n \pi y}{a}}\right)$

Take $c_{2} c_{3}=A, c_{2} c_{4}=B$.

$$
\begin{aligned}
& z(x, b)=0 \Rightarrow A e^{\frac{n \pi y}{a}}+B e^{\frac{-n \pi y}{a}}=0 \quad \Rightarrow A=\frac{-B \exp \left(\frac{-n \pi y}{a}\right)}{\exp \left(\frac{n \pi y}{a}\right)} . \\
& \therefore z(x, y)=\sin \frac{n \pi x}{a}\left[\frac{-B e^{\frac{-n \pi b}{a}}}{e^{\frac{n \pi z}{a}}} e^{\frac{n \pi y}{a}}+B e^{\frac{-n \pi y}{a}}\right] \\
& =\frac{-B}{e^{\frac{n \pi}{a}} \sin } \frac{n \pi x}{a}\left[e^{\frac{n \pi(y-b)}{a}}-e^{\frac{-n \pi(y-b)}{a}}\right] .
\end{aligned}
$$

So the general solution is now written as

$$
z(x, y)=\sum_{n=1}^{\infty} b_{n} \sin \frac{n \pi x}{a} \sinh \frac{n \pi(y-b)}{a} \text {, where } b_{n}=\frac{-2 B}{e^{\frac{n \pi b}{a}}}
$$

Now using the non-homogeneous condition

$$
z(x, 0)=x(a-x)=\sum_{n=1}^{\infty} b_{n} \sinh \frac{n \pi b}{a} \sin \frac{n \pi x}{a}=\sum_{n=1}^{\infty} B_{n} \sin \frac{n \pi x}{a} \text { (say) }
$$

the coefficient $B_{n}$ are found as $B_{n}=\frac{2}{a} \int_{0}^{x} x(a-x) \sin \frac{n \pi x}{a} d x$
$=\frac{4 a^{2}}{n^{3} \pi^{3}}(1-\cos n \pi)=\left\{\begin{array}{l}\frac{8 a^{2}}{n^{3} \pi^{3}}, n=2 m-1 \\ 0, n=2 m\end{array}, m\right.$ is a positive integer.
$\therefore z(x, y)=\frac{8 a^{2}}{\pi^{3}} \sum_{n=1,3,5, \ldots} \frac{\sin \frac{4 n \pi(b-y)}{a}}{n^{3} \sinh \frac{n \pi(b-y)}{a}} \sin \frac{n \pi x}{a}$
or $z(x, y)=\frac{8 a^{2}}{\pi^{3}} \sum_{n=0}^{\infty} \frac{\sinh \frac{(2 n+1) \pi(b-y)}{a}}{(2 n+1)^{3} \sinh \frac{(2 n+1) \pi(b-y)}{a}} \sin \frac{(2 n+1) \pi x}{a}$.

Keywords: Dirichlet Problem, Neumann Problem, Robin Problem

## Exercises 1

1. Solve $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0$ in $\mathbb{L} .0<x<\pi, 0<y<\pi$; with the conditions $z(0, y)=z(\pi, y)=z(x, \pi)=0 ; z(x, 0)=\sin ^{2} x$
2. A rectangular plate has sides $a$ and $b$. taking the side of length $a$ as $O X$ and that of length $b$ as $O Y$ and other sides to be $x=a$ and $y=b$, the sides $x=0, x=a, y=b$ are insulated and the edge $y=0$ is kept at temperature $u_{0} \cos \frac{\pi x}{a}$. Find the temperature $z(x, y)$ in the steady state.
3. Solve $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0$ subject to
(i) $z(0, y)=0 ; z(x, 0=0) ; z(1, y)=0 ; z(x, 1)=100 \sin \pi x$.
(ii) $z(0, y)=0 ; z(x, 0=0) ; z(1, y)=100 \sin \pi y ; z(x, 1)=0$.
(iii) $z(0, y)=0 ; \frac{\partial z}{\partial y}(x, 0)=0 ; \frac{\partial z}{\partial x}(1, y)=0 ; z(x, 1)=100$.
(iv) $z(0, y)=100 ; z(x, 0)=100 ; z(1, y)=200 ; z(x, 1)=100$.

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## Lesson 48

## Application of Laplace and Fourier Transforms to Boundary Value Problems in p.d.es

### 48.1 Introduction

One dimensional heat and wave equations in semi-infinite or infinite region can be solved using the Laplace transform or Fourier transform techniques. Application of these transforms on these one dimensional equations reduce the p.d.e to an ordinary differential equation. The solution of this o.d.e. involves the parameter that is associated with the transformation and on applying the inverse transformation, this solution gives the solution of the given p.d.e.

### 48.2 Selection of Transform Technique

If for a problem, $z(t, x=0)$ is given, then we make use infinite sine transform to remove $\frac{\partial^{2} z}{\partial x^{2}}$ form the differential equation. In case if $\frac{\partial z}{\partial x}(t, x=0)$ (flux condition) is given, then we employ infinite cosine transform to remove $\frac{\partial^{2} z}{\partial x^{2}}$. If in a problem $z(t, 0)$ and $z(t, L)$ (Dirichlet boundary condition) are given, then we use finite sine transform to remove $\frac{\partial^{2} z}{\partial x^{2}}$ in the p.d.e similarly if $\frac{\partial z}{\partial x}(t, 0)$ and $\frac{\partial z}{\partial x}(t, L)$ (Neumann boundary condition.) are given, then use finite cosine transform to remove $\frac{\partial^{2} z}{\partial x^{2}}$. Let us consider some examples to explain this technique. Let us now see the transform of the partial derivatives.

## Example 1:

find the Laplace transform $L$ of (i) $\frac{\partial z}{\partial t}$ (ii) $\frac{\partial^{2} z}{\partial t^{2}}$ (iii) $\frac{\partial z}{\partial x}$ (iv) $\frac{\partial^{2} z}{\partial x^{2}}$

## Solution:

(i) $L\left[\frac{\partial z}{\partial t}\right]=\int_{0}^{\infty} e^{-s t} \frac{\partial z}{\partial t} d t$
$=\lim _{p \rightarrow \infty} \int_{0}^{p} e^{-s t} \frac{\partial z}{\partial t} d t$
$=\lim _{p \rightarrow \infty}\left\{\left.e^{-s t} z(t, x)\right|_{0} ^{p}+s \int_{0}^{p} e^{-s t} z(t, x) d t\right\}$
$=s L[z(t, x)]-z(0, x)$
(ii) Show that $L\left[\frac{\partial^{2} z}{\partial t^{2}}\right]=s^{2} L[z(t, x)]-s[z(0, x)]-\frac{\partial z}{\partial t}(0, x)$ is left as an exercise.
(iii) $L\left[\frac{\partial z}{\partial x}\right]=\int_{0}^{\infty} e^{-s t} \frac{\partial z}{\partial x} d x=\frac{d}{d x} \int_{0}^{\infty} e^{-s t} z d x=\frac{d}{d x} L[z(t, x)]$
(iv) $L\left[\frac{\partial^{2} z}{\partial x^{2}}\right]=\frac{d^{2}}{d x^{2}} L[z(t, x)]$ is left as an exercise.

## Example 2:

Find (i) $\mathbb{F}\left[\frac{\partial^{2} z}{\partial x^{2}}\right]$ (ii) $\mathbb{F}_{s}\left[\frac{\partial^{2} z}{\partial x^{2}}\right]$ and (iii) $\mathbb{F}_{c}\left[\frac{\partial^{2} z}{\partial x^{2}}\right]$ where $\mathbb{F}$ is the Fourier transform,
$\mathbb{F}_{s}$ is the sine and $\mathbb{F}_{c}$ is the cosine Fourier transform.

## Solution:

By definition $\mathbb{F}\left[\frac{\partial^{2} z}{\partial x^{2}}\right]=\int_{-\infty}^{\infty} e^{-i s x} \frac{\partial^{2} z}{\partial x^{2}} d x$
$=\left.e^{-i s x} \frac{\partial^{2} z}{\partial x^{2}}\right|_{-\infty} ^{\infty}+\int_{-\infty}^{\infty} i s e^{-i s x} \frac{\partial z}{\partial x} d x$
$=\left.e^{-i s x} \frac{\partial z}{\partial x}\right|_{-\infty} ^{\infty}-\left.i s e^{-i s x} z\right|_{-\infty} ^{\infty}-s^{2} \int_{-\infty}^{\infty} e^{-i s x} z \cdot d x$
$=-s^{2} \mathbb{F}[z(t, x)] \quad$ provided $z$ and $\frac{\partial z}{\partial x}$ tend to zero as $x \rightarrow \pm \infty$.
$\therefore \mathbb{F}_{s}\left[\frac{\partial^{2} z}{\partial x^{2}}\right]=-s^{2} \mathbb{F}[z(t, x)]$ provided both $z$ and $\frac{\partial z}{\partial x} \rightarrow 0$ as $x \rightarrow \pm \infty$.
(ii) By definition, $\mathbb{F}_{s}\left[\frac{\partial^{2} z}{\partial x^{2}}\right]=\int_{0}^{\infty} \frac{\partial^{2} z}{\partial x^{2}} \sin s x d x$

$$
\begin{aligned}
& =\left.\sin s x \frac{\partial z}{\partial x}\right|_{0} ^{\infty}-\int_{0}^{\infty} s \cos x \frac{\partial z}{\partial x} d x \\
& =\left.\sin s x \frac{\partial z}{\partial x}\right|_{0} ^{\infty}-\left.s \cos x \cdot z\right|_{0} ^{\infty}-s^{2} \mathbb{F}_{s}[z(t, x)]
\end{aligned}
$$

$$
=\left.s \cdot z(t, x)\right|_{x=0}-s^{2} \mathbb{F}_{s}[z(t, x)] \text { provided } z \rightarrow 0, \frac{\partial z}{\partial x} \rightarrow 0 \text { as } x \rightarrow \infty
$$

(iii) By definition, $\mathbb{F}_{c}\left[\frac{\partial^{2} z}{\partial x^{2}}\right]=\int_{0}^{\infty} \frac{\partial^{2} z}{\partial x^{2}} \cos s x d x$

Integrating by parts, twice, we obtain

$$
=-\left.\frac{\partial z}{\partial x}\right|_{x=0}-s^{2} \mathbb{F}_{c}[z(t, x)] \text { provided } z \rightarrow 0 \text { as } x \rightarrow \infty
$$

Note: these results indicate that (i) if $z(t, x)$ is specified at $x=0 \forall t$, then Fourier sine transform is useful and (ii) if $\frac{\partial z}{\partial x}$ at $x=0 \forall t$ is specified, then the Fourier cosine is useful in semi-infinite region.

### 48.3 Heat Conduction - This Infinite Rod - Use of Fourier Transform Method

## Example 3:

Solve $\frac{\partial z}{\partial x}=k \frac{\partial^{2} z}{\partial x^{2}},-\infty<x<\infty ; t>0$; subject to $z(x, 0)=f(x),-\infty<x<\infty$ and $z(x, t)$ is bounded as $x \rightarrow \pm \infty$.

## Solution:

let $\bar{z}(t, s)$ indicate $\mathbb{F}[z(t, x)]$ i.e., $\mathbb{F}[z(t, x)]=\bar{z}(t, s)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} z(t, x) e^{-i s x} d x$

In the above we used the other form of Fourier transformation of a function. Apply Fourier transform to the equation $\frac{\partial z}{\partial t}=k \frac{\partial^{2} z}{\partial x^{2}}$
we get $\frac{\partial}{\partial t} \bar{z}(t, s)+k s^{2} \bar{Z}(t, s)=0$

Its solution is $\bar{z}(t, s)=A(s) e^{-k s^{2} t}$ where $A$ an arbitrary function to be determined from the initial condition. Applying transform to the initial condition,
$\mathbb{F}[z(0, x)]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} z(0, x) e^{i s x} d x$
or $\bar{z}(0, s)=\frac{\sqrt{2}}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(x) e^{i s x} d x \quad=F(s)$ (say)
$\Rightarrow A(s)=F(s)$.

Thus the solution is $\quad \bar{z}(t, s)=F(s) e^{-k s^{2} t}$.
Taking inverse Fourier transform to $\bar{z}(t, s)$, we get $\mathbb{F}^{-1}[\bar{z}(t, s)]=z(t, x)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} F(s) e^{-k s^{2} t} e^{-i s x} d s$
which is the solution of the given equation.

### 48.4 Heat Conduction Problem - Fourier Sine Transform

## Example 4:

Solve $\frac{\partial z}{\partial t}=\frac{\partial^{2} z}{\partial x^{2}}$ subject to $z(t, 0)=0$ for $t>0$,
$z(0, x)=\left\{\begin{array}{l}1,0<x<1 \\ 0, x>1\end{array}\right.$ and $z(t, x)$ is bounded.

## Solution:

Let $\bar{z}(t, s)$ denote $\mathbb{F}_{s}[\bar{z}(t, s)]$, apply Fourier sine transform to $\frac{\partial z}{\partial t}=\frac{\partial^{2} z}{\partial x^{2}}$
we get $\int_{0}^{\infty} \frac{\partial z}{\partial t} \sin s x d x=\int_{0}^{\infty} \frac{\partial^{2} z}{\partial x^{2}} \sin s x d x, \quad \frac{\partial}{\partial t} \bar{z}(t, s)=-s^{2} \bar{z}(t, s)+s z(t, 0)$
$\Rightarrow \frac{\partial \bar{z}}{\partial t}=-s^{2} \bar{z} \Rightarrow \bar{z}=A e^{-s^{2} t}$.

From the initial condition we have $\mathbb{F}_{s}[z(x, 0)]=\int_{0}^{1} 1 \cdot \sin s x \cdot d s=\frac{1-\cos s}{s}$

$$
\begin{aligned}
& \text { or } \bar{z}(s, 0)=A=\frac{1-\cos s}{s} \\
& \Rightarrow \bar{z}(s, t)=\left(\frac{1-\cos s}{s}\right) e^{-s^{2} t}
\end{aligned}
$$

Taking inverse Fourier sine transform, we get $\mathbb{F}_{s}^{-1}[z(x, 0)]=z(x, t)=\frac{2}{\pi} \int_{0}^{\infty} \frac{1-\cos s}{s} e^{-s^{2} t} \sin s x \cdot d s$ is the solution.

Note: The integral in general cannot be evaluated using simple integration rules with real variables, these involve complex integration techniques, so these integrals are left as they are.

### 48.5 Heat Conduction Equation - Fourier Cosine Transform Method:

## Example 3:

Using cosine Fourier transform technique
$\frac{\partial z}{\partial t}=k \frac{\partial^{2} z}{\partial x^{2}} 0 \leq x \leq \infty, t>0$ subject to $z(0, x)=0$ for $x \geq 0 \frac{\partial z}{\partial t}(t, 0)=-a$ (constant) $z(t, x)$ is bounded .

## Solution:

Let $\bar{z}(s, t)$ denote $\mathbb{F}_{c}[z(t, x)]$
Applying Fourier cosine transform to the equation $\frac{\partial z}{\partial t}=k \frac{\partial^{2} z}{\partial x^{2}}$
we get $\frac{\partial \bar{z}(t, s)}{\partial t}=k\left[-s^{2} \bar{z}-\frac{\partial z(t, 0)}{\partial t}\right]$ or $\frac{\partial \bar{z}}{\partial t}+k s^{2} \bar{z}=k a$
which has the solution $\bar{z}(t, s)=\left(k a \frac{e^{k s^{2} t}}{k s^{2}}+c\right) e^{-k s^{2} t}$
where $c$ is the arbitrary constant.

Taking transform to the initial condition, we get $\bar{z}(0, s)=0$
$\Rightarrow \bar{Z}(0, s)=\frac{k a}{k s^{2}}+c=0 \Rightarrow c=\frac{-a}{s^{2}}$
$\therefore \bar{z}(t, s)=\frac{a}{s^{2}}\left(1-e^{-k s^{2} t}\right)$.
Taking inverse Fourier cosine transform , we get $z(t, x)=\frac{2}{\pi} \int_{0}^{\infty} \frac{1}{s^{2}}\left(1-e^{-k s^{2} t}\right) \cos s x \cdot d s$

Keywords: Fourier Cosine Transform Method, Fourier Sine Transform, Laplace transform, Fourier transform, boundary value problem

## Exercises 1

1: Solve $\frac{\partial z}{\partial t}=k \frac{\partial^{2} z}{\partial x^{2}}, 0 \leq x \leq \infty, t>0$ Subject to
(i) $z(t, 0)=z_{0}, t>0$ (ii) $z(0, x)=0 ; 0<x<\infty$ and (iii) $z(t, x)$ is bounded.
2. Solve $\frac{\partial z}{\partial t}=\frac{\partial^{2} z}{\partial x^{2}} ; 0<x<\infty ; t>0$ subjected to the boundary conditions
(i) $\frac{\partial z}{\partial x}(t, 0)=0$ for $t>0$
(ii) $z(t, x)$ is bounded for $x>0, t>0$
and the initial condition (iii) $z(0, x)=\left\{\begin{array}{l}x, 0 \leq x \leq 1 \\ 0, x>1\end{array}\right.$.
3. Solve $\frac{\partial z}{\partial t}=2 \frac{\partial^{2} z}{\partial x^{2}}$ if $z(t, 0)=0 ; z(0, x)=e^{-x}$ and $z(t, x)$ is bounded where $x>0$ and $t>0$.

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## Lesson 49

## Laplace And Fourier Transform Techniques To Wave Equation And Laplace Equation

### 49.1 Introduction

In this lesson we see the utility o Fourier transform technique to the hyperbolic (wave) and elliptic (Laplace) equations.

## Example 1:

Solve the problem of vibrations of an infinite string governed by $\frac{\partial^{2} z}{\partial t^{2}}-c^{2} \frac{\partial^{2} z}{\partial x^{2}}=0 ;-\infty<x<\infty, t>0$ subjected to the initial conditions
(i) $z(0, x)=f(x)$
(ii)

$$
\frac{\partial z}{\partial t}(0, x)=g(x)-\infty<x>\infty
$$

and boundary conditions at far field given by $z(t, x), \frac{\partial z}{\partial x}(t, x) \rightarrow 0$ as $x \rightarrow \pm \infty$.
Solution: Taking Fourier transform to the governing equation and the initial conditions (i),(ii) we get
$\frac{\partial^{2} \bar{z}(t, s)}{\partial t^{2}}+c^{2} s^{2} \bar{z}(t, s)=0 \quad \bar{z}(0, s)=F(s), \quad \frac{\partial z}{\partial t}(0, s)=G(s)$
where $\mathbb{F}[z(t, x)]=\bar{z}(t, s)$ and $\mathbb{F}[f(x)]=F(s)$ and $\mathbb{F}[g(x)]=G(s)$.
Equation (iii) $\Rightarrow \bar{z}(t, s)=A e^{i s c t}+B e^{-i s c t}$.

From all these equations we get $A=\frac{1}{2}\left[F(s)+\frac{G(s)}{i c s}\right]$ and $B=\frac{1}{2}\left[F(s)-\frac{G(s)}{i c s}\right]$.
$\therefore \bar{z}(t, s)=\frac{1}{2}\left[F(s)+\frac{G(s)}{i c s}\right] e^{i s c t}+\frac{1}{2}\left[F(s)-\frac{G(s)}{i c s}\right] e^{-i s c t}$.

Now taking the inverse Fourier transform $z(t, x)=\mathbb{F}^{-1}[\bar{Z}(t, s)]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-i s x} \bar{Z}(t, s) d s$ to this solution, we get

$$
\begin{aligned}
& =\frac{1}{2}\left[\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-i s(x-c t)} F(s) d s+\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-i s(x+c t)} F(s) d s+\frac{1}{c \sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-i s(x-c t)} \frac{G(s)}{i s} d s-\frac{1}{c \sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-i s(x+c t)} \frac{G(s)}{i s} d s\right] \\
& =\frac{1}{2}[f(x-c t)+f(x+c t)]-\frac{1}{2 c} \int_{0}^{x-c t} g(\xi) d \xi+\frac{1}{2 c} \int_{0}^{x+c t} g(\xi) d \xi \\
& =\frac{1}{2}[f(x-c t)+f(x+c t)]+\frac{1}{2 c} \int_{x-c t}^{x+c t} g(\xi) d \xi .
\end{aligned}
$$

This is the D'Alembert's solution of the wave equation.

## Exercise: 1.

A tightly stretched flexible string has its ends fixed at $x=0$ and $x=L$. At time $t=0$, the string is given a shape defined by $f(x)=\mu x(l-x)$, where $\mu$ is a constant and then released. Find the displacement of any point $x$ of the string at any time $t>0$.

## Example 2

An infinitely long string having one end at $x=0$ is initially at rest along the x axis. The end $x=0$ is given a transverse displacement $f(x)$ when $t>0$. Find the displacement of any point of the string at any time.

Solution: Let $z(t, x)$ denote the displacement in the string at any point $x$ at any time $t$, the wave equation is given by
$\frac{\partial^{2} z}{\partial t^{2}}=c^{2} \frac{\partial^{2} z}{\partial x^{2}}, \quad 0<x<\infty ; t>0 \quad$ subject to the initial conditions $z(0, x)=0$; $\frac{\partial z}{\partial t}(0, x)=0$ and $z(t, 0)=f(t)$ and $z(t, x)$ is bounded.

Now taking the Laplace transform on both sides of the governing equation, we get $L\left[\frac{\partial^{2} z}{\partial t^{2}}\right]=c^{2} L\left[\frac{\partial^{2} z}{\partial x^{2}}\right] \Rightarrow s^{2} \tilde{z}-s z(0, x)-\frac{\partial z}{\partial t}(0, x)=c^{2} \frac{\partial^{2} z}{\partial x^{2}}$ where $L[z(t, x)]=\tilde{z}(s, x)$

Using the initial conditions, we get $\Rightarrow s^{2} \tilde{z}=c^{2} \frac{\partial^{2} z}{\partial x^{2}} \Rightarrow \frac{\partial^{2} \tilde{z}}{\partial x^{2}}=\left(\frac{s}{c}\right)^{2} \tilde{z}$
and its solution is $\tilde{z}(s, x)=A e^{\frac{s x}{c}}+B e^{-\frac{s x}{c}}$ where $A$ and $B$ are arbitrary constants.
We have $z(t, 0)=f(t), \quad L[z(t, 0)]=\tilde{z}(s, 0)=L[f(t)]=\tilde{f}(s)$ say at $x=0$
$z(t, x)$ is bounded as $t \rightarrow \infty, \Rightarrow A=0$ in equation.
$\therefore \tilde{z}(s, x)=B e^{-\frac{s x}{c}}$
$\operatorname{Now} \tilde{z}(s, 0) B=\tilde{f}(s)$
$\therefore$ The solution in terms of the transformed variables 's' is $\tilde{z}(s, x)=\tilde{f}(s) e^{\frac{-s x}{c}}$.

On finding the inverse Laplace transform to $\tilde{f}(s) e^{-\frac{-5 x}{c}}$, we obtain the required solution as $z(t, x)=f\left(t-\frac{x}{c}\right)$.

Note: Use the complex inverse formula for inversion as $z(t, x)=\frac{1}{2 \pi i} \int_{a-i \infty}^{a+i \infty} \tilde{f}(s) e^{\left(t-\frac{x}{c}\right) s} d s$

### 49.2 Solution of the Laplace Equation in the Upper Half Plane Dirichlet Problem

## Example 3:

Solve $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0-\infty<x<\infty ; y>0$
subject to the boundary conditions
$z(0, x)=f(x) ;-\infty<x<\infty ;$
$z(x, y \rightarrow \infty)$ is bounded $;-\infty<x<\infty$
both $z$ and $\frac{\partial z}{\partial x} \rightarrow 0$ as $x \rightarrow \pm \infty \ldots .$.

## Solution:

Let $\bar{z}(s, y)$ indicate the Fourier transform of $z(x, y)$. Observe $z$ is defined for $-\infty<x<\infty$ so the Fourier transform technique can be used. Finding the Fourier transform of $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0$,
we get $\frac{\partial^{2} \bar{z}(s, y)}{\partial x^{2}}-s^{2} \bar{z}(s, y)=0$
Its solution is given by $\bar{z}(s, y)=A(s) e^{s y}+B(s) e^{-s y} \ldots .$. (vi)
Given that $z(x, y)$ is bounded as $y \rightarrow \infty$
$\Rightarrow \bar{z}(s, y)$ must also be as $y \rightarrow \infty$

Now this means $A(s)=0$ for $s>0$ and
$B(s)=0$ for $s<0$
$\therefore z(s, y)=\bar{z}(s, 0) e^{-s s y}$
where $\bar{z}(s, 0)=\mathbb{F}[z(x, 0)]=\mathbb{F}[f(x)]=F(s)$
$\therefore z(s, y)=F(s) e^{-s \mid y} \ldots .$. (vii)

We note $\mathbb{F}\left(e^{-|s| y}\right)=\sqrt{\frac{2}{\pi}}\left(\frac{y}{y^{2}+x^{2}}\right)$

## Exercise 2

Taking inverse Fourier transform on both sides of equation (vii) and applying the conclusion theorem, we get $z(x, y)=f(x) \sqrt{\frac{2}{\pi}}\left(\frac{y}{y^{2}+(x-\xi)^{2}}\right)$
$=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(\xi) \sqrt{\frac{2}{\pi}}\left(\frac{y}{y^{2}+x^{2}}\right) d \xi=\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{y^{2}+(x-\xi)^{2}} d \xi$.
Thus $z(x, y)=\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{y^{2}+(x-\xi)^{2}} d \xi$ is the solution of the Laplace equation in the upper half plane.

Exercise: 1. Take $f(x)=1$ and obtain $z(x, y)$.
2. Take $f(x)=x$ and obtain $z(x, y)$.

### 49.3 Solution of Laplace Equation in the Upper Half Plane Neumann Problem

## Example 4:

Solve the Laplace equation with derivative boundary condition given by $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0,-\infty<x<\infty ; y>0$ subject to $\frac{\partial z(x, 0)}{\partial y}=g(x)-\infty<x<\infty$ with the condition that $z$ is bounded as $y \rightarrow \infty$ and $z$ and $\frac{\partial z}{\partial x}$ vanish as $x \rightarrow \pm \infty$ and $\int_{-\infty}^{\infty} g(x) d x=0$ which is the necessary condition for the existence of solution.

Solution: Use the transformation $v(x, y)=\frac{\partial z(x, y)}{\partial x}$
Then $z(x, y)=\int_{a}^{y} v(x, \eta) d \eta$.

Now $\frac{\partial^{2} v}{\partial x^{2}}+\frac{\partial^{2} v}{\partial y^{2}}=\frac{\partial}{\partial y}\left(\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}\right)=0$,
$v(x, 0)=\frac{\partial z(x, 0)}{\partial y}=g(x) \quad$ (given)
Thus we have $\frac{\partial^{2} v}{\partial x^{2}}+\frac{\partial^{2} v}{\partial y^{2}},-\infty<x<\infty$;
subject to $v(x, 0)=g(x) ;-\infty<x<\infty$ $v$ is bounded as $y \rightarrow \infty$, both $v$ and $\frac{\partial v}{\partial x}$ vanish as $x \rightarrow \pm \infty$.

Thus we have the Dirichlet problem in terms of the new dependent function $v(x, y)$. Its solution is given by
$v(x, y)=\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)}{y^{2}+(\xi-x)^{2}} d \xi$
$\Rightarrow z(x, y)=\frac{1}{\pi} \int_{a}^{y} \eta\left(\int_{-\infty}^{\infty} \frac{g(\xi)}{y^{2}+(\xi-x)^{2}} d \xi\right) d \eta$
$=\frac{1}{2 \pi} \int_{-\infty}^{\infty} g(\xi) \log \left[\frac{(\xi-x)^{2}+y^{2}}{(\xi-x)^{2}+a^{2}}\right] d \xi$ is the required solution.

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