Engineering Mathematics-III

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Engineering Mathematics-III

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Lesson 1

FiniteDifferences

1.1 Introduction

The analytical solution to a problem provides the value of the dependent variable for any value of the independent variable. Consider, for instance the simple Spring-mass system governed by the differential equation.

$$\ddot{y} + \lambda^2 y = 0 \tag{1.1}$$

whose analytical solution can be readily written as:

$$y(t) = c_1 \sin \lambda t + c_2 \cos \lambda t \tag{1.2}$$

on any interval. Solution (1.2) gives the function value at any point in the interval.If equation (1.1) is solved numerically, the time interval is first divided into a pre-determined finite number of non-overlapping equally spaced or otherwise subintervals and the numerical solution is obtained at the end points of these subintervals. If the interval considered is (say) $[t_0, t_n]$and if the interval is divided into N number of non-overlapping subintervals, say of equally spaced ones, then the numerical solution is obtained at the set of points $\{t_0, t_1, t_2, ..., t_{N-1}, t_N\}$, with the difference between t_{j+1} and t_j being constant for every j, j $= 0, 1, ..., N-1$.

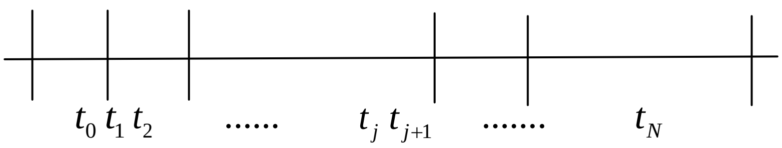

Fig.1.1: Equally spaced division of the interval $[t_0, t_N]$.

Similarly, this can be done with unequally spaced subintervals also

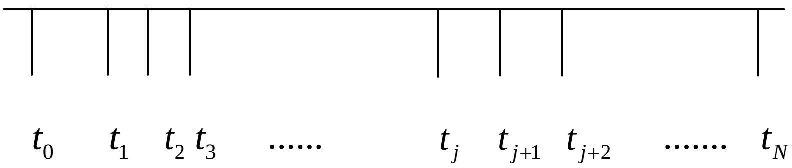

Fig.1.2: Unequally spaced division of the interval $[t_0, t_n]$.

The set of points $\{t_0, t_1, t_2, ..., t_N\}$ are called the nodal points. The numerical solution at non-nodal points can also be obtained either using the interpolation of the data at the nodal points. This concept is true when one is solving an ordinary or partial differential equation. Many of the Finite difference operators are applicable over equally spaced data points.

In the later part of this lesson, we shall define some of the finite differences operators and their utility and their relationships.

1.2 Difference Operators and their Utility

Let $y(t)$ be the variable depending on the independent variable t, consider the equally spaced $(N+1)$ data points, such that $t_{j+1} - t_j = h$, h being constant.

Table 1.1. Equally spaced data set

t_0	t_1	t_2	t_3	$...t_{j-1}$	t_j	t_{j+1}	......	t_N
y_0	y_1	y_2	y_3	$\cdots y_{j-1}$	y_j	y_{j+1}	......	y_N

The difference operator acts on the dependent function. Notation: $y_j = y(t_j)$

1.2.1 Shift Operator

It is denote by E, when it acts on y_j, it shifts the data solution at t_j to the solution at t_{j+1} (i.e., the data is moved by one spacing forward) i.e.,

$$Ey_j = y_{j+1} \tag{1.3}$$

Similarly, $$E^{-1}y_j = y_{j-1} \tag{1.4}$$

For any (positive or negative) integer n,

$$E^n y_j = y_{j+n} \tag{1.5}$$

1.2.2 Forward Difference Operator

It is denoted by Δ, which is defined as

$$\Delta y_i = y_{i+1} - y_i \tag{1.6}$$

This is called the first forward difference operator. The second forward difference operator is Δ^2, its action is defined as:

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i \tag{1.7}$$
$$= (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i)$$

Or $$\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i \tag{1.8}$$

In a similar way for any positive integer n, the n^{th} forward difference operator is defined as:

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i \tag{1.9}$$

1.2.3 Backward Difference Operator:

The first backward difference operator is denoted by ∇, and this is denoted as

$$\nabla y_i = y_i - y_{i-1} \tag{1.10}$$

The second order backward difference is ∇^2 and this is defined as:

$$\begin{aligned} \nabla^2 y_i &= y_i - y_{i-1} \\ &= y_i - 2y_{i-1} + y_{i-2} \end{aligned} \tag{1.11}$$

The n^{th} order backward difference of y_i is

$$\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1} \tag{1.12}$$

Observe that

$\nabla^n y_{i+1} = y_{i+1} - y_i = \nabla y_i$ (1.13)

This means their difference operators are related to each other.

1.2.4 The Central Difference Operator

It is denoted by δ and it is defined as:

$$\delta y_i = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}} \tag{1.14}$$

Here, $$y_{i+\frac{1}{2}} = y(t_i + \frac{h}{2}) \text{ and } y_{i-\frac{1}{2}} = y(t_i - \frac{h}{2}).$$

Also, $$\delta^2 y_i = \delta y_{i+\frac{1}{2}} - \delta y_{i-\frac{1}{2}}$$
$$= (y_{i+1} - y_i) - (y_i - y_{i-1})$$
$$= y_{i+1} - 2y_i + y_{i-1} \tag{1.15}$$

$$\delta^n y_i = \delta^{n-1} y_{i+\frac{1}{2}} - \delta^{n-1} y_{i-\frac{1}{2}} \tag{1.16}$$

Observe that, $$\delta^2 y_i = y_{i+1} - 2y_i + y_{i-1}$$
$$= \Delta^2 y_{i+1} \tag{1.17}$$

1.3 Construction of Difference Tables

Now, let us construct the forward, backward and central difference tables for the given data points.

Example 1: Construct the forward difference table for the following set of equally spaced data given by:

t_0	t_1	t_2	t_3	t_4	t_5
y_0	y_1	y_2	y_3	y_4	y_5

Solution:

t_0	y_0	Δ	Δ^2	Δ^3	Δ^4	Δ^5

t_1	y_1	$y_1 - y_0 = \Delta y_0$				
t_2	y_2	$y_2 - y_1 = \Delta y_1$	$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$			
t_3	y_3	$y_3 - y_2 = \Delta y_2$	$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$		
t_4	y_4	$y_4 - y_3 = \Delta y_3$	$\Delta y_3 - \Delta y_2 = \Delta^2 y_2$	$\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$	$\Delta^3 y_1 - \Delta^3 y_0 = \Delta^4 y_0$	
t_5	y_5	$y_5 - y_4 = \Delta y_4$	$\Delta y_4 - \Delta y_3 = \Delta^2 y_3$	$\Delta^2 y_3 - \Delta^2 y_2 = \Delta^3 y_2$	$\Delta^3 y_2 - \Delta^3 y_1 = \Delta^4 y_1$	$\Delta^4 y_1 - \Delta^4 y_0 = \Delta^5 y_0$

From the forward difference table constructed above it is noted that for a data with 6 data points, we have the maximum order forward difference is Δ^5 and all differences of order 6 and above are zero. The entries with the same subscript on y lie on sloping lines downward.

Example 2: Construct the backward difference table for the given data:

t_i	-1	0	1	2	3
y_i	9.1	8.2	7.3	6.4	5.5

Solution:

We know $\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}$. Given $y_0 = 9.1$, $y_1 = 8.2$, $y_3 = 7.3$, $y_4 = 6.4$, $y_5 = 5.5$.

Now the difference table is:

		∇	∇^2	∇^3	∇^4
-1	9.1				
0	8.2	-0.9=∇y_1			
1	7.3	-0.9=∇y_2	0=$\nabla^2 y_2$		
2	6.4	-0.9=∇y_3	0=$\nabla^2 y_3$	0=$\nabla^3 y_3$	
3	5.5	-0.9=∇y_4	0=$\nabla^2 y_4$	0=$\nabla^3 y_4$	0=$\nabla^4 y_4$

It is noted that for the given data, upto 4^{th} order backward differences exist. It is also evident that the second order differences onward the values are zero, the reason being all the first order differences are same.

Example 3: Compute a table of differences through Δ^3 and ∇^3 for the function $y(t) = t^2 - 1$ using the step size $h = 1$ and $t_0 = -1$.

Solution: For the given function, the data set is constructed as:

t	$t_0 = -1$	$t_1 = 0$	$t_2 = 1$	$t_3 = 2$
y(t)	$y_0 = 0$	$y_1 = -1$	$y_2 = 0$	$y_3 = 3$

For a second degree polynomial $t^2 - 1$, the third order (forward) difference is zero, that is the reason; we considered only 4 data points.

Now the forward difference table is:

		Δ	Δ^2	Δ^3
-1	0			
0	-1	-1=Δy_0		
1	0	1=Δy_1	2=$\Delta^2 y_0$	

2	3	3= Δy_2	2= $\Delta^2 y_1$	0= $\Delta^3 y_0$

Similarly the backward difference table is written as:

		∇	∇^2	∇^3
-1	0= y_0			
0	-1= y_1	-1= ∇y_1		
1	0= y_2	1= ∇y_2	2= $\nabla^2 y_2$	
2	3= y_3	3= ∇y_3	2= $\nabla^2 y_3$	0= $\nabla^3 y_3$

The illustration reveals that the difference table is the same for both forward and backward differences, but the entries are labelled differently.

Example 4: Form the central differences for the data given in the example 1.

Solution:

t	y	δ	δ^2	δ^3	δ^4
t_0	y_0				
t_1	y_1	$y_1 - y_0 = \delta y_{\frac{1}{2}}$			
t_2	y_2	$y_2 - y_1 = \delta y_{\frac{3}{2}}$	$\delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}} = \delta^2 y_1$		
t_3	y_3	$y_3 - y_2 = \delta y_{\frac{5}{2}}$	$\delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}} = \delta^2 y_2$	$\delta^3 y_{\frac{3}{2}}$	
t_4	y_4	$y_4 - y_3 = \delta y_{\frac{7}{2}}$	$\delta y_{\frac{7}{2}} - \delta y_{\frac{5}{2}} = \delta^2 y_3$	$\delta^3 y_{\frac{5}{2}}$	$\delta^4 y_2$

In the central difference table, the entries with the same script lie on the horizontal lines.

Keywords: Difference Tables, Central Difference Operator, Backward Difference Operator, Forward Difference Operator, Shift Operator, Finite Differences.

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Lesson 2

Relation between Difference Operators

2.1 Introduction

In the previous lecture, we have noticed from the difference table that these difference operators are related. In this lecture we establish the relations between these operators.

Example 1: Show that the shift operator is related to the forward difference operator as $\Delta = E - 1$ [1being the identity operator] and to the backward difference operator ∇ as $\nabla = 1 - E^{-1}$.

Solution:

By definition, the forward difference operator when operating over the function data y_i, Δy_i, it becomes

$$\begin{aligned} \Delta y_i &= y_{i+1} - y_i \\ &= Ey_i - y_i \\ &= (E-1)y_i \\ \therefore \Delta &= E - 1. \end{aligned}$$

Similarly,

$$\begin{aligned} \nabla y_i &= y_i - y_{i-1} \\ &= y_i - E^{-1}y_i \\ &= (1 - E^{-1})y_i \end{aligned}$$

From the above example, one can write that $E = \Delta + 1$ and $E^{-1} = 1 - \nabla$.

Example 2: Establish $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$.

Solution: We know $\delta y_i = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}$

$$= E^{\frac{1}{2}} y_i - E^{-\frac{1}{2}} y_i \quad = (E^{\frac{1}{2}} - E^{-\frac{1}{2}}) y_i$$

$$\therefore \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}.$$

From these examples, one can establish the relation between Δ, ∇ and δ as:

$$\delta = (1+\nabla)^{\frac{1}{2}} - (1-\nabla)^{\frac{1}{2}}.$$

Example 3: Verify $\nabla E = E\nabla = \Delta = \delta E^{\frac{1}{2}}$

Solution:

$$\nabla E y_i = \nabla\left(y_{i+1}\right) = y_{i+1} - y_i = \Delta y_i.$$

And $E\nabla y_i = E\left(y_i - y_{i-1}\right)$

$$= y_{i+1} - y_i$$

$$= \Delta y_i$$

Similarly, $\delta E^{\frac{1}{2}} y_i = \delta y_{i+\frac{1}{2}} = y_{i+1} - y_i = \Delta y_i$.

Thus we have $\nabla E = E\nabla = \Delta = \delta E^{\frac{1}{2}}$.

Example 4: Show that $2\mu\delta = \nabla + \Delta$ where μ is the averaging operator defined as $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$.

Solution:

$$2\mu\delta y_i = 2\mu\left[y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}\right]$$

$$= 2\mu y_{i+\frac{1}{2}} - 2\mu y_{i-\frac{1}{2}}$$

$$= \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)y_{i+\frac{1}{2}} - \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)y_{i-\frac{1}{2}}$$

$$= \left(y_{i+1} + y_i\right) - \left(y_i + y_{i-1}\right)$$

$$= y_{i+1} - y_{i-1}$$

Also $$\left(\nabla + \Delta\right)y_i = \left(y_i - y_{i-1}\right) + \left(y_{i+1} - y_i\right)$$

$$= y_{i+1} - y_{i-1}$$

$$\therefore 2\mu\delta = \nabla + \Delta .$$

Exercise 1: Show that $\Delta = \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}$ and $\nabla = \delta\sqrt{1 + \frac{\delta^2}{4}} - \frac{\delta^2}{2}$.

In example 3 of previous lesson, we observed that the third order difference for a second degree polynomial is zero. This is similar to the third derivative of a second degree polynomial is zero. This gives an intuition that the differential operator D is connected with the difference operator.

Let us denote D by $\frac{dy}{dt}$, D^2 by $\frac{d^2y}{dt^2}$ and so on.

Relation between Difference Operators

Consider the function $y(t)$ at a general node t_{i+1}. Now $y\left(t_{i+1}\right)=y(t+h)$ can be expanded in Taylor series about the nodal point t_i [assuming the continuity of the higher order derivative of $y(t)$ at t_i],

$$y(t_{i+1})=y(t_i+h)=y\left(t_i\right)+h\left.\frac{dy}{dt}\right|_{t_i}+\left.\frac{h^2}{2!}\frac{d^2y}{dt^2}\right|_{t_i}+..... \tag{2.1}$$

Using the operators E and D, the above equation (2.1) can be written as

$$Ey_i=\left[1+hD+\frac{h^2}{2!}D^2+\frac{h^3}{3!}D^3+.....\right]y_i \tag{2.2}$$

Or
$$Ey_i=(e^{hD})y_i$$

Or
$$E=e^{hD} \tag{2.3}$$

Also, we know that $E=\Delta+1$, using this equation (2.3) we have:

$$\Delta=e^{hD}-1=hD+\frac{h^2D^2}{2!}+\frac{h^3D^3}{3!}+..... \tag{2.4}$$

Thus the forward difference operator Δ is connected with the differential operator D. We can express D explicitly in terms of Δ.

From equation (2.4), we can write

$$hD=\ln\left(1+\Delta\right)$$

$$= \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - ....$$

Relation between Difference Operators

or
$$D = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - ....\right) \qquad (2.5)$$

Now, the second order difference Δ^2 can be written as:

$$\Delta^2 = \left(hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ...\right)^2 \qquad (2.6)$$

Or
$$\Delta^2 = h^2D^2 + h^3D^3 + \frac{7}{12}h^4D^4 + ...$$

Exercise 2: Show that $D^2 = \frac{1}{h^2}\left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 ....\right)$.

Exercise 3: Establish

(i) $E^{\frac{1}{2}} = e^{\frac{hD}{2}}$

(ii) $\nabla = hD - \frac{h^2}{2!}D^2 + \frac{h^3}{3!}D^3 + ...$

(iii) $D^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + ...\right]$

(iv) $\delta^2 = h^2D^2 + \frac{h^4D^4}{12} + \frac{h^6D^6}{360} + ...$

Thus the first and second order derivatives of a function $y(t)$ at t_i are written using Δ as:

$$\frac{dy_i}{dt} = Dy_i = \frac{1}{h}(\Delta y_i - \frac{\Delta^2 y_i}{2} + \frac{\Delta^3 y_i}{3} - .....) \qquad (2.7)$$

$$\frac{d^2 y_i}{dt^2} = D^2 y_i = \frac{1}{h^2}(\Delta^2 y_i - \Delta^3 y_i + \frac{11}{12}\Delta^4 y_i - .....) \qquad (2.8)$$

Relation between Difference Operators

Similarly, in terms of ∇ we can write the differential operator as:

$$Dy_i = \frac{1}{h}(\nabla y_i + \frac{\nabla^2 y_i}{2} + \frac{\nabla^3 y_i}{3} + ...) \qquad (2.9)$$

and $$D^2 y_i = \frac{1}{h^2}(\nabla^2 y_i + \nabla^3 y_i + \frac{11}{12}\nabla^4 y_i + ...) \qquad (2.10)$$

In terms of δ:

$$Dy_i = \frac{\mu}{h}(\delta y_i - \frac{\delta^3 y_i}{6} + \frac{\delta^5 y_i}{30} - .....) \qquad (2.11)$$

and $$D^2 y_i = \frac{1}{h^2}(\delta^2 y_i - \frac{\delta^4 y_i}{12} + \frac{\delta^6 y_i}{90} - .....) \qquad (2.12)$$

In the next example we illustrate the use of the relations among these operators.

Example 5: Consider the function $y(t) = t^3 + 2t^2 + 3t$ in the interval $[-2, 2]$ with step size $h = 1.0$. Find the approximate value for (i) $\frac{dy}{dt}$ at $t = -1$ and $\frac{d^2y}{dt^2}$ at $t = -1$ using the forward differences and (ii) $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ at $t = 2$ using the backward differences.

Solution:

Step1: Let us construct the data set for the given function: Given $h = 1$,

Relation between Difference Operators

$t_0 = -2$	$y_0 = -6$
$t_1 = -1$	$y_1 = -2$
$t_2 = 0$	$y_2 = 0$
$t_3 = 1$	$y_3 = 6$
$t_4 = 2$	$y_4 = 22$

t	-2	-1	0	1	2
y	-6	-2	0	6	22

Step 2: Construct the difference table. Note that for the given cubic, the third derivative is constant and the fourth and higher derivatives are zero. Similarly the third difference will be constant and all higher order differences will become zero. The expressions for $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ are as given in equations (2.7), (2.8), (2.9) and (2.10).

Difference Table:

t	y				
-2	-6				

-1	-2	$\Delta y_0 = 4$			
0	0	$\Delta y_1 = 2$	$\Delta^2 y_0 = -2$		
1	6	$\Delta y_2 = 6$	$\Delta^2 y_1 = 4$	$\Delta^3 y_0 = 6$	
2	22	$\Delta y_3 = 16$	$\Delta^2 y_2 = 10$	$\Delta^3 y_1 = 6$	$\Delta^4 y_0 = 0$

Note: $\Delta^3 y_0 = \Delta^3 y_1 = 6$ (constant), $\Delta^4 y_0 = 0 = \Delta^5 y_0 .....$

Relation between Difference Operators

Step 3: Calculations:

(i) $\left.\frac{dy}{dt}\right|_{t=-1} = \left.\frac{dy}{dt}\right|_{t=t_1} = \frac{1}{h}\left[\Delta y_1 - \frac{1}{2}\Delta^2 y_1 + \frac{1}{3}\Delta^3 y_1\right]$

$= \frac{1}{1}\left[2 - \frac{1}{2}\cdot 4 + \frac{1}{3}\cdot 6\right] = 2.$

$\left.\frac{d^2y}{dt^2}\right|_{t=t_1} = \frac{1}{h^2}\left[\Delta^2 y_1 - \Delta^3 y_1\right] = \frac{1}{1}[4-6] = -2$

Thus $y'(-1) = 2$ and $y''(-1) = -2$.

These approximations are coinciding with the exact values.

(ii) From the above table, we know $\nabla y_4 = 16$, $\nabla^2 y_4 = 10$, $\nabla^3 y_4 = 6$.

$\left.\frac{dy}{dt}\right|_{t=t_4} = \frac{1}{h}\left[\nabla y_4 + \frac{1}{2}\nabla^2 y_4 + \frac{1}{3}\nabla^3 y_4\right]$

$= \frac{1}{1}\left[16 + \frac{1}{2}\cdot 10 + \frac{1}{3}\cdot 6\right] = 23.$

$$\left.\frac{d^2y}{dt^2}\right|_{t=t_4} = \frac{1}{h^2}\left[\nabla^2 y_4 + \nabla^3 y_4\right]$$

$$= \frac{1}{1}\left[10+6\right] = 16.$$

These values also coincide with the exact values of $y'(2) = 23$ and $y''(2) = 16$. Thus these differences give us a way of evaluating the derivative values from the given data set.

Exercise 4: Find $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ at $t = 0$ from the data set.

t	0	1	2	3	4
y	-1	2	-3	4	5

Exercise 5: Find $\frac{d^2y}{dt^2}$ at $t = 2$ from the given data.

t	-1	0	1	2
y	-1	1	3	5

Exercise 6: Write D^3 in terms of (i) Δ (ii) ∇ (iii) δ and μ.

Keyword: Forward difference operator

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Lesson 3

Sources of Error and Propagation of Error in the Difference Table

3.1 Introduction

In a numerical procedure, many types of errors can occur, among which the prominent ones are (i) the truncation error and (ii) round-off error. We briefly introduce these two types of errors below. The other types of errors are not discussed in this lesson.

We noted that while writing the Taylor series expansion for a function about a point in its neighbourhood, the series is truncated after a finite number of terms in the series. This induces certain amount of error in the solution and this error is called the truncation error. Thus the truncation error is the quantity T (say) which must be added to the approximate solution in order to get the exact solution.

For example, consider the function $f(x) = (1+x)^{\frac{1}{2}}, x \in [0, 0.1]$. We now try to obtain a second degree polynomial approximation to this function by using the Taylor Series expansion about the point $x = 0$.

We have $f(0) = 1$, and a certain higher order derivatives of the function and their values at $x = 0$ are found as:

$$f'(x) = \frac{1}{2\sqrt{1+x}},\ f'(0) = \frac{1}{2};\ f''(x) = -\frac{1}{4(1+x)^{\frac{3}{2}}},\ f''(0) = -\frac{1}{4};\ f'''(x) = \frac{3}{8(1+x)^{\frac{5}{2}}}.$$

Now the second order approximation with the remainder term is written as:

$$\therefore (1+x)^{\frac{3}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}\frac{x^3}{\left[(1+\xi)^{\frac{1}{2}}\right]^5}, \text{for some } 0 < \xi < 0.1.$$

Thus the truncation error after 2^{nd} degree approximation is $T = \frac{1}{16}\frac{x^3}{\left[(1+\xi)^{\frac{5}{2}}\right]}$.

If we compute the value of $f(x)$ at $x = 0.05$, using this approximation we get the value as $f(0.5) \approx 1.0246875$. The exact value is 1.0246951077.

The upper bound for the Truncation error for $x \in [0, 0.1]$ is given by

$$|T| = \max_{x\in[0,0.1]} \frac{(0.1)^3}{16\left[(1+x)^{\frac{5}{2}}\right]} \leq \frac{(0.1)^3}{16} = 0.000625.$$

Thus the approximate value of $f(x)$ at 0.05 is 1.0246875 which is with a maximum error of 0.000625. This error can be further lowered by considering a higher order approximation for $f(x)$.

There is another type of error called the round off error which is due to the precision of the computer while doing the arithmetic operations among the number. The round off error is the quantity denoted by R which must be added to the finite representation of a computed number in order to make it the true representation of that number. This is due to the rounding of numbers after a certain decimal place during computation. The present day computers use the double precession, because of which the round off error is minimum.

In general, the error is defined as the difference between the True value and the Approximate value.

$Error = True\ value - Approximate\ value.$

$Absolute\ error = |Error|.$

The Relative error is defined as

$$Relative\ error = \frac{Absolute\ error}{|True\ Value|}.$$

Also one uses the Percentage Relative error which is given by

$$Percentage\ Relative\ error = 100 \times \frac{|Error|}{|True\ Value|}.$$

While dealing with finite difference operators, usually, the solution is associated with a certain amount of Truncation error. This error can be minimized by increasing the order of approximation in the Taylor Series expansion. The difference operators are magnifiers of the error that occurred in the initial data set, in turn affecting the approximate solution. The following example illustrates how a small amount of error $\in$ induced in the initial data is increasing with higher order differences.

Example 1: Tabulate the propagation of initial error $\in$ with higher order forward difference operators for the data:

x	-3	-2	-1	0	1	2	3
$f(x)$	0	0	0	$\in$	0	0	0

Solution: The forward difference table for the above given data set is:

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
-3	0						
		0					
-2	0		0				

				$\in$			
-1	0		$\in$		$-4\in$		
		$\in$		$-3\in$		$10\in$	
0	$\in$		$-2\in$		$6\in$		$-20\in$
		$-\in$		$3\in$		$-10\in$	
1	0		$\in$		$-4\in$		
		0		$-\in$			
2	0		0				
		0					
3	0						

From the above we note that

i) The magnitude of error increases with the order of the differences

ii) The error for any order difference is the binomial coefficients with alternating signs.

iii) The algebraic sum of the errors in any column is zero.

Exercises:

1. Find the approximate value of $f(2)$ from $f(x)=ln(x)$, $x\in[1,3]$ by considering a 3^{rd} order Taylor series approximation. Obtain the local truncation error and the percentage relative error in this approximation.

2. If the number $\pi=4\tan^{-1}(1)$ is approximated using 5 decimal digit, find the percentage relative error due to rounding.

3. Tabulate the propagation of initial error $\in$ in the following data set with increasing order differences.

x	0	1	2	3	4
$f(x)$	-1	1	$\in$	2	2

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Lesson 4

Solutions of Non-Linear Equation

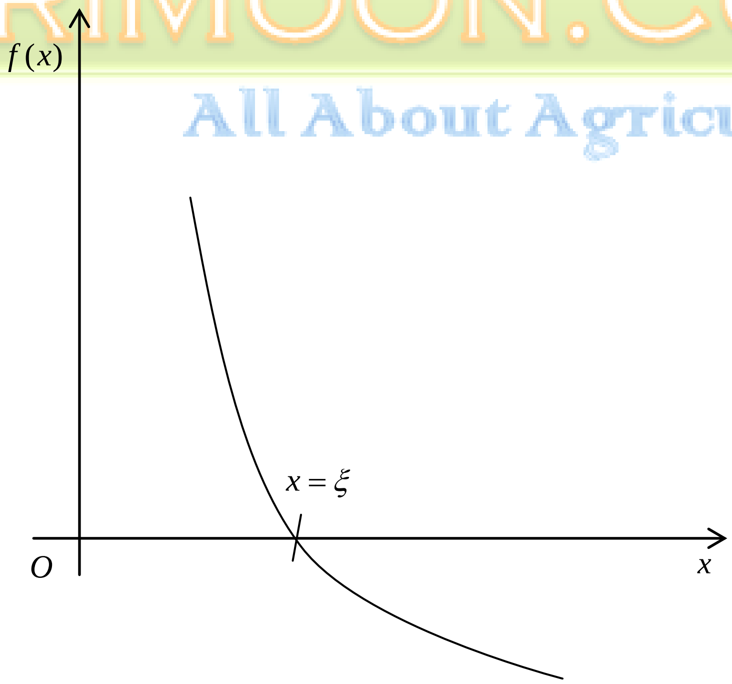
4.1 Introduction

In this lesson, we learn to find the roots of a given non-linear equation involving a polynomial and Transcendental functions.

For example (i) $x^3 - 4x^2 - \frac{1}{x} = 0$ is a polynomial equation while

(ii) $\sin x - x\cos\frac{1}{x} = 0$ is a transcendental equation.

Definition: A number ξ is said to be the root of (or solution of) the equation $f(x) = 0$ if $f(\xi) \equiv 0$. It is also called a zero of $f(x) = 0$. A root of $f(x) = 0$ is the value of x at which the graph of $y = f(x)$ intersects the $x-$axis.



If the equation $f(x)=0$ can be expressed as $f(x)=(x-\xi)^n g(x)=0$ for some $g(x)$ such that $g(\xi)\neq 0$ and $g(x)$ is bounded for all x in the domain of definition of the function $f(x)$, then $x=\xi$ is called a multiple root of $f(x)=0$ with multiplicity n. If $n=1$, then it is called a simple root.

The root of $f(x)=0$ is found using either a **direct method**, which gives the exact value of $x=\xi$ or by writing $f(x)=0$ as an **iterative procedure** such as $x_{n+1}=g(x_n),\ n=0,1,2,.....$. Root of cubic or higher degree polynomial equations and transcendental equations cannot be found using the direct methods whereas the iterative methods are quite handy though they provide approximate value to the root of $f(x)=0$.

4.2 Iterative Methods

These methods are based on the idea of successive approximations.

Consider for example the equation $f(x)=x^2+2x-4=0$. This can be written as

(i) $x=\dfrac{4}{x+2}$ or (ii) $x=\dfrac{1}{2}\left(4-x^2\right)$ or (iii) $x=\sqrt{4-2x}$.

Here we expressed the given $f(x)=0$ in three different forms as $x=\varphi_i(x)$, $i=1,2,3$ where $\varphi_1(x)=\dfrac{4}{x+2}$, $\varphi_2(x)=\dfrac{1}{2}\left(4-x^2\right)$ and $\varphi_3(x)=\sqrt{4-2x}$.

Thus any given $f(x)=0$ is written as $x=\varphi(x)$, then the root of $f(x)=0$ is the point of intersection of the function $y=\varphi(x)$ with then line $y=x$.

The function $\varphi(x)$ is called the iteration function. This is obtained by writing an iterative method as $x_{n+1}=\varphi(x_n)$ and generate a sequence of iterates $\{x_k\}_{k=1}^{\infty}$ by starting with a suitable initial approximation x_0, i.e., start with x_0, generate $x_1 = \varphi(x_0)$, using x_1, generate x_2, repeat this finitely many times until we notice that the condition $|x_{k+1} - x_k| < \in$ is satisfied, where x_k, x_{k+1} are two consecutive iterates and $\in$ is the pre assigned error tolerance. We note that with different initial approximations and different initial iterative function, the sequence of approximations generated will be different but in all these cases, these sequences always give its limit as the root of the equation $f(x) = 0$.

Definition: A sequence of iterates $\{x_k\}$ is said to converge to the root $x = \xi$, if $\lim_{k \to \infty} x_k = \xi$. The convergence of an iteration method depends on the suitable choice of the function $\varphi(x)$ and also on the initial approximation x_0 to the root. Below we state a sufficient condition on $\varphi(x)$ for the convergence of $\{x_k\}$ to the root ξ.

Result: If $\varphi(x)$ is a continuous function in some closed interval $I \subset \mathbb{R}$ that contains the root ξ of $x = \varphi(x)$ and $|\varphi'(x)| \le a < 1, \ \forall \ x \in I$, then for any choice of $x_0 \in I$, the sequence $\{x_k\}$ generated from $x_{k+1} = \varphi(x_k), k = 0, 1, 2, \ldots,$ converges to the root ξ of $x = \varphi(x)$.

Example: Find the root of $f(x) = \cos x + 3 - 2x = 0$ correct to two decimal places.

Solution:

Given $2x = \cos x + 3$ or $x = \frac{1}{2}(\cos x + 3)$

Choose $\varphi(x) = \frac{1}{2}(\cos x + 3)$

Write $x_{n+1} = \frac{1}{2}(\cos x + 3)$

Since $|\varphi'(x)| = \frac{1}{2}|\sin x| < 1$,

we start with $x_0 = \frac{\pi}{2}$, and generate the sequence of iterates as

$$x_1 = \frac{1}{2}\left(\cos\frac{\pi}{2} + 3\right) = 1.5$$

$$x_2 = \frac{1}{2}\left(\cos(1.5) + 3\right) = 1.535$$

$$x_3 = \frac{1}{2}\left(\cos(1.535) + 3\right) = 1.518$$

$$x_4 = \frac{1}{2}\left(\cos(1.518) + 3\right) = 1.526$$

$$x_5 = \frac{1}{2}\left(\cos(1.526) + 3\right) = 1.522$$

The root of the above equation correct to two decimal places is taken as 1.522 since $|x_5 - x_4| = |1.522 - 1.526| = 0.004$. We can improve the accuracy in this approximation by considering more iteration.

4.2.1 How to make a good guess:

Convergence of the successive iterations $\{x_k\}$ obtained for a given iterative method $x_{k+1} = \varphi(x_k)$ depends on a good guess for the initial approximation x_0 for the root $x = \xi$. The initial approximation is usually obtained from the physical considerations of the problem and also based on the graphical representation of the given function $f(x)$. For the solution of algebraic equation $f(x) = 0$, an elegant method to choose the initial approximation is by using the intermediate value theorem which is stated below:

Intermediate value Theorem:

If $f(x)$ is continuous on some interval $I=[a,b]\subset\mathbb{R}$ and the product $f(a)\cdot f(b)<0$, then the equation $f(x)=0$ has at least one real root or an odd number of real roots in the interval (a,b).

Now it amounts to choosing two real values a and b such that $f(a)\cdot f(b)<0$, and then choose the initial approximation between a and b.

Example 1: Find an interval in which the root of $x^3+4x-5=0$ lies in.

Solution:

Given $f(x)=x^3+4x-5$.

Take $x=0$, $f(x)=-5$.

Again when $x=2$, $f(2)=11$.

Now $f(0)$ and $f(2)$ are having opposite signs. Hence $f(0)\cdot f(2)<0$.

The intermediate value theorem guarantees that a root of $x^3+4x-5=0$ will lie in the interval $(0,2)$.

[Exercise: Plot this function – it cuts the x-axis at $x=1$].

Example 2: Find the interval which contains the smallest positive root of $xe^x-\cos x=0$.

Solution:

Take $x = 0$, $f(0) = 0 \cdot e^0 - \cos 0 = -1$, and $x = 1$, $f(1) = 2.718 - 0.54 = 2.178$. Since $f(0) \cdot f(1) < 0$, the root is in $(0,1)$.

4.3 Bisection Method

This method is based on the repeated application of the intermediate value theorem. Let the root of the equation $f(x) = 0$ be in the interval $I_0 = (a_0, b_0)$. Bisect the interval I_0 and let m_1 be the midpoint of (a_0, b_0), i.e., $m_1 = \frac{1}{2}(a_0 + b_0)$.

Check the following conditions:

i) if $f(a_0) \cdot f(m_1) < 0$, then root lies between a_0 and m_1.

ii) if $f(m_1) \cdot f(b_0) < 0$, then the root lies between m_1 and b_0.

Accordingly take I_1 as either (a_0, m_1) or (m_1, b_0), as one of these contains the root. Bisect I_1 and let m_2 be the midpoint of I_1. The above conditions are checked for these bisected intervals to get the new subinterval that contains the root.

Keep bisecting this way until we get a smallest subinterval containing the root. Then the midpoint of that smallest interval is taken as the approximation for the root of $f(x) = 0$.

The following example illustrates the Bisection method:

Example 3: Find a positive root of the equation $f(x) = xe^x - 1$ which lies in $(0,1)$.

Solution:

Given $a_0 = 0, b_0 = 1$,

$f(a_0) = -1,\ f(b_0) = 1.718,\ f(0)\cdot f(1) < 0$.

Step 1: Bisecting $(0,1)$ gives $(0,0.5)$ and $(0.5,1)$ as two subintervals. $f(0.5) < 0$ and $f(0) < 0$, so the interval $(0,0.5)$ is discarded as $f(0)f(0.5) > 0$. Denote $I_1 = (0.5,1)$.

Step 2: Bisecting $I_1 = (0.5,1)$ gives $(0.5,0.75)$ and $(0.75,1)$ as the new subintervals. $f(0.75) > 0$, $f(0.5) < 0 \Rightarrow f(0.5)f(0.75) < 0$,

also $f(0.75) > 0$, $f(1.0) > 0 \Rightarrow f(0.75)f(1) > 0$.

Discard the interval $(0.75,1.0)$ and denote $I_2 = (0.5,0.75)$.

Step 3: Bisecting I_2 gives $(0.5,0.625)$ and $(0.625,0.75)$. Note $f(0.5).f(0.625) < 0$ and $f(0.625).f(0.75) > 0$ so discard the interval $(0.625,0.75)$ and take $I_3 = (0.5,0.625)$.

Step 4: Bisecting I_3, we get $(0.5,0.5625)$ and $(0.5625,0.625)$, we see $f(0.5625)f(0.625) < 0$, take $I_4 = (0.5625,0.625)$ keep doing this process for more steps to get a better approximation to the root. After step 4, the root lies in I_4 and we take the approximate value of the root as the midpoint of this interval I_4, i.e., $\xi = 0.59375$.

This is a crude approximation to the root.

Note: Both the iteration method and Bisection method take more steps to give a solution with reasonable accuracy.

Exercises:

1. Use bisection method to obtain a real root for the following equations correct to three decimal places:

   i) $x^3 + x^2 - 1 = 0$

   ii) $e^{-x} = 10x$

   iii) $4x - 1 = 4\sin x$

   iv) $x + \log x = 2$.

2. By constructing a proper iterative function for the given equation, find an approximation to its root.

   i) $\sin x = 1 - x$

   ii) $e^x = \cot x$

   iii) $x^2 - x^3 = -1$

   iv) $\sin x = \frac{x}{2}$

   v) $x^4 + x^2 - 80 = 0$

   vi) $\sin^2 x = x^2 - 1$.

Keywords: Polynomial and Transcendental Functions, Direct Method, Iterative Methods, Iterative Procedure

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Lesson 5

Secant and Regula-Falsi Methods

5.1 Introduction

A root of the algebraic equation $f(x)=0$ can be obtained by using the iteration methods based on the first degree equation. We approximate the given $f(x)$ by a first degree equation in the neighborhood of the root of $f(x)=0$.

We may write the first degree equation as:

$$f(x)=a_1x+a_0 \tag{5.1}$$

with $a_1 \neq 0$ and a_0 and a_1 are arbitrary parameters to be determined by posing two conditions on $f(x)$ or its derivative at two different x locations on the curve.

5.2 Scant Method

Take two approximations x_{k-1} and x_k to the root ξ of $f(x)=0$. We determine the constants a_0 and a_1 in equation (5.1) by using the conditions

$$f_{k-1}=a_1x_{k-1}+a_0 \tag{5.2}$$

and

$$f_k=a_1x_k+a_0 \tag{5.3}$$

Where, $f(x_{k-1}) = f_{k-1}$ and $f(x_k) = f_k$.

Solving equations (5.2) and (5.3), we get,

$$a_0 = \frac{(x_k f_{k-1} - x_{k-1} f_k)}{(x_k - x_{k-1})} \tag{4}$$

and
$$a_1 = \frac{(f_k - f_{k-1})}{(x_k - x_{k-1})} \tag{5}$$

The solution of (5.1) is

$$x = -\frac{a_0}{a_1} \tag{6}$$

Now the new approximation x_{k+1} based on two initial approximations x_{k-1} and x_k is written as equation (5.6) as:

$$x_{k+1} = \frac{(x_k f_{k-1} - x_{k-1} f_k)}{(f_k - f_{k-1})}$$

This may be rewritten as:

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{(f_k - f_{k-1})} f_k,\ k = 1, 2, \ldots \tag{7}$$

Here iteration function $\varphi(x_k, x_{k-1}) = x_k - \frac{(x_k - x_{k-1})}{(f_k - f_{k-1})} f_k$ generates a sequence of iterates $\{x_{k+1}\}_{k=1}^{\infty}$.

5.2.1 Algorithm

1. Start with (x_0, f_0) and (x_1, f_1).
2. Generate x_2 using (7) and compute $f_2 = f(x_2)$.
3. Use (x_1, f_1) and (x_2, f_2) to compute x_3.
4. Repeat steps (2) & (3) until the difference between two successive approximations x_{k+1} and x_k is less than the error tolerance.

The method fails if at any stage of computation $f(x_k) = f(x_{k-1})$ (see equation 5.7).

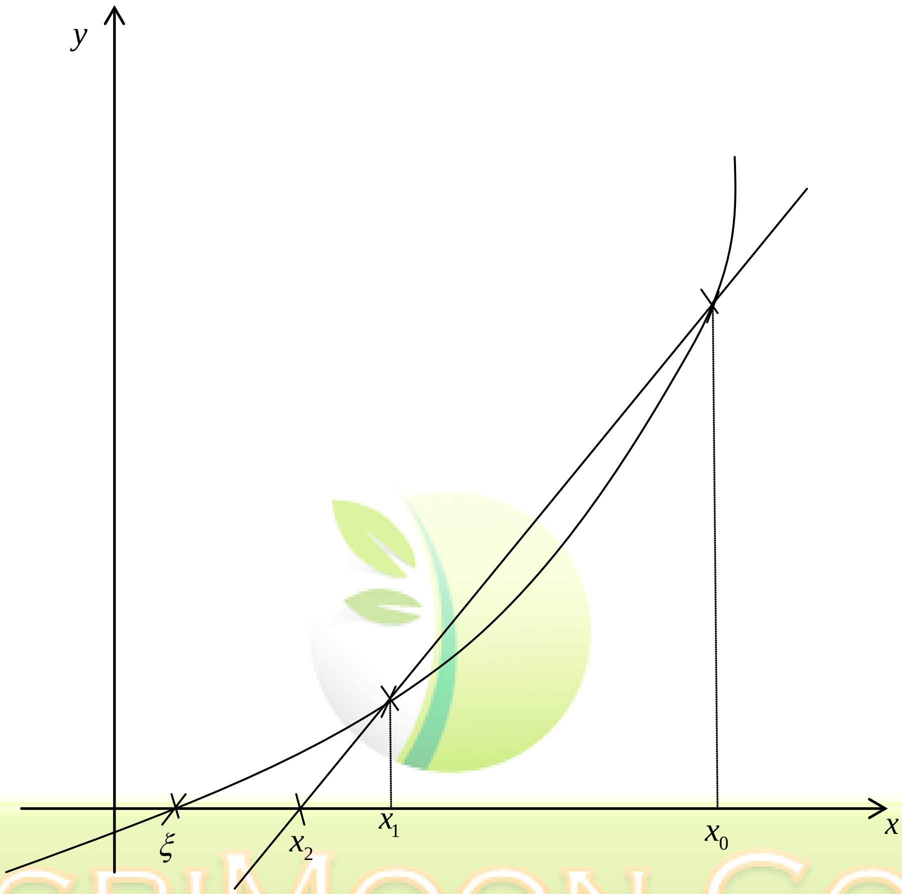

Fig. 5.1. Secant Method – a Graphical Representation.

5.3 Regula-Falsi Method

The initial approximations x_{k-1} and x_k taken in the secant method are arbitrary. If we impose a condition on these initial approximations such that $f(x_k).\,f(x_{k-1})<0$ the method given by equation (5.7) satisfying this condition is called the Regula-falsi method. Then the root of $f(x)=0$ lies in between these two values x_k and x_{k-1}. Now draw a chord joining the points $(x_{k-1},\, f_{k-1})$ and $(x_k,\, f_k)$. This chord intersects the $x-$ axis, say at $x=x_{k+1}$. Now look for $f(x_{k+1}).\,f(x_k)<0$ or $f(x_{k+1}).\,f(x_{k-1})<0$. One of these two will hold. Without loss of generality, say $f(x_{k+1}).\,f(x_k)<0$, then join the points (x_k, f_k) and (x_{k+1}, f_{k+1}) by a chord, this chord intersects $x - axis$ say at $x = x_{k+2}$. The above procedure is repeated till we reach the root of $f(x) = 0$. In this procedure we are sure of convergence of the sequence of iterates generated using equation (5.7).

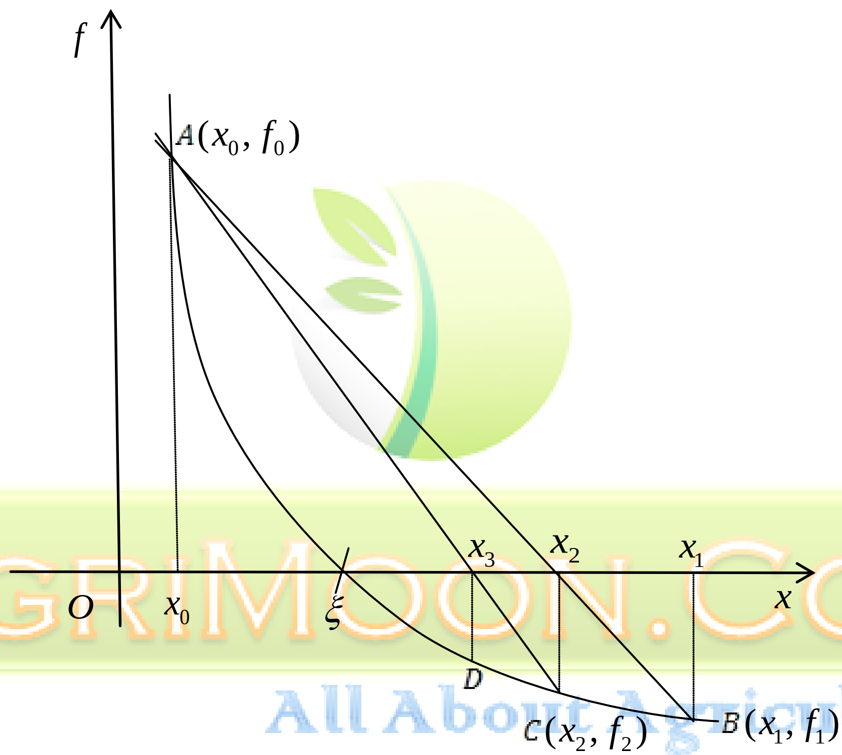

Fig. 5.2. Regula - Falsi Method – a Graphical Representation.

Example 1: Find the real root of $x^3 - 2x - 5 = 0$ using the Regula-falsi method.

Solution:

Given $f(x) = x^3 - 2x - 5$. We compute that $f(2) = -1 < 0$ and $f(3) = 16 > 0$.

Take $x_0 = 2, f_0 = -1, x_1 = 3, f_1 = 16$.

Using the method given by (7),

$$x_2 = 3 - \frac{3-2}{16+1}16 = 2.0813, \quad f(x_2) = -0.1472 < 0.$$

The root lies in the interval $(2.083, 3)$, join the points $(2.0813, -0.1472)$ and $(3, 16)$ by the chord, which cuts the $x-$axis at $x_3 = 2.08964$.

Proceeding in this way, we obtain

$x_4 = 2.09274$, $x_5 = 2.09388$, $x_6 = 2.0943$ etc.

The approximate value for the root is taken as $x = 2.0943$.

Now we illustrate this method to find a root of the transcendental equation.

Example 2: Find the root of $\cos x - xe^x = 0$ using the Regula-falsi method correct to four decimal places.

Solution:

Let $f(x) = \cos x - xe^x$, $f(0) = 1$, $f(1) = -2.17798$.

Clearly $f(0)\,.\,f(1) < 0$.

We obtain $x_2 = x_0 - \dfrac{x_1 - x_0}{f_1 - f_0} f_0 \; = 0.31467$ and $f(0.31467) = 0.51987 > 0$.

The root lies between 0.31467 and 1. Repeating the procedure we obtain the approximations to the roots as

$x_3 = 0.44673$,

$x_4 = 0.49402$,

$x_5 = 0.50995$

$x_6 = 0.51520$, $x_7 = 0.51692$

$x_8 = 0.51748, x_9 = 0.51767$,

$x_{10} = 0.51775, \ldots$

The root correct to 4 decimal places as $\xi = 0.5177$.

Exercises

1. Find a real root of the following equations using the Regula-falsi method:

a) $x \log_0 x = 1.2$

b) $x^4 - 32 = 0$

c) $x^3 - 2x - 5 = 0$

d) $sinx = \dfrac{1}{x}$

2. Solve the above problems using the Secant method.

Keywords: Scant Method, Regula-Falsi Method

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Lesson 6

Newton-Raphson Method

6.1 Introduction

Both the Bisection and Regula-fasi methods give a rough estimate for the root of the equation $f(x) = 0$, but these methods take many iterations to get a reasonably accurate approximation to the root. An elegant method of obtaining the root of $f(x) = 0$ is discussed below which is known as Newton-Raphson method.

Let x_0 be an approximate root of the equation $f(x) = 0$. Let x_1 be a neighbouring point of x_0, such that for every small $h > 0$, $x_1 = x_0 + h$. Also let x_1 be the exact root of $f(x) = 0$.

Then $f(x_1) = 0$.

Expanding $f(x_0 + h)$ in Taylor Series about x_0, weget $f(x_1) = f(x_0 + h) = f(x_0) + hf'(x_0) + ... = 0$.

Since h is very small, we neglected h^2 and higher powers of h in the above Taylor series expansion.

Thus we have $f(x_0) + hf'(x_0) = 0$ giving $h = -\dfrac{f(x_0)}{f'(x_0)}$.

Then we have $x_1 = x_0 + h = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

∴ a closer approximation to the root of the given $f(x) = 0$ is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{6.1}$$

A better approximation x_2 can be obtained by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{6.2}$$

We generalize this procedure and write a general iteration method as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0,1,2,\ldots \tag{6.3}$$

This is called the Newton-Raphson iteration method. Comparing this method with the general iteration method

$$x_{n+1} = \varphi(x_n) \tag{6.4}$$

the iteration function is given as $\varphi(x) = x - \dfrac{f(x)}{f'(x)}$.

For convergence of this iteration method the sufficient condition is $|\varphi'(x)| < 1$

which becomes $\left|1 + \dfrac{f'}{f'} + \dfrac{f \cdot f''}{f'^2}\right| = \left|\dfrac{f(x)\cdot f''(x)}{[f'(x)]^2}\right| < 1$ or $|f(x)\cdot f''(x)| < \left|[f'(x)]^2\right|$.

If we choose the initial approximation x_0 in the interval I such that for all $x \in I$,

$\left|f(x)\cdot f''(x)\right| < \left|\left[f'(x)\right]^2\right|$ is satisfied, then the sequence of iterations generated by iteration method (3) will converge to the root of $f(x) = 0$.

Example 1: Using the Newton-Raphson method, find a real root of $f(x) = x^4 - 11x + 8 = 0$.

Solution:

Given $f(x) = x^4 - 11x + 8 = 0$

$f'(x) = 4x^3 - 11$.

Choose $x_0 = 2$. This initial approximation can be obtained by using the intermediate value theorem.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2}{11} = 1.90476$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.90476 - \frac{(1.90476)^4 - 11(1.90476) + 8}{4(1.90476)^3} = 1.89209,$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.89188$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.89188.$$

We now accept the numerical approximation to the root as $\xi = 1.89188$, correct to 5 decimal places.

Note that this method requires evaluation of $f'(x)$ at every stage. Also if $f'(x)$ is very large, then $h\left(= -\frac{f(x)}{f'(x)}\right)$ will be very small. Thus x_0 is close to x_1 (which is assumed to be the root of $f(x) = 0$ in the derivation of the method) and makes the convergence of the successive iterations to the root faster.

Example 2: Evaluate $\sqrt{29}$ to four decimal places using the Newton-Raphson method.

Solution:

Let $x = \sqrt{29}$. Then $x^2 - 29 = 0$

Consider $f(x) = x^2 - 29$, then $f'(x) = 2x$.

$$\therefore x_{n+1} = x_n - \frac{x_n^2 - 29}{2x_n}, \; n = 0,1,2,\dots.$$

Take $x_0 = 3.3$, we get $x_1 = 5.3858$, $x_2 = 5.3851$ and $x_3 = 5.3851$.

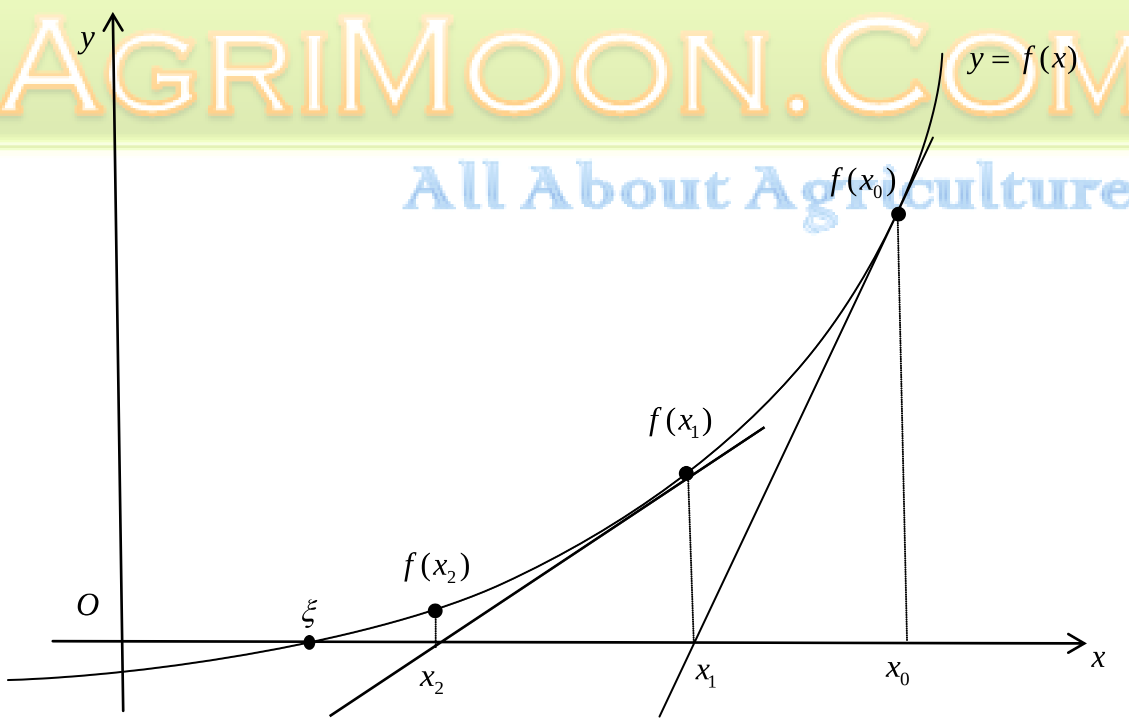

Fig. 6.1. Newton-Raphson method – a schematic representation; $\xi \approx f(x_n)$

Example 3: Find a root of $f(x)=3x-\cos x-1=0$ lying between 0 and 1.

Solution: Given $f(x)=3x-\cos x-1=0$

$$f'(x)=3+\sin x \quad \text{choose } x_0=0.6.$$

$$x_{n+1}=x_n-\frac{3x_n-\cos x_n-1}{3+\sin x_n}$$

$$x_{n+1}=\frac{x_n\sin x_n+\cos x_n+1}{3+\sin x_n}$$

This gives $x_1=0.6071$, $x_2=0.6071$. Thus the approximation to the root is 0.6071.

Definition: An iterative method $x_{n+1}=\varphi(x_n)$ is said to be of order n or has the rate of convergence n if n is the largest positive real number for which there exists a finite constant $M\neq 0$ such that $|\epsilon_{k+1}|\le M|\epsilon_k|^n$, where $\epsilon_k=x_k-\xi$ is the error in the k^{th} iterate. The constant M is called the asymptotic error constant.

Exercise 4: Determine the order of the Newton-Raphson method.

Solution:

The Newton-Raphson method is $x_{k+1}=x_k-\dfrac{f(x_k)}{f'(x_k)}$.

Take $\epsilon_k=x_k-\xi$ or $x_k=\xi+\epsilon_k$ and expanding $f(\xi+\epsilon_k)$ and $f'(\xi+\epsilon_k)$ in Taylor Series about the root ξ, we obtain.

$$\epsilon_{k+1}=\epsilon_k-\frac{\left[\epsilon_k f'(\xi)+\frac{1}{2}\epsilon_k^2 f''(\xi)+\ldots\right]}{\left[f'(\xi)+\epsilon_k f''(\xi)+\ldots\right]}$$

$$= \in_k - \left[\in_k + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)} \in_k^{\;2} + \ldots \right]\left[1 + \frac{f''(\xi)}{f'(\xi)} \in_k + \ldots \right]^{-1}$$

$$\text{or } \in_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)} \in_k^{\;2} + O(\in_k^{\;3}).$$

On neglecting $\in_k^{\;3}$ and higher power $\in_k$, we get $\in_{k+1} = C \in_k^{\;2}$ where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$.

Thus the Newton-Raphson method is a second order method.

Exercises:

Solve the following equations $f(x) = 0$ using Newton-Raphson method:

i) $x - e^{-x} = 0$

ii) $\cos x - xe^{-x} = 0$

iii) $x^3 - 5x + 1 = 0$

iv) $\sin x + \frac{\cos x}{x} = 0$

v) $x^3 - 2x - 5 = 0$

Keywords: Sufficient Condition, Newton-Raphson Iteration Method

References

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Lesson 7

Linear System of Algebraic Equations – Jacobi Method

7.1 Introduction

In this lesson we discuss an iterative method which gives the solution of a system linear algebraic equations.

Consider a system of algebraic equations as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots \quad \ldots \quad \ldots & \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned} \tag{7.1}$$

This is a linear system of n-algebraic equations in n unknowns. In general, the number of equations is not equal to the number of unknowns. In (7.1), $a_{ij}, i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$ are the given coefficients, $b_i, i = 1, 2, \ldots, n$ are the known numbers and $x_i, i = 1, 2, \ldots, n$ are the unknown numbers to be determined.

The system given in (7.1) can also be written in the matrix vector form as:

$$\underline{A}\underline{x} = \underline{b} \tag{7.2}$$

where $\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$, $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$

A is called the coefficient matrix.

The solution of this system can be obtained using direct methods such as Gauss Elimination and also by finding the inverse of the matrix A directly. When the size of the matrix A is very large, applicability of these direct methods is limited as finding the inverse of a large size matrix is not so easy. Also, Gauss elimination demands the diagonal dominant structure for A. These difficulties are overcome in the other class of methods called the iterative methods. In what follows, we discuss two useful iterative methods namely the Jacobi and Gauss-Seidel methods.

The iterative methods are based on the idea of successive approximations. Initially, the system of equations is written as:

$$\underline{X}_{n+1} = H\underline{X}_n + \underline{C}\,,\ n = 0,1,2,\ldots \tag{7.3}$$

where H is the Iterative matrix which depends on the matrix A and $\underline{C}$ is a column vector which depends on both A and $\underline{b}$. We start with an initial approximation to the solution vector $\underline{x} = \underline{x}_0$ and obtain a sequence of approximation to $\underline{x}$ as $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, \ldots$, this sequence, in the limit as $n \to \infty$, converge to the exact solution vector $\underline{x}$. We stop the iteration process when the magnitude of the two successive iterates of $\underline{x}$ i.e., $\underline{x}_{n+1}$ and $\underline{x}_n$ is smaller than the pre-assigned error tolerance $\in$, $|\underline{x}_{n+1} - \underline{x}_n| \le \in$ for all elements of $\underline{x}$.

The procedure of obtaining the iterative matrix H is given below.

Let the diagonal elements $a_{ii},\ i = 1, 2, \ldots, n$ in the linear system (1) do not vanish. We now rewrite the system (1) as:

$$x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2 - \frac{a_{13}}{a_{11}} x_3 - \ldots - \frac{a_{1n}}{a_{11}} x_n$$

$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1 - \frac{a_{23}}{a_{22}} x_3 - \ldots - \frac{a_{2n}}{a_{22}} x_n$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1 - \frac{a_{n3}}{a_{22}} x_2 - \ldots - \frac{a_{n\cdot n-1}}{a_{nn}} x_{n-1} \tag{7.4}$$

The system of equations (7.4) is meaningful only if all a_{ii} (diagonal elements) are non-zero. If some of the diagonal elements in the system of equations given in (7.1) are zero, than the equations should be rearranged so that this condition satisfied. We now form an iterative mechanism for the equation (7.4) by writing

$$x_1^{(n+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \frac{a_{13}}{a_{11}} x_3^{(n)} - \ldots - \frac{a_{1n}}{a_{11}} x_n^{(n)}$$

$$x_2^{(n+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n)} - \frac{a_{23}}{a_{22}} x_3^{(n)} - \ldots - \frac{a_{2n}}{a_{22}} x_n^{(n)}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$x_n^{(n+1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n)} - \frac{a_{n2}}{a_{nn}} x_2^{(n)} - \ldots - \frac{a_{n\cdot n-1}}{a_{nn}} x_{n-1}^{(n)}. \tag{7.5}$$

Choose the set of initial approximations as: $\underline{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}\right)$ and generate a sequence of iterates $\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(k)}, \underline{x}^{(k+1)}, \ldots,$ until the convergence condition $\left|\underline{x}^{(k+1)} - \underline{x}^{(k)}\right| \leq \in$ is satisfied.

Equation (7.5) is written in the matrix vector form as $\underline{X}^{(n+1)} = H\underline{X}^{(n)} + \underline{C}$

where $H = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & \cdots & -\frac{a_{2n}}{a_{11}} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n1}}{a_{nn}} & \cdots & 0 \end{bmatrix}$, $C = \begin{bmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \cdots \\ \frac{b_n}{a_{nn}} \end{bmatrix}$.

In matrix vector form, the Jacobi method is derived as follows:

Given $A\underline{x} = \underline{b}$, decompose A as the sum of lower Triangular, Diagonal and upper triangular matrices. This decomposition is always possible.

i.e., $A = L + D + U$

$Ax = (L + D + U)\underline{x} = b$

or $D\underline{x} = -(L+U)\underline{x} + \underline{b}$

or $\underline{x} = -D^{-1}(L+U)\underline{x} + D^{-1}\underline{b}$

or $\underline{x}^{k+1} = -D^{-1}(L+U)\underline{x}^{k} + D^{-1}b$

or $\underline{x}^{k+1} = H\underline{x}^{k} + \underline{C}$

where the Iteration matrix H in Jacobi method is $-D^{-1}(L+U)$ and C is $D^{-1}b$.

Example 1: Solve the following system of equations using Jacobi method

$$4x_1 + x_2 + x_3 = 2$$

$$x_1 + 5x_2 + 2x_3 = -6$$

$$x_1 + 2x_2 + 3x_3 = -4$$

by taking the initial approximation as $\underline{x}^{(0)} = [0.5, -0.5, -0.5]^T$.

Solution:

Given $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Note that $a_{ii} \neq 0$ for $i = 1,2,3$.

First write A as $A = L + D + U$, where

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \; L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \text{ and } U \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The iteration matrix H is

$$H = -D^{-1}(L+U) = -\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix}$$

and the column vector

$$C = D^{-1}b = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{6}{5} \\ -\frac{4}{3} \end{bmatrix}.$$

We now write the **Jacobi iterative method** as:

$$\underline{x}^{(n+1)} = \begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \\ x_3^{(n+1)} \end{pmatrix} = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \\ x_3^{(n)} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{6}{5} \\ -\frac{4}{3} \end{bmatrix}, n = 0,1,2,\ldots \qquad \ldots\ldots \text{(i)}$$

Start with the given initial approximation $\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$, we generate from (i)

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix}.$$

Using this, we generate the next approximation as $\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.85 \end{bmatrix}$, using this, $\begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0.9933 \\ -1.0733 \\ -1.1 \end{bmatrix}$, ...

The exact solution of this system is $x_1 = 1, x_2 = -1, x_3 = -1$.

Example 2: Solve the following system of linear algebraic equations using the Jacobi method by writing the iterative system directly:

$20x + y - 2z = 17$, $3x + 20 - z = -18$, $2x - 3y + 20z = 25$.

Solution:

We can directly write the iterative system as

$$x_1^{(n+1)} = \frac{1}{20}\left(17 - y^{(n)} + 2z^{(n)}\right)$$

$$y_1^{(n+1)} = \frac{1}{20}\left(-18 - x^{(n)} + z^{(n)}\right)$$

$$z_1^{(n+1)} = \frac{1}{20}\left(25 - 2x^{(n)} + 3y^{(n)}\right) n = 0,1,2,\ldots$$

Start with $x^{(0)} = y^{(0)} = z^{(0)} = 0$.

$$x^{(1)} = 0.85, y^{(1)} = -0.9, z^{(1)} = 1.25.$$

Using this, we generate

$$x^{(2)} = 1.02, y^{(2)} = -0.965, z^{(2)} = 1.1515.$$

In the same way, we generate few more successive approximations as

$$x^{(3)} = 1.0134, y^{(3)} = -0.9954, z^{(3)} = 1.0032$$

$$x^{(4)} = 1.0009, y^{(4)} = -1.0018, z^{(4)} = 0.9993$$

$$x^{(5)} = 1.0, y^{(5)} = -1.0002, z^{(5)} = 0.9996$$

$$x^{(6)} = 1.0, y^{(6)} = -1.0, z^{(6)} = 1.0$$

Thus the solution is: $x = 1.0$, $y = -1.0$, $z = 1.0$.

This is an alternative way of writing the iteration procedure used in the earlier example.

Exercises:

1. Solve the equations $10x_1 - 2x_2 - x_3 - x_4 = 3, -2x_1 + 10x_2 - x_3 - x_4 = 15, -x_1 - x_2 + 10x_3 - 2x_4 = 27$ $-x_1 - x_2 - 2x_3 + 10x_4 = -9$ using the Jacobi method by taking $x_1^{(0)} = 0.3, x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$.

2. Write L, D, U, H and C for the following system of equations: $54x + y + z = 110, 2x + 15y + 6z = 72, -x + 6y + 27z = 85$. Solve this system using the Jacobi iterative method by taking $x^{(0)} = 2, y^{(0)} = 0, z^{(0)} = 0$.

Keywords: System of Linear Algebraic Equations, Jacobi Iterative Method

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Lesson 8

Gauss-Seidel Iteration Method

8.1 Introduction

This method is applied to the linear system of algebraic equation $Ax = b$ for which the diagonal elements of A are larger in absolute value than the sum of other elements in the each of its row in.

One should arrange, by row and column interchange that larger elements fall along the diagonal, to the maximum possible extent. This method may be seen as an improvement to the Jacobi method, where the available values for the unknowns in a particular iteration are used in the same iteration. Consider the system of linear algebraic equations (5) as given in the lesson 7.

$$x_1^{(1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(0)} - \frac{a_{13}}{a_{11}} x_3^{(0)} - \ldots - \frac{a_{1n}}{a_{11}} x_n^{(0)}$$

$$x_2^{(1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(0)} - \frac{a_{23}}{a_{22}} x_3^{(0)} - \ldots - \frac{a_{2n}}{a_{22}} x_n^{(0)}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$x_n^{(1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(0)} - \frac{a_{n2}}{a_{nn}} x_2^{(0)} - \ldots - \frac{a_{n \cdot n-1}}{a_{nn}} x_{n-1}^{(0)}.$$

This is the first step of the Jacobi iteration method.

In the first step of the iteration, we make use of the initial approximations $x_2^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ in the first of the above equation to evaluate $x_1^{(1)}$ using,

$x_1^{(1)} = \frac{b_1}{a_{11}} - \frac{1}{a_{11}}\left(a_{12}x_2^{(0)} + a_{13}x_3^{(0)} + \ldots + a_{1n}x_n^{(0)}\right)$. This approximation for $x_1^{(1)}$ is used in approximating $x_2^{(1)}$ as shown below:

$$x_2^{(1)} = \frac{b_2}{a_{22}} - \frac{1}{a_{22}}\left(a_{21}x_2^{(1)} + a_{23}x_3^{(0)} + \ldots + a_{2n}x_n^{(0)}\right)$$

Likewise, the other unknown are found as shown below:

$$\begin{array}{l} x_3^{(1)} = \frac{b_3}{a_{33}} - \frac{1}{a_{33}}\left(a_{31}x_1^{(1)} + a_{32}x_2^{(1)} + \ldots + a_{3n}x_n^{(0)}\right) \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n^{(1)} = \frac{b_n}{a_{nn}} - \frac{1}{a_{nn}}\left(a_{n1}x_1^{(1)} + a_{n2}x_2^{(1)} + \ldots + a_{n\cdot n-1}x_n^{(1)}\right) \end{array}$$

We proceed to find the second approximation for the solution in the same manner. The iteration process is terminated using the same criterion that was discussed in the case of Jacobi method.

We express the Gauss-Seidel method in the matrix form as follows for $Ax = b$.

Let $A = L + D + U$, where L and U are the lower and upper triangular matrices with zeros for the diagonal entries and D is the diagonal matrix.

Then, $Ax = (L + D + U)x = b$.

$$(L + D)\underline{x}^{(k+1)} = b - U\underline{x}^{(k)}$$

$$\underline{x}^{(k+1)} = (D + L)^{-1}b - (D + L)^{-1}U\underline{x}^{(k)}$$

or $\underline{x}^{(k+1)} = H\underline{x}^{(k)} + C$

where $H = -(D+L)^{-1} \cdot U$ and $C = (D+L)^{-1} b$.

Let us illustrate the use of Gauss-Seidel method below.

Example 1: Solve the system of equations using the Gauss-Seidel method correct to three decimal places:

$x + 2y + z = 0$ ...... (i)

$3x + y - z = 0$ ...... (ii)

$x - y + 4z = 3$ ...... (iii)

Solution:

In the above equations, note that

i. $|1| > |2| + |1|$, i.e., $|1| > |3|$ is false.

ii. $|1| > |3| + |-1|$ i.e., $1 > 4$ is false.

iii. $|4| > |1| + |-1|$ i.e., $4 > 2$ is true.

Thus in the first two equations, the diagonal dominance is not present.
We now rearrange the order of the given system as:

$3x + y - z = 0$

$x + 2y + z = 0$

$x - y + 4z = 3$

Now this system shows diagonal dominance as $3 \geq 2, 2 \geq 2, 4 > 2$.

We now write the iterative process for the above as:

$$x^{(k+1)} = \frac{1}{3}\left[-y^{(k)} + z^{(k)}\right]$$

$$y^{(k+1)} = \frac{1}{2}\left[-x^{(k+1)} - z^{(k)}\right]$$

$$z^{(k+1)} = \frac{1}{4}\left[3 - x^{(k+1)} + y^{(k+1)}\right],\ k = 0,1,2,\ldots$$

We start with the initial guess values as $x^{(0)} = 1, y^{(0)} = 1, z^{(0)} = 1$.

This gives,

$$x^{(1)} = \frac{1}{3}\left[z^{(0)} - y^{(0)}\right] = 0$$

$$y^{(1)} = \frac{1}{2}\left[-x^{(1)} - z^{(0)}\right] = -0.5$$

$$z^{(1)} = \frac{1}{4}\left[3 - x^{(1)} + y^{(1)}\right] = 0.625.$$

The second iteration is given by

$$x^{(2)} = \frac{1}{3}\left[z^{(1)} - y^{(1)}\right] = 0.375$$

$$y^{(2)} = \frac{1}{2}\left[-x^{(2)} - z^{(1)}\right] = -0.5$$

$$z^{(2)} = \frac{1}{4}\left[3 - x^{(2)} + y^{(2)}\right] = 0.53125.$$

Proceeding in this way, we generate the sequence of approximations as:

$$x^{(3)} = 0.34375, y^{(3)} = -0.4375, z^{(3)} = 0.55469$$

$$x^{(4)} = 0.33075, y^{(4)} = -0.44271, z^{(4)} = 0.55664$$

$$x^{(5)} = 0.33312, y^{(5)} = -0.4449, z^{(5)} = 0.5555.$$

The approximate solution correct to 3 decimal places is take as:

$x = 0.333, y = -0.445, z = 0.555.$

Example 2: Solve the system of equations

$2x - y = 7, -x + 2y - z = 1, -y + 2z = 1$ using the Gauss-Seidel method by taking the initial approximations as $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$.

Solution:

Given $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$, we find the decomposition of A as

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Gauss-Seidel method is $\underline{x}^{(k+1)} = -(D+L)^{-1} \cdot U\underline{x}^{(k)} + (D+L)^{-1}b$.

$$(D+L) = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}, (D+L)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{bmatrix},$$

$$(D+L)^{-1}U=\begin{bmatrix}0 & -\frac{1}{2} & 0\\ 0 & -\frac{1}{4} & -\frac{1}{2}\\ 0 & -\frac{1}{8} & -\frac{1}{4}\end{bmatrix},\quad (D+L)^{-1}b=\begin{bmatrix}\frac{7}{2}\\ \frac{9}{4}\\ \frac{13}{8}\end{bmatrix}.$$

Applying the above method,

$$\begin{pmatrix}x^{(1)}\\ y^{(1)}\\ z^{(1)}\end{pmatrix}=\begin{bmatrix}0 & \frac{1}{2} & 0\\ 0 & \frac{1}{4} & \frac{1}{2}\\ 0 & \frac{1}{8} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}+\begin{bmatrix}\frac{7}{2}\\ \frac{9}{4}\\ \frac{13}{8}\end{bmatrix}=\begin{bmatrix}3.5\\ 2.25\\ 1.625\end{bmatrix}$$

Using $\begin{pmatrix}x^{(1)}\\ y^{(1)}\\ z^{(1)}\end{pmatrix}$, we generate $\begin{pmatrix}x^{(2)}\\ y^{(2)}\\ z^{(2)}\end{pmatrix}=\begin{pmatrix}4.625\\ 3.625\\ 2.3125\end{pmatrix}$,

Similarly, $\begin{pmatrix}x^{(3)}\\ y^{(3)}\\ z^{(3)}\end{pmatrix}=\begin{pmatrix}5.3125\\ 4.3125\\ 2.6563\end{pmatrix}$.

Note that these iterates are approaching the exact solution $x=6,\ y=5,\ z=3$.

Exercises: By choosing suitable initial approximations, solve the following system of linear algebraic equations using Gauss-Seidel method.

1. $8x+2y-2z=8,\ x-8y+3z=-4,\ 2x+y+9z=12.$
2. $2x+10y+z=51,\ x-2y+8z=5,\ 15x+3y-2z=85.$
3. $x+2y+z=0,\ 3x+y-5=0,\ x-y+4z=3.$

4. $10x_1 - 2x_2 - x_3 - x_4 = 3, -2x_1 + 10x_2 - x_3 - x_4 = 15, -x_1 - x_2 + 10x_3 - 2x_4 = 27,$
$-x_1 - x_2 - 2x_3 + 10x_4 = -9$.

Keywords: Linear Algebraic Equations, Gauss-Seidel Method, Diagonal Dominance

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Lesson 9

Decomposition Methods

9.1 Introduction

LU and Cholesky Decomposition methods:

Gauss Elimination method, Gauss-Jordan method and LU and Cholesky decomposition methods are the direct methods to solve the system of linear algebraic equations $Ax = b$. Decomposition methods are also known as Factorization methods. In this lesson, we discuss the LU and Cholesky Decomposition methods. In this, the basic idea is to write the coefficient matrix A as the product of a Lower triangular matrix L and an upper triangular matrix U.

9.2 LU Decomposition Method

Given the matrix A as $A = (a_{ij}),\ i, j = 1, 2, \ldots, n$

in general or

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, in particular, the decomposition is possible if all the

principal minors of A, i.e., $a_{11}, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ are non-singular.

Let $A = LU$

with a choice for $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$.

One can also choose 1's along the diagonal elements for U and l_{ii} as diagonal elements for L. We now determine the element of L and U as follows.

Consider $LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A.$

Equating the corresponding elements on both sides, we get

$u_{11} = a_{11}, u_{12} = a_{12}, u_{13} = a_{13},$

$l_{21}u_{11} = a_{21} \Rightarrow l_{21} = \dfrac{a_{21}}{a_{11}},$

$l_{31}u_{11} = a_{31} \Rightarrow l_{31} = \dfrac{a_{31}}{a_{11}},$

$l_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}}a_{12},$

$l_{21}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - \dfrac{a_{21}}{a_{11}}a_{13},$

$l_{31}u_{12} + l_{32}u_{22} = a_{32} \Rightarrow l_{32} = \dfrac{a_{32} - \left(\dfrac{a_{31}}{a_{11}}\right)a_{12}}{a_{22} - \left(\dfrac{a_{21}}{a_{11}}\right)a_{12}},$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$

$\Rightarrow u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}.$

Once we compute l_{ij}'s and u_{ij}'s, we obtain the solution of $Ax = b$ as given below.

Consider $LU\underline{x} = b$

Call $U\underline{x} = \underline{z} \Rightarrow L\underline{z} = b$.

i.e., $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Now Forward substitution gives

$\Rightarrow z_1 = b_1, l_{21}z_1 + z_2 = b_2 \Rightarrow z_2 = b_2 - l_{21}b_1$

$l_{31}z_1 + l_{32}z_2 + z_3 = b_3 \Rightarrow z_3 = b_3 - l_{31}z_1 - l_{32}(b_2 - l_{21}b_1)$.

This gives $\underline{z}$ in terms of the elements of $\underline{b}$. Having found z, $Ux = z$ gives the unknown as

$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$.

Using backward substitution process,

$u_{33}x_3 = z_3 \Rightarrow x_3 = \dfrac{z_3}{u_{33}}$,

$u_{22}x_2 + u_{23}x_3 = z_2$

$\Rightarrow x_2 = \dfrac{\left[z_2 - u_{23}\dfrac{z_3}{u_{33}}\right]}{u_{22}}$,

$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = z_1$,

$\Rightarrow x_1 = z_1 - u_{13}x_3 - u_{12}x_2$,

$[x_1, x_2, x_3]^T$ is the required solution.

Example 1: Solve the following equations by LU decomposition

$2x + y + 4z = 12,\ 8x - 3y + 2z = 20,\ 4x + 11y - z = 33.$

Solution:

Given $A = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}, b = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$.

Clearly $a_{11} = 2 \neq 0, \begin{bmatrix} 2 & 1 \\ 8 & -3 \end{bmatrix} = -14 \neq 0, |A| \neq 0$.

We find a decomposition of A as LU as follows

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -\frac{9}{7} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{bmatrix}.$$

$A\underline{x} = LU\underline{x} = b$.

Take $U\underline{x} = \underline{z}$.

Then $L\underline{z} = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -\frac{9}{7} & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$

$\Rightarrow z_1 = 12, z_2 = 20 - 4 \times 12 = -28,$

$z_3 = 33 + \frac{9}{7}(-28) - 2(12) = -27$.

Now $U\underline{x} = \underline{z} \Rightarrow \begin{bmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -28 \\ -27 \end{bmatrix}$

$$z = \frac{-27}{-27} = 1,$$

$$y = -\frac{1}{7}\left[-28 + 14.1\right] = 2$$

$$x = \frac{1}{2}\left[12 - 2 - 4.1\right] = 3$$

Thus the solution of the given system of equation is $x = 3$, $y = 2$ and $z = 1$.

The advantage of direct methods is that we obtain exact solution while the iterative methods give an approximate solution. For the generalization of the LU decomposition procedure to a system of n equations in n-unknowns, the reader is referred to any standard text book on *Numerical methods*.

Exercise 1: Solve the equations using LU decomposition.

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

9.3 Cholesky Method

If the coefficient matrix A is symmetric (i.e., $A = A^T$) and all leading order principal minors are non-singular (as in case of LU decomposition method), then the matrix A can be decomposed as

$$A = L \cdot L^T$$

where $L = l_{ij}, l_{ij} = 0$ for $i < j$.

Then $A\underline{x} = b$ becomes

$L \cdot L^T \underline{x} = b$

Take $L^T \underline{x} = \underline{z}$

$\Rightarrow L\underline{z} = b$.

The intermediate solution z_i, $i = 1,2,\dots,n$ is obtained by forward substitution (as described earlier) and solution x_i, $i = 1,2,\dots,n$ is determined by the back substitution.

Example 2: Solve the system of equation using the Cholesky decomposition method

$x + 2y + 3z = 5$

$2x + 8y + 22z = 6$

$3x + 22y + 82z = -10.$

Solution:

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$.

Write $A = L \cdot L^T$

or $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$

$\Rightarrow l_{11}^2 = 1 \Rightarrow l_{11} = 1,$

$l_{11} l_{21} = 2 \Rightarrow l_{21} = 2,$

$l_{11}l_{31} = 3 \Rightarrow l_{31} = 3,$

$l_{21}^2 + l_{22}^2 = 8 \Rightarrow l_{22} = 2,$

$l_{31}l_{21} + l_{32}l_{22} = 22 \Rightarrow l_{32} = 8,$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 82 \Rightarrow l_{33} = 3.$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}.$$

Now $A\underline{x} = b \Rightarrow L \cdot L^T \underline{x} = b$.

Put $L^T \underline{x} = \underline{u}$ where $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ (say)

$$\Rightarrow L\underline{u} = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$\Rightarrow u_1 = 5, u_2 = -2, u_3 = -3$.

Solving $L^T \underline{x} = \underline{u}$ gives $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$

$\Rightarrow z = -1, y = 3$ and $x = 2$ as the required solution.

Exercises 2:

1. Solve the system of equations

   $4x + y + z = 4,\ x + 4y - 2z = 4,\ 3x + 2y - 4z = 6$ using the (i) LU and Cholesky decomposition methods.

2. Solve the system of equation

$$\begin{bmatrix} 2 & 1 & -4 & 1 \\ -4 & 3 & 5 & -2 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & -3 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 2 \\ -1 \end{bmatrix}$$

by LU decomposition method.

3. Solve $4x - y = 1$, $-x + 4y - z = 0$, $-y + 4z = 0$ by the Cholesky method.

Keywords: Cholesky Decomposition Method, LU Decomposition Method

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Lesson 10

Interpolation

10.1 Introduction

Let $f(x)$ be a continuous function defined on some interval $[a,b]$. Consider a partition of this interval $[a,b]$ as $a = x_0 < x_0 < ... < x_{i-1} < x_i < ... < x_N = b$, having $(N+1)$ nodal points. These nodes are either equally spaced ($x_i - x_{i-1} = h$ a constant, $i = 0,1,2,...,N-1$) or unequally spaced. Let $f(x_i) = f_i, i = 0,1,2,...,N$. The process of approximating the given function $f(x)$ on $[a,b]$ OR a set of $(N+1)$ data points (x_i, f_i), where the function $f(x)$ is not given explicitly using polynomial functions $\{x^0, x^1, x^2, ..., x^N, ...\}$ is known as polynomial interpolation. The problem of polynomial approximation is to find a polynomial $P_n(x)$ of degree n or less than n, which satisfies the condition $P_n(x_i) = f(x_i), i = 0,1,2,...,N$.

In such a case, the polynomial $P_n(x)$ is called the interpolating polynomial. The advantage of polynomial interpolation is of two folds. The first use is in reconstructing the approximation to the function $f(x)$ when it is not given explicitly. The second use is to replace the function $f(x)$ by an interpolating polynomial $P_n(x)$ using which differentiation and integration operations can be easily performed. In general, if these are $N+1$ distinct points $a = x_0 < x_0 < ... < x_{i-1} < x_i < ... < x_N = b$, then the problem of interpolation is to obtain $P_n(x)$ satisfying the conditions $P_n(x_i) = f(x_i), i = 0,1,2,...,N$. We discuss below the methods in which these interpolating polynomials can be obtained.

10.2 Linear Interpolation

Consider two data point (x_0, f_0) and (x_1, f_1). We wish to determine a polynomial

$$P_1(x) = a_1 x + a_0, \tag{10.1}$$

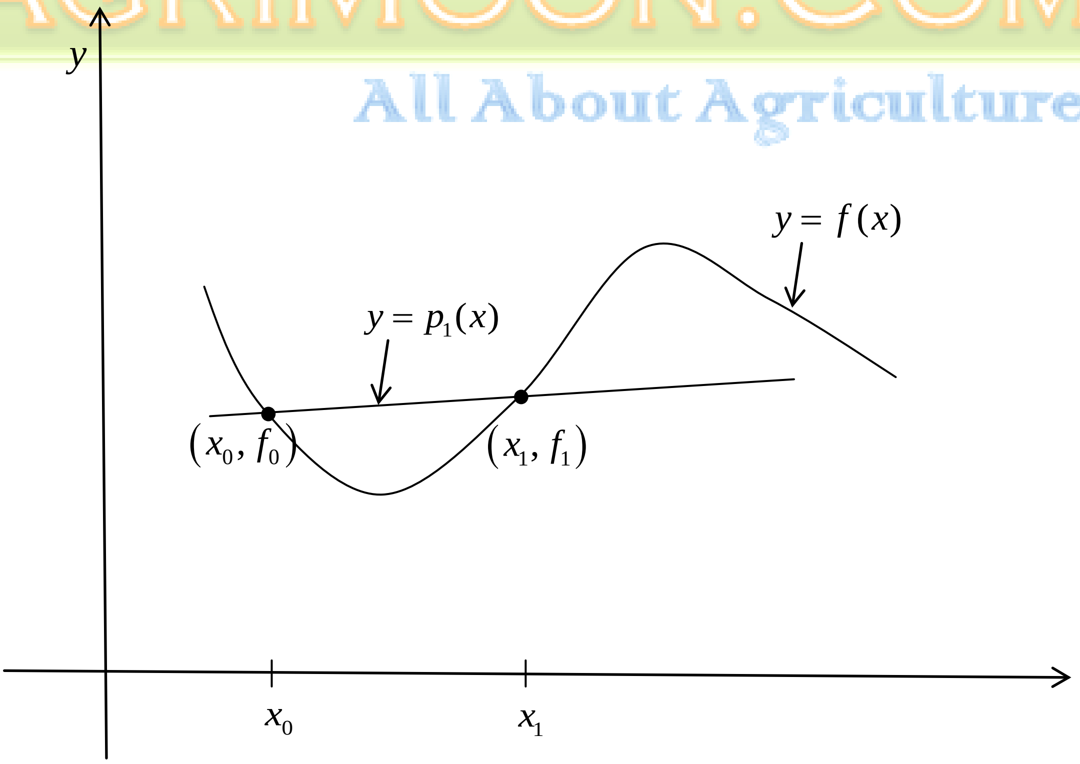
where a_0 and a_1 are arbitrary constants, satisfying the interpolating conditions

$$P_1(x_i) = f(x_i), i = 0,1 \tag{10.2}$$

$f(x_0) = P_1(x_0) = a_1 x_0 + a_0$ and $f(x_1) = P_1(x_1) = a_1 x_1 + a_0$.

Eliminating a_0 and a_1, we obtain the interpolating polynomial as:

$$\begin{bmatrix} P_1(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{bmatrix} = 0.$$



Linear interpolation

Expanding the determinant, we obtain

$$P_1(x)(x_0 - x_1) - f(x_0)(x - x_1) + f(x_1)(x - x_0) = 0$$

or $P_1(x) = \dfrac{x - x_1}{x_0 - x_1} f(x_0) + \dfrac{x - x_0}{x_1 - x_0} f(x_1)$.

This gives the linear interpolating polynomial.

Write $$P(x) = l_0(x) f(x_0) + l_1(x) f(x_1) \qquad (10.3)$$

where $l_0(x) = \dfrac{x - x_1}{x_0 - x_1}$, $l_1(x) = \dfrac{x - x_0}{x_1 - x_0}$

The functions $l_0(x)$ and $l_1(x)$ are linear functions and are called the Lagrange fundamental polynomials. They satisfy the conditions (i) $\sum_{i=0}^{1} l_i(x) = 1$ and (ii) $l_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.

Equation (10.3) is called the Linear Lagrange interpolating polynomial.

10.3 Quadratic Interpolation

We now wish to determine a polynomial

$$P_2(x) = a_0 + a_1 x + a_2 x^2, \qquad (10.4)$$

where a_0, a_1 and a_2 are arbitrary constants, satisfying the conditions

$$f(x_0)=P_2(x_0),\ f(x_1)=P_2(x_1)\text{ and }\ f(x_2)=P_2(x_2).(10.5)$$

Thus we are determining $P_2(x)$ passing through 3 data points

$$(x_0, f_0),(x_1, f_1),(x_2, f_2).$$

The arbitrary constants a_0, a_1 and a_2 can be determined from the three conditions:

$$f(x_0)=a_0+a_1x_0+a_2x_0^2$$

$$f(x_1)=a_0+a_1x_1+a_2x_1^2$$

$$f(x_0)=a_0+a_1x_2+a_2x_2^2$$

Eliminating a_0, a_1, a_2 we obtain:

$$\begin{bmatrix} P_2(x) & 1 & x & x^2 \\ f(x_0) & 1 & x_0 & x_0^2 \\ f(x_1) & 1 & x_1 & x_1^2 \\ f(x_2) & 1 & x_2 & x_2^2 \end{bmatrix}=0$$

$$\Rightarrow P_2(x)\begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_2^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} - f(x_0)\begin{vmatrix} 1 & x & x^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} + f(x_1)\begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} - f(x_2)\begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \end{vmatrix}=0$$

Expanding the determinants and simplifying, we obtain

$$P_2(x)=l_0(x)f(x_0)+l_1(x)f(x_1)+l_2(x)f(x_2) \qquad (10.6)$$

Where, $$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)},$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)},$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

These $l_i(x)$, $i = 0,1,2$ are Lagrange fundamental polynomials of second degree, which satisfy (i) $\sum_{i=0}^{2} l_i(x) = 1$ and (ii) $l_i(x_j) = \delta_{ij}$.

Example 1: Obtain the value of $f(0.15)$ using Lagrange linear interpolating polynomial for the data:

x:	0.1	0.2
$f(x)$:	0.09983	0.19867

Solution:

Put $x = 0.15$ in (3), we get

$$P_1(0.15) = \frac{0.15-0.2}{0.1-0.2}(0.09983) + \frac{0.15-0.1}{0.2-0.1}(0.19867)$$

$$= 0.14925$$

Example 2: Find $P_2(x)$ for the following unequally spaced data set:

x:	0	1	3
$f(x)$	1	3	55

Hence find $P_2(x)$.

Solution:

We compute the fundamental polynomial as:

$$l_0(x) = \frac{(x-1)(x-3)}{(-1)(-3)} = \frac{1}{3}\left(x^2 - 4x + 3\right)$$

$$l_2(x) = \frac{(x-0)(x-3)}{(1)(-2)} = \frac{1}{2}\left(3x - x^2\right)$$

$$l_3(x) = \frac{(x-0)(x-1)}{(3)(2)} = \frac{1}{6}\left(x^2 - x\right)$$

Now $$P_2(x) = \frac{1}{3}\left(x^2 - 4x + 3\right)\cdot 1 + \frac{1}{2}\left(3x - x^2\right)\cdot 3 + \frac{1}{6}\left(x^2 - x\right)\cdot 55$$

$$= 8x^2 - 6x + 1.$$

$$\therefore P_2(x) = 8\cdot(2)^2 - 6(2) + 1$$

$$= 32 - 12 + 1$$

$$= 21.$$

Example 3: Find $P_2(x)$ approximating the given data

x:	2	2.5	3
$f(x)$:	0.69315	0.91629	1.09861

Hence find $y(2.7)$.

Solution:

Let us find the fundamental polynomials $l_0(x), l_1(x)$ and $l_2(x)$:

$$l_0(x)=\frac{(x-2.5)(x-3)}{(-0.5)(-1.0)}=2x^2-11x+15$$

$$l_1(x)=\frac{(x-2)(x-3)}{(0.5)(-0.5)}=-4x^2+20x-24$$

$$l_2(x)=\frac{(x-2)(x-2.5)}{(1.0)(0.5)}=2x^2-9x+10$$

Hence,

$$P_2(x)=\left(2x^2-11x+15\right)(0.69315)-\left(4x^2-20x+24\right)(0.91629)$$
$$+\left(2x^2-9x+10\right)(1.09861)$$

Simplifying, we get

$$P_2(x)=-0.08164x^2+0.81366x-0.60761.$$

Putting $x=2.7$, $P_2(2.7)=0.99412$.

Exercises:

1. Find the Lagrange quadratic interpolating polynomial $P_2(x)$ for the data: $f(0)=1$, $f(1)=3$, $f(3)=5$.

2. The function $f(x)=sinx+cosx$ is represented by the data given by:

x:	10°	20°	30°
$f(x)$:	1.1585	1.2817	1.366

Find $P_2(x)$ for this data. Hence find $P_2\left(\frac{\pi}{12}\right)$. Compare it with the exact value $f\left(\frac{\pi}{12}\right)$.

Keywords: Equally Spaced, Nodal Points, Unequally Spaced, Linear Interpolation, Quadratic Interpolation.

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Lesson 11

Higher Order Lagrange Interpolation

11.1 Introduction

Consider the data set $\{(x_0, f_0),(x_1, f_1),...,(x_N, f_N)\}$ corresponding to the function $f(x)$ on the interval $[x_0, x_N]$. Also, the partition of the interval $[x_0, x_N]$ is $x_0 < x_1 < x_2 < ... < x_N$, need not be equally spaced. We now derive an interpolating polynomial $P_N(x)$ for the above data, satisfying $P_N(x_i) = f_i, i = 0,1,..., N$.

11.2 Lagrange Interpolating Polynomial

Result: Show that the N^{th} degree Lagrange interpolating polynomial for the data set $\{(x_0, f_0),(x_1, f_1),...,(x_N, f_N)\}$ is given by

$$P_N(x) = \frac{(x-x_1)(x-x_2)...(x-x_N)}{(x_0-x_1)(x_0-x_2)...(x_0-x_N)} f_0 + \frac{(x-x_0)(x-x_2)...(x-x_N)}{(x_1-x_0)(x_1-x_2)...(x_1-x_N)} f_1 + ...$$

$$... + \frac{(x-x_0)(x-x_1)...(x-x_{N-1})}{(x_N-x_0)(x_N-x_1)...(x_N-x_{N-1})} f_N$$

This is written in the compact form as:

$$P_N(x) = \sum_{i=0}^{N} l_i(x) f_i$$

where $l_i(x) = \dfrac{w(x)}{(x-x_i)w'(x_i)}$

where $w(x) = (x-x_0)(x-x_1)...(x-x_N)$

and $w'(x_i)=(x_i-x_0)(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_N)$.

These $l_i(x)$ are the n^{th} degree fundamental polynomials satisfying (i) $\sum_{i=0}^{N} l_i(x)=1$ and (ii) $l_i(x_j)=\delta_{ij}$.

The Truncation error in Lagrange interpolation is given by $T_n(f,x)=\dfrac{w(x)}{(n+1)!}f^{(n+1)}(\xi)$

where ξ is some point from the discrete data set $\{x_0,x_1,...,x_N\}$ such that $\min\{x_0,x_1,...,x_N,x\}<\xi<\max\{x_0,x_1,...,x_N,x\}$.

The derivatives of this truncation error is not done here, the reader is referred to the reference books.

Proof: Let $P_N(x)$ be of the form

$$P_N(x)=f(x)=a_0(x-x_1)(x-x_2)...(x-x_N)+a_1(x-x_0)(x-x_2)...(x-x_N)+$$
$$...+a_n(x-x_1)(x-x_2)...(x-x_{N-1}) \qquad (11.1)$$

Use the condition at $x=x_0,\ f(x_0)=f_0$ in (11.2)

$$f_0=a_0(x_0-x_1)(x_0-x_2)...(x_0-x_N)$$

$$\Rightarrow a_0=\frac{f_0}{(x_0-x_1)(x_0-x_2)...(x_0-x_N)}$$

Similarly, use the other N conditions, we get

$$a_1 = \frac{f_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_N)},$$

$$a_2 = \frac{f_2}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_N)}, \ldots a_N = \frac{f_N}{(x_N - x_0)(x_N - x_1)\ldots(x_N - x_{N-1})}.$$

Substituting a_i's in (11.2), we get $P_N(x)$ as given in equation (11.1).

Example 1: For the below given unequally spaced data find the interpolating polynomial with highest degree:

$x:$	0	1	3	4
$f(x)$	-20	-12	-20	-24

Then compute $f(2)$.

Solution:

Given 4 data points, we can find utmost a third degree polynomial of the form $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where these a_i's are determined using the given data.

Now let us use the Lagrange Interpolation formula

$$P_3(x) = l_0(x)\, f_0(x) + l_1(x)\, f_1(x) + l_2(x)\, f_2(x) + l_3(x)\, f_3(x)$$

where $l_i(x)$ are the fundamental polynomials;

$$\therefore P_3(x) = \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-20) + \frac{(x-0)(x-3)(x-4)}{(1)(-2)(-3)}(-12) +$$

$$\frac{(x-0)(x-1)(x-4)}{(3)(2)(-1)}(-20)+\frac{(x-0)(x-1)(x-3)}{(4)(3)(1)}(-24)$$

or $P_3(x)=x^3-8x^2+15x-20$ is the highest degree polynomial that satisfies the given data set.

Now $f(2)\approx P_3(2)=(2)^3-8(2)^2+15(2)-20$

$$=-14.$$

11.3 Inverse Interpolation

In interpolation, we find the function value of $f(x)$ at some non-nodal point x in the interval. On the other hand, the process of estimating the value x for a value of $f(x)$ which is not among the tabulated values is called the inverse interpolation. The inverse interpolating polynomial is obtained by interchanging the roles of x_i and $f(x_i)$ in the Lagrange interpolating polynomial.

For $y_i = f(x_i)$ and for the given data set $\{(x_0,y_0)(x_1,y_1)...(x_N,y_N)\}$, it is given by:

$$x=\frac{(y-y_1)(y-y_2)...(y-y_N)}{(y_0-y_1)(y_0-y_2)...(y_0-y_N)}x_0+\frac{(y-y_0)(y-y_1)...(y-y_N)}{(y_1-y_0)(y_1-y_2)...(y_1-y_N)}x_1+...$$

$$...+\frac{(y-y_0)(y-y_1)...(y-y_{N-1})}{(y_N-y_0)(y_N-y_2)...(y_N-y_{N-1})}x_N.$$

Example 2: Find the value of x, if $f(x)=7$ from the given table.

x	1	3	4
$f(x)$	4	12	19

Solution:

Using the above inverse interpolating polynomial with $N = 2$, we get

$$x = \frac{(7-12)(7-19)}{(4-12)(4-19)}(1) + \frac{(7-4)(7-19)}{(12-4)(12-19)}(3) + \frac{(7-1)(7-3)}{(19-4)(19-12)}(4)$$

$$= \frac{1}{2} + \frac{27}{14} - \frac{4}{7}$$

$$= 1.860.$$

Observe that the function representing the above data set is $y(x) = x^2 + 3$.

Exercises:

1. Find the value of $f(7)$ from the following data using the Lagrange interpolation.

x	5	6	9	11
$f(x)$	380	-2	196	508

2. Find a polynomial which passes through the points $(0,-12)$, $(1,0)$, $(3,6)$, $(4,12)$.

3. Find x if $f(x) = 6$ from the below given table

x:	0	1	3	4
$f(x)$:	-12	0	12	24

using the inverse interpolating polynomial.

4. Find the value of x corresponding to $f(x)=12$ from the following data set $\{(2.8, 9.8), (4.1, 13.4), (4.9, 15.5), (6.2, 19.6)\}$ using the inverse interpolating polynomial.

Keywords: Lagrange Interpolation, Inverse Interpolating Polynomial

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Lesson 12

Newton's Forward Interpolation Formula with Equal Intervals

12.1 Introduction

Let $y = f(x)$ be a continuous function defined on the interval $[a,b]$. Consider the partition of the interval into equally spaced subintervals as:

$$a = x_0 < x_1 < x_2 < \ldots < x_{j-1} < x_j < x_{j+1} \ldots < x_N = b$$

where $x_j - x_{j-1} = h$ for each $j = 0,1,2,\ldots,N$.

Thus the nodes $x_0, x_1, \ldots, x_N$ are such that $x_j = x_0 + jh$ for $j = 0,1,2,\ldots,N$.

We have the data set as:

$$\{(x_0, f_0),(x_1, f_1),\ldots,(x_N, f_N)\}.$$

12.2 Gregory-Newton Forward difference Interpolating Polynomial

Suppose it is required to evaluate $f(x)$ for some $x = x_0 + ph$ where p is any real number. Newton's forward difference interpolation makes use of the forward difference operator Δ on the given data set to generate a polynomial. For any real number p, the shift operator E gives $E^p f(x_0) = f(x_0 + ph)$.

Also we know $E = 1 + \Delta$.

$$\therefore f(x_0 + ph) = (1+\Delta)^p f(x_0) = (1+\Delta)^p y_0$$

$$=\{y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\ldots$$

$$\ldots+\frac{p(p-1)(p-2)\ldots(p-N+1)}{N!}\Delta^N y_0+\ldots\}$$

(using Binomial theorem).

For an N-data set, the $(N+1)^{\text{th}}$ and higher order forward differences became zero, so the infinite series in the above equation becomes a polynomial of degree N.

Note that $p=\dfrac{x-x_0}{h}$ is a linear function of x.

Thus $\dfrac{p(p-1)(p-2)\ldots(p-N+1)}{N!}$ will be a polynomial of degree N is x.

Thus we have

$$y_p=f(x_0+ph)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\ldots$$

$$\ldots+\frac{p(p-1)(p-2)\ldots(p-N+1)}{N!}\Delta^N y_0 \qquad (12.1)$$

as the n^{th} degree polynomial interpolating the given equally spaced data set. This is called the *Gregory-Newton Forward difference interpolating polynomial*.

The local truncation error in (12.1) becomes

$$T_N(f;x)=\frac{p(p-1)(p-2)\ldots(p-N)}{(N+1)!}h^{(n+1)}f^{(N+1)}(\xi)$$

where $\min\{x_0, x_1, \ldots, x_N, x\} < \xi < \max\{x_0, x_1, \ldots, x_N, x\}$.

Example 1: Find the Newton Forward interpolating polynomial for the equally spaced data points

x:	1	2	3	4
$f(x)$:	-1	-2	-1	-2

Compute $f(1.5)$.

Solution:

Given $x_0 = 1, h = 1, f_0 = -1$.

The difference table for the given data is:

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
1	-1			
		-1		
2	-2		2	
		1		0
3	-1		-2	
		-1		
4	-2			

Clearly $\Delta f = -1, \Delta^2 f = 2$ and $\Delta^3 f = 0$.

Using the Newton's forward interpolating polynomial given by (12.1), we have

$$f(x) = -1 + \left(\frac{x-1}{1}\right)(-1) + \frac{(x-1)(x-2)}{2!}(-2) + \frac{(x-1)(x-2)(x-3)}{3!}(0)$$

$$= -1 + (x-1)(-1) + \frac{(x-1)(x-2)(-2)}{2}$$

$$= x^2 - 4x + 2.$$

Now $f\left(x = \frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 4\left(\frac{3}{2}\right) + 2 = -\frac{7}{4} = -1.75.$

$\therefore f(1.5) = -1.75.$

Example 2: Interpolate at $x = 0.25$ from the data given without writing the polynomial.

x:	0.1	0.2	0.3	0.4	0.5
$f(x)$:	1.4	1.56	1.76	2.0	2.28

Solution:

The function value is to be found at $x = 0.25$ which is nearer to the node $x = 0.2$.

So choose $x_0 = 0.2, h = 0.1, x = 0.25$.

$$x_p = 0.25 = x_0 + ph = 0.2 + p(0.1)$$

$$\Rightarrow p = \frac{0.25 - 0.2}{0.1} = 0.5.$$

The forward difference table for the given data is written as

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$

0.1	1.4				
		0.16			
0.2	1.56		0.04		
		0.2		0.0	
0.3	1.76		0.04		0.0
		0.24		0.0	
0.4	2.0		0.04		
		0.28			
0.5	2.28				

From the above, $x_0 = 0.2$ (chosen), $f_0 = 1.56, \Delta f_0 = 0.2, \Delta^2 f_0 = 0.4$ and $\Delta^3 f_0 = \Delta^4 f_0 = 0$.

The Newton's forward interpolating polynomial becomes

$$f(0.25) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0.$$

Thus $f(0.25) = 1.56 + (0.5)(0.2) - (0.125)(0.04)$

$= 1.655.$

Note: Newton's forward interpolation formula is used if the function evaluation is desired near the beginning of the tabulated values.

Exercises:

Use Newton's forward interpolation formula to find

1. $f(-0.5)$ and $f(0.5)$ from

x:	-2	-1	0	1	2
$f(x)$:	9	16	17	18	24

2. $f(45)$ from

x:	40	50	60	70	80
$f(x)$:	31	73	124	159	190

3. Find the number of persons getting salary below Rs. 300 per day from the following data.

Wages in Rs.	$200-250$	$250-300$	$300-350$	$350-400$	$400-450$
Frequency	9	16	35	70	20

Keywords: Newton's Forward Interpolation Formula, Forward Difference Operator

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Lesson 13

Newton's Backward Interpolation Polynomial

13.1 Introduction

Consider the discrete data set for the continuous function $f(x)$ on the interval $[a,b]$ as (as considered in the lesson 12)

$$\{(x_0, f_0),(x_1, f_1),...,(x_N, f_N)\}$$

Let us now try to evaluate the function $f(x)$ at some location x near the end nodes x_{N-1} or x_N.

Write $x = x_N + ph$, p is any real number.

Then $y_p = f(x) = f(x_N + ph) = E^p f(x_N)$.

13.2 Gregory-Newton Backward Interpolating Polynomial

Use the relation between E and the backward difference operator ∇ given as $E \equiv (1-\nabla)^{-1}$.

Now $E^p \equiv (1-\nabla)^{-p}$.

Thus $f(x_N + ph) = (1-\nabla)^{-p} y_N$.

Expanding $(1-\nabla)^{-p}$ using binomial expansion, we write

$$f\left(x_N + ph\right) = \{y_N + p\nabla y_N + \frac{p(p+1)}{2!}\nabla^2 y_N + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_N + \ldots$$

$$\ldots + \frac{p(p+1)(p+2)\ldots(p+N-1)}{N!}\nabla^N y_N + \ldots\}.$$

For the given N-data set, the $(N+1)^{\text{th}}$ and higher order backward differences become zero, so the infinite series above becomes a polynomial of degree N.

Note that $x = x_N + ph$

$\Rightarrow p = \dfrac{x - x_N}{h}$ is a linear function of x, and the product $\dfrac{p(p+1)\ldots(p+N-1)}{N!}$ is a polynomial of degree N in x.

Thus we get the N^{th} degree interpolating polynomial in terms of the backward differences at x_N as:

$$f\left(x_N + ph\right) = y_N + p\nabla y_N + \frac{p(p+1)}{2!}\nabla^2 y_N + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_N + \ldots$$

$$\ldots + \frac{p(p+1)(p+2)\ldots(p+N-1)}{N!}\nabla^N y_N \qquad (13.2)$$

This is called the Gregory-Newton Backward interpolating polynomial.

The local truncation error in (13.2) is

$$T_N(f;x) = \frac{p(p+1)\ldots(p+N)}{\left(N+1\right)!}h^{(N+1)} f^{(N+1)}(\xi)$$

where $\min\{x_0, x_1, ..., x_N, x\} < \xi < \max\{x_0, x_1, ..., x_N, x\}$.

Note: Newton backward interpolating polynomial is more efficient when applied to interpolate the data at the end of the data set.

Let us illustrate this method.

Example 1: The following data represents the relation between the distance as a function of height:

x =height	150	200	250	300	350	400
y =distance	13.03	15.04	16.81	18.42	19.90	21.27

Find $\mathrm{y}(410)$.

Solution:

Let $x = 410$, chose $x_N = 400$, $h = 50$, $y_N = 21.27$.

$$\therefore p = \frac{x - x_N}{h} = \frac{10}{50} = 0.2.$$

We now find $\nabla y_N, \nabla^2 y_N, \nabla^3 y_N, \nabla^4 y_N$ by constructing the difference table:

x	f	∇	∇^2	∇^3	∇^4
150	13.03				
		2.01			
200	15.04		-0.24		
		1.77		0.08	

250	16.81		-0.16		-0.05
		1.61		0.03	
300	18.42		-0.13		-0.01
		1.48		0.02	
350	19.90		-0.11		
		1.37			
400	21.27				

Using (2), we obtain

$$f(410) = y_N + p\nabla y_N + \frac{p(p+1)}{2!}\nabla^2 y_N + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_N$$

$$= 21.27 + (0.2)(1.37) + \frac{(0.2)(1.2)}{2}(-0.11) + \frac{(0.2)(1.2)(2.2)}{6}(0.02)$$

$$= 21.53.$$

Thus $f(410) = 21.53$.

Note that, when $p = 0.2$, p^3, p^4 only correct the solution at fourth and fifth decimal places.

Example 2: Find the cubic polynomial which takes the following data:

x:	0	1	2	3
$f(x)$:	1	2	1	10

Solution:

Let us now form the difference table for the given data:

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
$x_0 = 0$	$1 = f_0$			
		$1 = \Delta f_0$		
1	2		-2 $= \Delta^2 f_0$	
				$12 \begin{cases} = \Delta^3 f_0 \\ = \Delta^3 f_N \end{cases}$
		-1		
2	1		$10 = \Delta^2 f_N$	
		$9 = \Delta f_N$		
$x_N = 3$	$9 = f_N$			

Case 1: Take $x_0 = 0, p = \dfrac{x - x_0}{h} = \dfrac{x - 0}{1} = x$.

Let us now find the Newton's forward interpolating polynomial:

$$f(x) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 f_0$$

$$= 1 + x \cdot 1 + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1 \qquad \text{...... (i)}$$

Case 2: Let us now find the Newton's backward interpolating polynomial:

Take $x_N = 3, f_N = 10, h = 1$.

$$p = \frac{x - x_N}{h} = \frac{(x-3)}{1} = (x-3).$$

$$f(x) = f_N + p\nabla f_N + \frac{p(p+1)}{2}\nabla^2 f_N + \frac{p(p+1)(p+2)}{6}\nabla^3 f_N.$$

$$p = (x-3), p(p+1) = (x-3)(x-2), p(p+1)(p+2) = (x-3)(x-2)(x-1)$$

Thus $f(x) = 10 + (x-3)(9) + \frac{(x-3)(x-2)}{2}(10) + \frac{(x-3)(x-2)(x-1)}{6}(12)$

$= 2x^3 - 7x^2 + 6x + 1.$ …… (ii)

(i) and (ii) clearly indicate that the interpolating polynomial for the given data is the same though we use different methods.

Exercises:

1. Use the Lagrange interpolating polynomial for the data:

x:	0	1	2	3
$f(x)$:	1	2	1	10

Show that the interpolating polynomial is $p_3(x) = 2x^3 - 7x^2 + 6x + 1.$

2. Find f(2) from the data

x:	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

3. If $f(1.15) = 1.0723$, $f(1.2) = 1.0954$, $f(1.25) = 1.118$ and $f(1.3) = 1.1401$ find $f(1.35)$.

4. Find the number of men getting wages below Rs. 35 from the data:

Wages in Rs:	0 – 10	10 – 20	20 – 30	30 – 40
Frequency	9	30	35	42

Keywords: Backward Difference Operator, Local Truncation Error, Newton's Backward Interpolation Polynomial

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Lesson 14

Gauss Interpolation

14.1 Introduction

Newton's Forward and Backward interpolating polynomials are used to interpolate the function values at the starting or end of the data respectively. We now see the central difference formulas which are most suited for interpolation near the middle of a tabulated set.

Consider the data points as:

$$\left\{(x_{-3},y_{-3}),(x_{-2},y_{-2}),(x_{-1},y_{-1}),(x_0,y_0),(x_1,y_1),(x_2,y_2),(x_3,y_3)\right\}$$

14.2 Gauss-Forward Interpolation Formula

The difference table for the above data is:

The newton's Forward interpolation formula is:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots \qquad (14.1)$$

We know $\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1} \Rightarrow \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$.

Similarly, $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$, $\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$.

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2} \Rightarrow \Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$.

Similarly, $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc.

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \ldots$ in equation (1), and rearranging, we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)(p)(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \tag{14.2}$$

This is called Gauss-Forward interpolation formula.

		1st difference	**2nd difference**	**3rd difference**	**4th difference**	**5th difference**
x_{-3}	y_{-3}					
		$\Delta y_{-3}\left(=\delta y_{-\frac{5}{2}}\right)$				
x_{-2}	y_{-2}		$\Delta^2 y_{-3}\left(=\delta^2 y_{-2}\right)$			
		$\Delta y_{-2}\left(=\delta y_{-\frac{3}{2}}\right)$		$\Delta^3 y_{-3}\left(=\delta^3 y_{-\frac{3}{2}}\right)$		
x_{-1}	y_{-1}		$\Delta^2 y_{-2}\left(=\delta^2 y_{-1}\right)$		$\Delta^4 y_{-3}\left(=\delta^4 y_{-1}\right)$	
		$\Delta y_{-1}\left(=\delta y_{-\frac{1}{2}}\right)$		$\Delta^3 y_{-2}\left(=\delta^3 y_{-\frac{1}{2}}\right)$		$\Delta^5 y_{-3}\left(=\delta^5 y_{-\frac{1}{2}}\right)$
x_0	$y_0 \rightarrow$	Central Line …	$\Delta^2 y_{-1}\left(=\delta^2 y_0\right)$		$\Delta^4 y_{-2}\left(=\delta^4 y_0\right)$	
		$\Delta y_0\left(=\delta y_{\frac{1}{2}}\right)$		$\Delta^3 y_{-1}\left(=\delta^3 y_{\frac{1}{2}}\right)$		$\Delta^5 y_{-2}\left(=\delta^5 y_{\frac{1}{2}}\right)$
x_1	y_1		$\Delta^2 y_0\left(=\delta^2 y_1\right)$		$\Delta^4 y_{-1}\left(=\delta^4 y_1\right)$	
		$\Delta y_1\left(=\delta y_{\frac{3}{2}}\right)$		$\Delta^3 y_0\left(=\delta^3 y_{\frac{3}{2}}\right)$		

x_2	y_2		$\Delta^2 y_1\left(=\delta^2 y_2\right)$			
		$\Delta y_2\left(=\delta y_{\frac{5}{2}}\right)$				
x_3	y_3					

We know $\Delta y_0 = \delta y_{\frac{1}{2}}, \Delta^2 y_{-1} = \delta^2 y_0, \Delta^3 y_{-1} = \delta^3 y_{\frac{1}{2}}$ and $\Delta^4 y_{-2} = \delta^4 y_0$;using these in (2),

we write the equation in (2) in terms of the central differences as:

$$y_p = y_0 + p\delta y_{\frac{1}{2}} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)(p)(p-1)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{(p+1)(p)(p-1)(p-2)}{4!}\delta^4 y_0 + \dots \tag{14.3}$$

This formula can be used directly to interpolate the function at the centre of the data i.e., for values of p, $0 < p < 1$.

14.3 Gauss-Backward Interpolation Formula

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

$\Rightarrow \Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$,

$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$, $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ etc

Also, $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$,

$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$, etc.

Substituting these in equation (1), we get

$$y_p = y_0 + p\left(\Delta y_{-1} + \Delta^2 y_{-1}\right) + \frac{p(p-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \frac{p(p-1)(p-2)}{3!}\left(\Delta^3 y_{-1} + \Delta^4 y_{-1}\right) + \dots$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\left(\Delta^3 y_{-2} + \Delta^4 y_{-2}\right) + \dots$$

or

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots$$

(14.4)

This is called Gauss-Backward interpolation formula. This is written using the central differences as:

$$y_p = y_0 + p\delta y_{-\frac{1}{2}} + \frac{p(p+1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{-\frac{1}{2}} + \frac{(p+2)(p+1)p(p-1)}{4!}\delta^4 y_0 + \ldots$$

(14.5)

Formula given in (2) and (4) or (3) and (5) have limited utility, but are useful in deriving the important method known as Stirling's method.

Keywords: Stirling's Method, Central Differences, Gauss-Forward Interpolation Formula, Interpolation near the Middle of a Tabulated Set

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Lesson 15

Everett's Central Difference Interpolation

15.1 Introduction

We have the Gauss forward interpolation formula as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+\frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \ldots \qquad (15.1)$$

15.2 Everett's Formula

Eliminating odd differences $\Delta y_0, \Delta^3 y_{-1}, \Delta^5 y_{-2}$ etc. by

$\Delta y_0 = y_1 - y_0, \Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}, \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$ etc., then (1) becomes

$$y_p = y_0 + p\left(y_1 - y_0\right) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\left(\Delta^2 y_0 - \Delta^2 y_{-1}\right) +$$

$$\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\left(\Delta^4 y_{-1} - \Delta^4 y_{-2}\right) + \ldots$$

$$= (1-p)y_0 + py_1 - \frac{p(p-1)(p-2)}{3!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^2 y_0 -$$

$$\frac{(p+1)p(p-1)(p-2)(p-3)}{5!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^4 y_{-1} + \ldots$$

$$(15.2)$$

This is known as Everett's formula.

This formula is extensively used as it involves only even differences in and below the central line.

Example 1: Below given data represents the function $f(x) = log x$. Use Everett's formula to find $f(337.5)$:

x:	310	320	330	340	350	360
$f(x)$:	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

Take the data as:

$x_{-2} = 310,\ f_{-2} = 2.49136,$

$x_{-1} = 320,\ f_{-1} = 2.50515,$

$x_0 = 330,\ f_0 = 2.51851,$

$x_1 = 340,\ f_1 = 2.53148,$

$x_2 = 350,\ f_2 = 2.54407,$

$x_3 = 360,\ f_3 = 2.5630,$

$$h = 10,\ p = \frac{x - 330}{10}.$$

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.49136					
	0.01379				
2.50515		-0.00043			
	0.01336		0.00004		
2.51881		-0.00039		-0.00003	

	0.01297		0.00001		0.00004
2.53148		-0.00038		0.00001	
	0.01259		0.00002		
2.54407		-0.00036			
	0.01223				
2.55630					

Take $x = 337.5,\ p = \dfrac{337.5 - 330}{10} = 0.75$.

To change the terms with negative sign, putting $\mathrm{p} = 1 - \mathrm{q}$ in equation (1), we get

$$y_p = qy_0 + \frac{q(q^2 - 1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2 - 1^2)(q^2 - 2^2)}{5!}\Delta^4 y_{-2} + \ldots$$

$$+ py_1 + \frac{p(p^2 - 1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2 - 1^2)(p^2 - 2^2)}{5!}\Delta^4 y_{-1} + \ldots$$

$q = 1 - p = 0.25$.

$$\therefore y_p = 0.62963 + 0.00002 - 0.0000002 + 1.89861 + 0.00002 + 0.00000001 = 2.52828$$

Exercise:

1. Find $f(25)$ from the data

x:	20	24	28	32
f:	854	3162	3544	3992

using Everett's formula.

Keywords: Everett's Formula, Gauss Forward Interpolation

References

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Lesson 16

Stirling's and Bessel's Formula

16.1 Stirling's Formula

This is obtained by taking the mean of the Gauss Forward and Backward interpolation formulae.

This is written as:

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p\left(p^2-1\right)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2\left(p^2-1\right)}{4!}\Delta^4 y_{-2} + \dots \tag{16.1}$$

Writing this using central differences, we obtain

$$y_p = y_0 + \frac{p}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) + \frac{p^2}{2!}\delta y_0 + \frac{p\left(p^2+1^2\right)}{3!\cdot 2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right) + \frac{p^2\left(p^2-1^2\right)}{4!}\delta^4 y_0 + \dots \tag{16.2}$$

This is called the Stirling's formula.

Example 1: Find the value of e^x when $x = 0.644$ from the below given table:

x	0.61	0.62	0.63	0.64	0.65
$y = e^x$	1.840431	1.858928	1.87761	1.896481	1.91554

Solution:

$$x = 0.644, x_0 = 0.64, p = \frac{0.644 - 0.64}{0.01} = 0.4, y_0 = 1.896481.$$

By forming the difference table (left as an exercise!) we note that $\Delta y_{-1} = 0.018871, \Delta y_0 = 0.01906, \Delta^2 y_{-1} = 0.000189$ and all higher order differences are approximately zero. Substituting these in the Stirling's formula given in (1), we get

$y(0.64) = 1.896481 + 0.0075862 + 0.00001512 = 1.904082.$

16.2 Bessel's Formula

We know $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

$\Rightarrow \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}.$

Similarly $\Delta^4 y_{-1} - \Delta^4 y_{-2} = \Delta^5 y_{-2}$

$\Rightarrow \Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$

Using these in the Gauss forward interpolation formula, we obtain

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \ldots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}\left(\Delta^2 y_0 - \Delta^3 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2}+\frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\left(\Delta^4 y_{-1}-\Delta^5 y_{-2}\right)+\ldots$$

$$\text{or } y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}\right)+\frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}\right)+\ldots \tag{16.3}$$

This is known as the Bessel's formula.

Example 2: Using Bessel's formula, obtain $y(25)$,given$(20) = 2854, y(24) = 3162, y(28) = 3544, y(32) = 3992$.

Solution:

Taking $x_0 = 24, h = 4, y_0 = 3162$.

We have $p = \frac{1}{4}(x-24)$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	2854			
		308		
24	3162		74	
		382		-8
28	3544		66	
		448		

32 3992

Taking $x = 25, p = \frac{1}{4}$.

Bessel's formula is:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1} + \ldots$$

$$\therefore y(25) = 3162 + (0.25)(382) + \frac{(0.25)(-0.75)}{2}\left(\frac{74+66}{2}\right) + \frac{(-0.25)(0.25)(-0.75)}{6}(-8)$$

$= 3250.87$.

Note:

1. If the value of p lies between $-\frac{1}{4}$ and $\frac{1}{4}$, prefer Stirling's formula, it gives a better approximation.
2. If p lies between $\frac{1}{4}$ and $\frac{3}{4}$, Bessel's or Everett's formula gives better approximation.

<u>Exercises</u>:

1. Using Stirling's formula, find $y(35)$ from the data $y(20) = 512, y(30) = 439, y(40) = 346, y(50) = 243$.

2. Find $f(34)$ using Bessel's formula from

x:	20	25	30	35	40
$f(x)$:	11.47	12.78	13.76	14.49	15.05

3. Tabulate $f(x) = e^{-x}$ in $[1.72, 1.78]$ with $h = 0.01$. Find $f(1.7475)$ using (i). Bessel's and (ii) Everett's formula.

Keywords: Bessel's Formula, Stirling's Formula

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Jain, M. K., Iyengar. S.R.K., Jain. R.K. (2008). Numerical Methods. Fifth Edition, New Age International Publishers, New Delhi.

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Lesson 17

Newton's Divided Difference Interpolation

17.1 Introduction

In Lagrange interpolation, the fundamental polynomials are constructed for writing the interpolating polynomial. Suppose we found the fundamental polynomials for the given N data point set. If a data point is added to this set, then the fundamental polynomials are reconstructed. This makes the process laborious. An easier way of finding an interpolating polynomial is given by constructing the divided differences.

17.2 Divided Difference

For the set $\{(x_0, f_0),(x_1, f_1)\}$, the linear interpolating polynomial $a_0 + a_1 x$ is given by:

$$\begin{vmatrix} p(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0.$$

Expand the determinant in term of the first row, we get

$$p(x)\left(x_0 - x_1\right) - x\left[f(x_0) - f(x_1)\right] + 1\left[x_1 f(x_0) - x_0 f(x_1)\right] = 0$$

or it is rewritten as

$$p(x) = f(x_0) + (x - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= f(x_0) + (x - x_0) f\left[x_0, x_1\right] \tag{17.1}$$

where $f[x_0, x_1]$ is defined as the first divided difference of $f(x)$ relative to x_0 and x_1, given as:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Example 1: Given $f(2) = 4, f(2.5) = 5.5$, find the linear interpolating polynomial using Newton's divided difference interpolation.

Solution:

Given $x_0 = 2,\ f_0 = 4,\ x_1 = 2.5,\ f_1 = 5.5.$

Newton's first divided difference

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{5.5 - 4}{2.5 - 2} = \frac{1.5}{0.5} = 3.$$

The interpolating polynomial is

$$p_1(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$

$$= 4 + (x - 2)(3)$$

$$= 3x - 2.$$

17.3 Generalization to $(N+1)$ Data Points

The second divided difference of $f(x)$ relative to the points x_0, x_1, x_2 is written as:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

where $f[x_1, x_2]$ is the first divided difference of $f(x)$ relative to x_1 and x_2 is given by:

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The third divided difference of $f(x)$ relative to x_0, x_1, x_2, x_3 is given by

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{(x_3 - x_0)}.$$

The same way, the n^{th} divided difference is written as

$$f[x_0, x_1, ..., x_N] = \frac{f[x_1, x_2, ..., x_N] - f[x_0, x_1, x_2, ..., x_{N-1}]}{(x_N - x_0)}.$$

Let $p(x) = a_0 + (x - x_0)a_1 + ... + (x - x_0)(x - x_1)...(x - x_{N-1})a_N$ be the interpolating polynomial for the $(N+1)$ distinct points $\{(x_0, f_0), (x_1, f_1), ..., (x_N, f_N)\}$.

Substituting $x = x_0$ in the above, we get

$$p(x_0) = a_0 = f(x_0) = f[x_0].$$

Put $x = x_1$, we obtain

$p(x_1) = a_0 + (x - x_0)a_1 = f[x_1]$ (say)

$\Rightarrow a_1 = \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)} = f[x_0, x_1].$

Put $x = x_2$, we get

$$p(x_2) = f(x_0) + (x - x_0)\, f[x_0, x_1] + (x_2 - x_0)(x_2 - x_1)\, a_2 = f(x_2).$$

On simplifying, we get

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}.$$

Proceeding in this way, we show that

$$a_n = f[x_0, x_1, ..., x_N].$$

Then we obtain the divided difference interpolating polynomial as

$$p_N(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + ... + (x - x_0)(x - x_1)...(x - x_{N-1})\, f[x_0, x_1, ..., x_N]. \quad (17.2)$$

This formula is easily extended to (say) $(N + 2)$ data. (i.e., addition of one more data point to the previous data set) as

$$p_{N+1}(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + ... + (x - x_0)...(x - x_{N-1})\, f[x_0, x_1, ..., x_N] +$$
$$(x - x_0)...(x - x_N)\, f[x_0, x_1, ..., x_N, x_{N+1}] \quad (17.3)$$

It amounts to finding the next divided difference and adding it to previously obtained interpolating polynomial as shown above.

Example 2: Find the Newton divided difference interpolating polynomial for the data.

x:	0	1	3
f:	1	3	5.5

Solution:

$x_0 = 0, x_1 = 1, x_2 = 3, f_0 = 1, f_1 = 3, f_2 = 55$.

$$f[0,1] = \frac{f(1) - f(0)}{1-0} = \frac{3-1}{1} = 2,$$

$$f[1,3] = \frac{55-3}{3-1} = \frac{52}{2} = 26,$$

$$\therefore f[0,1,3] = \frac{26-2}{3-0} = \frac{24}{3} = 8.$$

$\therefore$ Newton's divided difference interpolating polynomial is

$$p_2(x) = f(0) + (x-0) f[0,1] + (x-0)(x-1) f[0,1,3]$$

$$= 1 + x \cdot 2 + (x)(x-1)8$$

$$= 8x^2 - 6x + 1.$$

Exercises:

1. Find $f(7)$ from the following data using the Newton's divided difference interpolation.

x:	1.5	3.0	5.0	6.5	8.0
f:	5.0	31.0	131.0	282.0	521.0

2. If $f(x) = \frac{1}{x^2}$, find the divided difference $f[x_0, x_1, x_2]$.

3. From the data

x:	-1	1	4	7
$f(x)$:	-2	0	63	342

(i) Find the Lagrange interpolating polynomial.

(ii) Find the Newton divided difference interpolating polynomial.

Keywords: Divided Difference

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Lesson 18

Numerical Differentiation

18.1 Introduction

Given a continuous function $f(x)$ on an interval $[a,b]$, it can be differentiated on (a,b). However, when $f(x)$ is a complicated function or when it is given in a data set, we use numerical methods to find its derivatives. There are many ways of finding derivatives of a function when given in its data form. Important among these methods are methods based on interpolation, method based on finite difference operators. We discuss these methods through examples.

18.2 Methods based on Interpolation

If (x_i, f_i), $i = 0,1,2...$, N are the $(N+1)$ data points representing a function $y = f(x)$. The Lagrange interpolating polynomial for above set of data points is given by

$$P_N(x) = \sum_{k=0}^{N} l_k(x) f_k \tag{18.1}$$

where $l_k(x) = \dfrac{\pi(x)}{(x - x_k)\pi'(x_k)}$.

and $\pi(x) = (x - x_0)(x - x_1)...(x - x_N)$.

Differentiating (1) w.r.t. x, we obtain

$$P_N'(x) = \sum_{k=0}^{N} l_k'(x) f_k .$$

Case 1: Linear interpolation: For $(x_0, f_0),(x_1, f_1)$:

We have $P_1(x) = l_0(x) f_0 + l_1(x) f_1$

where $l_0(x) = \dfrac{x - x_1}{x_0 - x_1}$ and $l_1(x) = \dfrac{x - x_0}{x_1 - x_0}$.

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 .$$

The derivative of this is

$$P_1'(x) = \frac{1}{x_0 - x_1} f_0 + \frac{1}{x_1 - x_0} f \tag{18.2}$$

Case 2: Quadratic interpolation: For the data $\{(x_0, f_0), (x_1, f_1), (x_2, f_2)\}$, we have the interpolating polynomial as $P_2(x) = l_0(x) f_0 + l_1(x) f_1 + l_2(x) f_2$.

Its derivative is

$$P_2'(x) = l_0'(x) f_0 + l_1'(x) f_1 + l_2'(x) f_2$$

where $l_0'(x) = \dfrac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)}$, $l_1'(x) = \dfrac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)}$, $l_2'(x) = \dfrac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)}$.

$$\therefore P_2'(x) = \left[\frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f_1 + \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} f_2 \right] \tag{18.3}$$

At $x = x_1$, $P_2'(x) = \dfrac{x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f_0 + \dfrac{2x_1 - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f_1 + \dfrac{x_1 - x_0}{(x_2 - x_0)(x_2 - x_1)} f_2$.

Similarly we can write $P_2'(x)$ at any nodal point or a non-nodal point.

In the same manner, $l_0''(x) = \dfrac{2}{(x_0 - x_1)(x_0 - x_2)}$, $l_1''(x) = \dfrac{2}{(x_1 - x_0)(x_1 - x_2)}$ and

$$l_2''(x) = \frac{2}{(x_2 - x_0)(x_2 - x_1)}$$

and the second derivative of $P_2(x)$ i.e.,

$$P_2''(x) = 2\left[\frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)} \right] \tag{18.4}$$

This gives a way of finding an approximation to $f'(x)$ at every $x \in [x_0, x_N]$ by finding the interpolating polynomial $P_n(x)$ for the given set of data points.

Example 1: Find $f'(2)$ and $f''(2)$ for the below given data set.

x_i	2	2.2	2.6
f_i	0.69315	0.78846	0.95551

Solution:

We have

$$f'(x_0)=f'(2)=\frac{2x_0-x_1-x_2}{(x_0-x_1)(x_0-x_2)}f_0+\frac{(x_0-x_2)}{(x_1-x_0)(x_1-x_2)}f_1+\frac{(x_0-x_1)}{(x_2-x_0)(x_2-x_1)}f_2$$

$$=\frac{4-2.2-2.6}{(-0.2)(-0.6)}(0.69315)+\frac{2-2.6}{(0.2)(-0.4)}(0.78846)+\frac{2-2.2}{(0.6)(0.4)}(0.95551)$$

$$=0.49619.$$

$$f''(x_0)=2\left[\frac{f_0}{(x_0-x_1)(x_0-x_2)}+\frac{f_1}{(x_1-x_0)(x_1-x_2)}+\frac{f_2}{(x_2-x_0)(x_2-x_1)}\right].$$

$$\therefore\ f''(2)=2\left[\frac{0.69315}{(-0.2)(-0.6)}+\frac{0.78846}{(0.2)(-0.4)}+\frac{0.95551}{(0.6)(0.4)}\right]$$

$$=-0.19642.$$

Thus $f'(2)=0.49619$ and $f''(2)=-0.19642$.

In the above, we used Lagrange interpolation method. Similarly one can use Newton interpolation methods also.

18.3 Methods based on Finite Differences

For a given equally spaced data set, we have learnt that $Ef(x)=e^{hD}f(x)$

i.e., $e^{hD}\equiv E\Rightarrow hD\equiv\log E$ (18.5)

where E is the shift operator, D is the differentiation operator, h being the constant step size. Using the relation between E, Δ and ∇, we write

$$hD \equiv \log E \equiv \begin{cases} \log(1+\Delta) \equiv \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \ldots \\ -\log(1-\nabla) \equiv \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots \end{cases} \tag{18.6}$$

Then $$h\frac{df}{dx}(x_k) \equiv hDf(x_k) \equiv \begin{cases} \Delta f_k - \frac{1}{2}\Delta^2 f_k + \frac{1}{3}\Delta^3 f_k - \ldots \\ \nabla f_k + \frac{1}{2}\nabla^2 f_k + \frac{1}{3}\nabla^3 f_k + \ldots \end{cases} \tag{18.7}$$

In general we can write the higher order derivatives in terms of the higher order differences, the n^{th} derivative operator $\frac{d^n}{dx^n}$ can be written as

$$h^n D^n \equiv \begin{cases} \Delta^n - \frac{n}{2}\Delta^{n+1} + \frac{n(3n+5)}{24}\Delta^{n+2} - \ldots \\ \nabla^n + \frac{n}{2}\nabla^{n+1} + \frac{n(3n+5)}{24}\nabla^{n+2} + \ldots \end{cases} \tag{18.8}$$

In particular, when $n=2$ and at $x = x_k$,

$$h^2 \frac{d^2 f}{dx^2}(x_k) \equiv \begin{cases} \Delta^2 f_k - \Delta^3 f_k + \frac{11}{12}\Delta^4 f_k - \ldots \\ \nabla^2 f_k + \nabla^3 f_k + \frac{11}{12}\nabla^4 f_k + \ldots \end{cases} \tag{18.9}$$

Example 2: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.2$ using forward differential from the following table:

x:	1	1.2	1.4	1.6	1.8	2.0
$y(x)$:	2.7183	3.3201	4.0552	4.953	6.0496	7.3891

Solution:

Take $x_0 = 1.2$, $y_0 = 3.3201$, $h = 0.2$. We form the difference table for the given data set as:

$$\left.\frac{dy}{dx}\right|_{x=1.2} = \frac{1}{h}\left[\Delta f(1.2) - \frac{1}{2}\Delta^2 f(1.2) + \frac{1}{3}\Delta^3 f(1.2) - \ldots\right]$$

$$= \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \ldots\right]$$

≈ 3.3205.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1.2	3.3201					
		0.7351				
1.4	4.0552		0.1627			
		0.8978		0.0361		
1.6	4.9530		0.1988		0.008	
		1.0966		0.0441		0.0001
1.8	6.0496		0.2429		0.0094	
		1.3395		0.0535		
2.0	7.3891		0.2964			
		1.6359				
2.4	9.025					

$$\left.\frac{d^2y}{dx^2}\right|_{x=1.2} = \frac{1}{h^2}\left[\Delta^2 f_k - \Delta^3 f_k + \ldots\right]$$

$$= \frac{1}{0.04}\left[0.1627 - 0.0361\right] \approx 3.165.$$

Example 3: Given: (x_i, y_i), $i = 1, 2, 3, 4, 5, 6$ as:

x	3	4	5	6	7	8
y	0.205	0.24	0.259	0.262	0.25	0.224

Find the value of x for which y is minimum.

Solution:

The difference table is:

x	y	Δ	Δ^2	Δ^3
3	0.205			
		0.035		
4	0.24		-0.016	
		0.019		0.0
5	0.259		-0.016	
		0.003		0.001
6	0.262		-0.015	
		-0.012		0.001
7	0.25		-0.014	
		-0.026		
8	0.224			

Take $x_0 = 3,\ y_0 = 0.205,\ h = 1.0$.

Let us now obtain the interpolating polynomial using Newton's forward difference interpolation. It is $y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0$ where $p = \frac{x - x_0}{h}$

or $y(x) = 0.205 + (0.035)p - \frac{(0.016)}{2}p(p-1)$.

The minimum value of $y(x)$ is obtained by solving $\frac{dy}{dp} = 0$

i.e., $(0.035) - \frac{(2p-1)}{2}(0.016) = 0$

$\Rightarrow 0.035 - 0.008(2p-1) = 0$

$\Rightarrow (2P-1) = \frac{0.035}{0.008}$

$\Rightarrow p = 2.6875$

Thus $x = x_0 + ph = 3 + 2.6875 = 5.6875$.

Hence minimum value of $y(x)$ is attained at $x = 5.6875$ and the minimum value is 0.2628.

Exercises

1. Find the value of $\cos 1.747$ using the below table:

x:	1.7	1.74	1.78	1.82
$\sin x$:	0.9916	0.9857	0.9781	0.9691

2. Given $\sin 0^o = 0, \sin 10^o = 0.1736, \sin 20^o = 0.3420, \sin 30^o = 0.5, \sin 40^o = 0.6428$.

Find

(i) $\sin 23^o$

(ii) $\cos 10^o$

(iii) $-\sin 20^o$ using the method based on the finite differences.

3. Find the value of x for which y is maximum from the below tabulated values for $y(x)$.

x:	1.2	1.3	1.4	1.5	1.6
y:	0.93	0.96	0.98	0.99	1.0

Keyword: Finite differences, Lagrange interpolation,

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Lesson 19

Numerical Integration

19.1 Introduction

Consider the data set S for a given function $y = f(x)$ which is not known explicitly where $S = \{(x_0, y_0), (x_1, y_1), \ldots (x_N, y_N)\}$.

It is required to compute the value of the definite integral

$$I = \int_a^b y(x)dx; \tag{19.1}$$

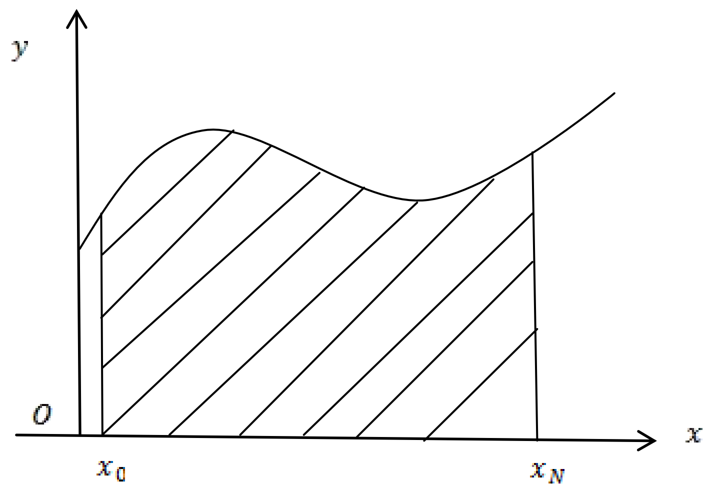
The Lagrange interpolating polynomial for the above data is given by

$$y(x) = \sum_{i=0}^{N} l_i(x) f_i + \frac{\pi(x)}{(N+1)!} f^{(N+1)}(\xi)$$

where $\pi(x) = (x - x_0)(x - x_1)\ldots(x - x_N)$; $x_0 < \xi < x_N$

and $l_i(x)$ is the Lagrange fundamental polynomial.

Replace the function $y(x)$ by (19.2) in the integral (19.1) we obtain



$$I = \int_a^b \left[\sum_{i=0}^{N} l_i(x) f_i \right] dx + \int_a^b \left[\frac{\pi(x)}{(N+1)!} f^{(N+1)}(\xi) \right] dx \tag{19.2}$$

$$= \sum_{i=0}^{N} \left(\int_a^b l_i(x)dx \right) f_i + R_n$$

or $I = \lambda_i f_i + R_n$ (19.3)

where λ_i are $\int_a^b l_i(x)dx$ and R_n is the remainder given by $\frac{1}{(N+1)!}\int_a^b \pi(x) f^{(N+1)}(\xi)dx$.

Equation (3)gives an approximation for the integral value.

19.2 Newton-Cotes Formulae

Using Newton's, forward difference interpolation polynomial for the given data set *S*, we now derive a general formula for numerical integration of

$$I = \int_a^b y(x)dx \qquad (19.4)$$

Consider the partition of the integral [*a*, *b*] as

$a = x_0 < x_1 ... < x_N = b$ such that $x_N = x_0 + Nh$ i.e., $h = \frac{b-a}{N}$.

Using Newton's forward interpolation formula in the above integral; we obtain

$$I = \int_{x_0}^{x_N} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{2!}\Delta^3 y_0 + ... \right] dx \qquad (19.5)$$

where $p = \frac{x - x_0}{h}$.

The above integral can be written as

$$I = h\int_0^N \left[y_0 + p\Delta y_0 + \frac{p^2 - p}{2!}\Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!}\Delta^3 y_0 + ... \right] dp \qquad (19.6)$$

or $I = \left[hy_0 + \frac{p^2}{2}\Delta y_0 + \left(\frac{p^3}{6} - \frac{p^2}{4} \right)\Delta^2 y_0 + \left(\frac{p^4}{24} - \frac{p^3}{6} + p^2 \right)\Delta^3 y_0 + ... \right]\Bigg|_{p=0}^{N}$

$$= Nh\left[y_0 + \frac{N}{2}\Delta y_0 + \frac{N(2N-3)}{12}\Delta^2 y_0 + \frac{N(N-2)^2}{24}\Delta^3 y_0 + \ldots\right] \qquad (19.7)$$

Case 1: With $N = 1$, we have

$$I = \int_{x_0}^{x_1} y(x)dx$$

$$= 1 \cdot h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= \frac{h}{2}(y_0 + y_1) \qquad (19.8)$$

Case 2: With $N = 2$, we have

$$I = \int_{x_0}^{x_2} y(x)dx$$

$$= 2h\left[y_0 + \frac{2}{2}\Delta y_0 + \frac{2(1)}{12}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[2y_1 + \frac{1}{3}y_2 - \frac{2}{3}y_1 + \frac{1}{3}y_0\right]$$

$$= \frac{1}{3}[y_0 + 4y_1 + y_2] \qquad (19.9)$$

Case 3: With N=3, we have

$$I = \int_{x_0}^{x_3} y(x)dx$$

$$= 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

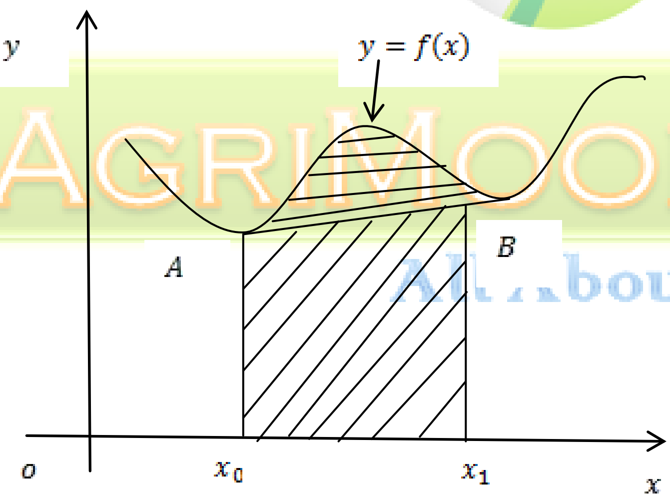
$$=\frac{3h}{8}[y_0+3y_1+3y_2+y_3] \tag{20.1}$$

We now discuss the use of the case1 for evaluating the integral, $I=\int_{x_0}^{x_N} y(x)dx$.

We can write $I=I_1+I_2+I_3+...+I_N=\sum_{j=1}^{N} I_j$

where $I_j=\int_{x_{j-1}}^{x_j} y(x)dx$.

We take two consecutive data points at once and apply the formula given in (19.5) for every pair of data points. Equation (19.4) is known as the Newton-Cotes quadrate (Interpolation) formula. With different values of $N=1,2,3,...$,we derive different integration methods.



We obtain

$$I_1=\int_{x_0}^{x_1} y(x)dx=h\left[y_0+\frac{1}{2}\Delta y_0\right]=\frac{h}{2}[y_0+y_1]$$

$$I_2=\int_{x_0}^{x_2} y(x)dx=h\left[y_1+\frac{1}{2}\Delta y_1\right]=\frac{h}{2}[y_1+y_2]$$

$$I_3 = \int_{x_0}^{x_3} y(x)dx = h\left[y_2 + \frac{1}{2}\Delta y_2\right] = \frac{h}{2}[y_2 + y_3]$$

……

$$I_N = \int_{x_{N-1}}^{x_N} y(x)dx = h\left[y_{N-1} + \frac{1}{2}\Delta y_{N-1}\right] = \frac{h}{2}[y_{N-1} + y_N].$$

Now

$$I = \int_{x_0}^{x_N} y(x)dx = \int_{x_0}^{x_1} y(x)dx + \int_{x_1}^{x_2} y(x)dx + ... + \int_{x_{N-1}}^{x_N} y(x)dx$$

$$= I_1 + I_2 + ... + I_N$$

$$= \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + ... + \frac{h}{2}[y_{N-1} + y_N]$$

$$= \frac{h}{2}\left[y_1 + 2\left(y_2 + y_3 + ... + y_{N-1}\right) + y_N\right] \qquad (20.2)$$

This is known as the Trapezoidal rule. Since the method involves finding the sum of the areas of these *N*-trapezoids, this method is named as Trapezoidal rule.

Example 1: Evaluate $\int_0^{1.2} 2e^x dx$ using trapezoidal rule by taking $h=0.2$.

Solution:

$a=0$, $b=1.2$, $h=0.2$, we tabulate the function e^x at the nodal points as:

x:	0	0.2	0.4	0.6	0.8	1.0	1.2
$y = e^x$:	1	1.221	1.492	1.822	2.226	2.718	3.32

The trapezoidal rule is given by

$$I = 2\frac{h}{2}\left[\left(y_0 + y_6\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right)\right]$$

$$= 0.2\left[(1+3.32)+2(1.221+1.492+1.822+2.226+2.718)\right].$$

$\therefore I = 4.656$.

Example 2: Evaluate $\int_0^{1.0} y(x)dx$, where $y(x)$ is tabulated as:

x:	0	0.25	0.5	0.75	1.0
y:	1	0.8	0.6667	0.5714	0.5

Solution:

The trapezoidal rule gives

$$I = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$= \frac{0.25}{2}[(1+0.5)+2(0.8+0.6667+0.5714)]$$

$$= 0.697.$$

Exercises:

1 A solid of revolution formed by rotating about the x –axis, the area between the x –axis, the lines $x = 0$ and $x = 1$ and a curve through the points with the following coordinates:

x:	0	0.25	0.5	0.75	1.0
y:	1.0	0.9896	0.9589	0.9089	0.8415

Find the volume of the solid formed using the Trapezoidal rule.

2. Evaluate $\int_0^1 \frac{1}{1+x}dx$ using the Trapezoidal rule by taking $h = 0.1$.

Keywords: Newton-Cotes Formulae, Numerical Integration,

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Lesson 20

Simpson's one Third and Simpson's Three Eighth Rules

20.1 Simpson's One-Third Rule

It is obtained by taking $N = 2$ in the Newton-Cotes formula (4). We described this in case (2) of the lesson (19). We divide the interval $[a, b]$ into an even number of subintervals of equal length having odd number of abscissas.

We divide the interval $[a, b]$ into $2k$ subintervals each of length $h = \frac{b-a}{2k}$, we then get $2k + 1$ abscissas as

$a = x_0 < x_0 < x_0 < \cdots < x_{2k} = b, \qquad x_i = x_0 + ih, \qquad i = 1,2, \dots ,2k - 1.$

Now $I = \int_a^b y(x)dx$

$$= \int_{x_0}^{x_2} y(x)dx + \int_{x_2}^{x_4} y(x)dx + \cdots + \int_{x_{2k-2}}^{x_{2k}} y(x)dx.$$

Now using the Newton-Cotes formula with $N = 2$ for each of the above integrals, we get

$$\int_{x_0}^{x_2} y(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

$$\int_{x_2}^{x_4} y(x)dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

...

$$\int_{x_{2k-2}}^{x_{2k}} y(x)dx = \frac{h}{3}[y_{2k-2} + 4y_{2k-1} + y_{2k}]$$

Add all these values, we obtain

$$I = \int_a^b y(x)dx$$

$$=\frac{h}{3}[y_0+4(y_1+y_3+\cdots+y_{2k-1})+2(y_2+y_4+\cdots+y_{2k-2})+y_{2k}]$$

or,

$$I=\frac{h}{3}[X+4.0+2E] \qquad (20.1)$$

where X = sum of the function values at the end points, O= sum of the function values at odd numbered abscissas, and E = sum of the function values at even numbered abscissas.

This formula is known as the Simpson's $\frac{1}{3}$rd rule of integration.

Example: Evaluate $\int_0^{12}\frac{1}{1+x^2}dx$ by Simpson's $\frac{1}{3}$rd rule taking $k=6$.

Solution:

$y(x)=\frac{1}{1+x^2}$, $a=0, b=12, k=6, h=2$

$y(x_i)=y_i; i=0,1,2,3,4,5.$

x:	0	2	4	6	8	10	12
$y(x)$:	1	0.2	0.05882	0.02703	0.01538	0.0099	0.0069

By Simpson's $\frac{1}{3}$rd rule:

$$I=\frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$=\frac{2}{3}[(1+0.0069)+4(0.2+0.02703+0.0099)+2(0.05882+0.01538)]$$

$$=1.40201.$$

20.2 Simpson's Three-Eighth Rule

Take $N = 3$ in the Newton-Cotes formula (4). This is described in Case (3) of the lesson (19).To apply this method the number of subintervals should be taken as multiples of 3.

The integral

$$I = \int_{x_0}^{x_N = x_0 + Nh} y(x)dx$$

$$= \int_{x_0}^{x_0+3h} y(x)dx + \int_{x_0+3h}^{x_0+6h} y(x)dx + \ldots + \int_{x_{N-3}}^{x_N} y(x)dx \qquad (20.2)$$

This is known as the Simpson's $\frac{3}{8}^{\text{th}}$ rule.

Example 2: Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by Simpson's $\frac{3}{8}^{\text{th}}$ rule.

Solution:

Take $h = 1, x_0 = 0, x_6 = 6$, $f(x) = \dfrac{1}{1+x^2}$.

The number of subintervals is 6, is a multiple of 3. So we can use the Simpson's $\frac{3}{8}^{\text{th}}$ rule.

$x:$	0	1	2	3	4	5	6
$y:$	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.027 y_6

$$I = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 3\left(y_1 + y_2 + y_4 + y_5\right) + 2\left(y_3\right)\right] = 1.3571.$$

Example 3: Using Simpson's $\frac{1}{3}$rd rule, find $\int_0^{0.6} e^{-x^2}dx$ by taking 6 subintervals.

Solution:

Evaluation of e^{-x^2} is not a simple function, that cannot be integrated directly. In such a situation using numerical integration it can be easily evaluated. Let us construct the data:

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$x:$	0	0.1	0.2	0.3	0.4	0.5	0.6
$x^2:$	0	0.01	0.04	0.09	0.16	0.25	0.36
$y=e^{-x^2}$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$rd rule, we have

$$\int_0^{0.6} e^{-x^2}dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$=\frac{0.1}{3}\left[(1+0.6977)+4(0.99+0.9139+0.8521)+2(0.9608+0.8521)\right]$$

$$=\frac{0.1}{3}[1.6977+10.7308+3.6258]$$

$$=0.5351.$$

Example 4: A solid of revolution is formed by rotating about the x-axis the area bounded by the x-axis, the lines $x=0$ and $x=1$, and the curve through the point with the following data:

$x:$	0	0.25	0.5	0.75	1.0
$y:$	1.0	0.9896	0.9589	0.9089	0.8415

	y_0	y_1	y_2	y_3	y_4

Estimate the volume of the solid using Simpson's $\frac{1}{3}$rd rule.

Solution:

Here $h = 0.25$.

The volume of the solid of revolution is

$$I = \int_0^1 \pi y^2 dx$$

Using the Simpson's $\frac{1}{3}$rd rule:

$$I = \frac{h}{3}\pi\left[\left(y_0^2 + y_4^2\right) + 4\left(y_1^2 + y_3^2\right) + 2\left(y_2^2\right)\right]$$

$$= \frac{0.25}{3}\frac{22}{7}\left[\left\{(1)^2 + (0.8415)^2\right\} + 4\left\{(0.9896)^2 + (0.9089)^2\right\} + 2(0.9589)^2\right]$$

$$= 0.2618[10.7687]$$

$$= 2.8192.$$

Exercises:

1. Evaluate $\int_4^{5.2} \log x \cdot dx$

Using (i) Trapezoidal rule

(ii) Simpson's $\frac{1}{3}$rd rule

(iii) Simpson's $\frac{3}{8}$th rule

by taking 12 subintervals. Then compare your results.

2. A curve is given by

x	0	1	2	3	4	5	6
y	0	2	2.5	2.3	2	1.7	1.5

Evaluate (i) the area below the given curve

(ii) $\int_0^6 xy \cdot dx$ using Simpson's $\frac{1}{3}^{rd}$ rule.

3. Estimate the length of the arc of the curve $3y = x^3$ from $(0,0)$ to $(1,3)$ using Simpson's $\frac{1}{3}^{rd}$ rule by taking 8 subintervals.

4. Evaluate $I = \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta} \cdot d\theta$ by using Simpson's $\frac{1}{3}^{rd}$ rule using 11 ordinates.

Keyword: Simpson's one-Third rule, Simpson's three-eighth rule, Even number of subintervals.

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Lesson 21

Boole's and Weddle's Rules

21.1 Introduction

Boole's and Weddle's rules are higher order integration methods. These methods use higher order differences as explained below. By taking $N = 4$ in Newton-Cotes formula, we obtain Boole's rule as:

$$\int_{x_0}^{x_4} y(x)dx = 4h\left[y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0 \right]$$

$$= \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

Similarly for the next set of data points between $(x_4, y_4)$and (x_8, y_8), we write the integral as

$$\int_{x_4}^{x_8} y(x)dx = \frac{2h}{45}[7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

By taking the number of subintervals as a multiple of 4, we obtain

$$I = \int_{x_0}^{x_N} y(x)dx$$

$$= \frac{2h}{45}\left[7y_0 + 32(y_1 + y_3 + y_5 + y_7 + \ldots) + 12(y_2 + y_6 + y_{10} + \ldots) + 14(y_4 + y_8 + y_{12} + \ldots) + 7y_N\right] \quad (21.1)$$

To use this method, the number of subintervals should be taken as a multiple of 4. By taking $N = 6$ in the Newton-Cotes integration formula, we obtain the Weddle's Rule. Here, the number of subintervals should be taken as a multiple of 6.

For $\int_{x_0}^{x_6} y(x)dx = 6h\left[y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0 \right]$

$= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6].$

For $\int_{x_6}^{x_{12}} y(x)dx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}].$

Proceeding this way, we write

$I = \int_{x_0}^{x_N} y(x)dx$

$= \frac{3h}{10}[y_0 + 5(y_1 + y_5 + y_7 + y_{11}...) + (y_2 + y_4 + y_8 + y_{10} + ...)$

$+6(y_3 + y_9 + y_{15} + ...) + 2(y_3 + y_9 + y_{15} + ...) + y_N]$ (21.2)

Weddle's rule is found to be more accurate than all the methods discussed earlier. This is because higher order approximation is used for the integration. The below given table gives the error estimates involved in the integration methods.

Summary of Newton-Cotes Methods

S.No.	Name	Integral	Formula	Error
1	Trapezoidal Rule	$\int_{x_0}^{x_1} y(x)dx$	$\frac{h}{2}(y_0+y_1)$	$-\frac{h^3}{12}y''(\xi); x_0<\xi<x_1$
2	Simpson's $\frac{1}{3}$rd Rule	$\int_{x_0}^{x_2} y(x)dx$	$\frac{h}{2}(y_0+4y_1+y_2)$	$-\frac{h^5}{90}y^{iv}(\xi); x_0<\xi<x_2$
3	Simpson's $\frac{3}{8}$th Rule	$\int_{x_0}^{x_3} y(x)dx$	$\frac{3h}{8}(y_0+3y_1+3y_2+y_3)$	$-\frac{3h^5}{90}y^{iv}(\xi); x_0<\xi<x_3$
4	Boole's Rule	$\int_{x_0}^{x_4} y(x)dx$	$\frac{2h}{45}(7y_0+32y_1+12y_2+32y_3+7y_4)$	$-\frac{8h^7}{945}y^{vi}(\xi); x_0<\xi<x_4$
5	Weddle's Rule	$\int_{x_0}^{x_4} y(x)dx$	$\frac{3h}{10}(y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6)$	$-\frac{h^7}{140}y^{vi}(\xi); x_0<\xi<x_6$

Example 1: Evaluate $\int_0^{1.2} e^x dx$ using Boole's rule by taking $h = 0.3$.

Solution:

The function $y(x) = e^x$ is tabulated at the nodes $x_0 = 0$ to $x_N = 1.2$ with $x_i = x_0 + ih, i = 1,2,3,4,\ldots$ as:

$x:$	0	0.3	0.6	0.9	1.2
$e^x:$	1 y_0	1.34986 y_1	1.82212 y_2	2.4596 y_3	3.32012 y_4

Using this data in Boole's rule

$$\int_0^{1.2} y(x)dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

$$= \frac{2(0.3)}{45}[7(1) + 32(1.34986) + 12(1.82212) + 32(2.4596) + 7(3.32012)]$$

$$= 2.31954.$$

Example 2: Evaluate $\int_0^{12} \frac{1}{1+x^2} dx$ by using Weddle's rule with $h = 2$.

Solution:

The function $y(x) = \frac{1}{1+x^2}$ is calculated

at $x_0 = 0, x_1 = 2, x_2 = 4, x_3 = 6, x_4 = 8, x_5 = 10$ and $x_6 = 12$

as $y_0 = 1, y_1 = 0.2, y_2 = 0.05882, y_3 = 0.02703, y_4 = 0.01538, y_5 = 0.0099$ and

$y_6 = 0.0069$.

Using this data in the Weddle's rule,

$$\int_0^{12} \frac{1}{1+x^2}dx = \frac{3h}{10}\left(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right)$$

$$= \frac{3(2)}{10}\left[1 + 5(0.2) + (0.05882) + 6(0.02703) + (0.01538) + 5(0.0099) + (0.0069)\right]$$

$$= 1.37567\,.$$

Exercises:

1. Evaluate $\int_4^{5.2} \log x \cdot dx$ using (i) Boole's rule with $h = 0.3$; (ii) Weddle's rule by taking $h = 0.2$. Compare these two values.

2. Evaluate $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx$ using Weddle's rule.

3. Evaluate $\int_0^1 \left(1 + e^{-x}\sin 4x\right)dx$ using Boole's rule with $h = \frac{1}{8}$.

4. Evaluate $\int_0^2 \frac{1}{1+x^2}dx$ using Weddle's rule taking 12 intervals.

Keyword: Boole's Rule, Higher order integration methods, Weddle's rule.

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Lesson 22

Gaussian Quadrature

22.1 Introduction: The problem of numerical integration is to find an approximate value for

$$I = \int_a^b w(x) f(x) dx \qquad (22.1)$$

where $w(x)$ is a positive valued continuous function defined on $[a,b]$ called the weight function. The function $w(x)f(x)$ is assumed to be integrable. The limits a and b are finite, semi-infinite or infinite. The integral (22.1) is approximated by a finite linear combination of $f(x_k)$ in the form

$$I = \int_a^b w(x) f(x) dx \;=\; \sum_{k=0}^{N} \lambda_k f_k \qquad (22.2)$$

where $x_k, k = 0,1,\ldots,N$ are called the nodes which are distributed within the limits of integration $[a,b]$ and $\lambda_k, k = 0,1,\ldots,N$ are called the discrete weights.

The formula (22.2) is also known as the quadrature formula.

The error in this approximation is given as

$$R_N = \int_a^b w(x) f(x)\, dx \;-\; \sum_{k=0}^{N} \lambda_k f_k \qquad (22.3)$$

An integration method of the form (22.2) is said to be order p if it produces exact results i.e.; $R_N = 0$ for all polynomials of degree less than or equal to p.

In evaluating the integral (22.1) using (22.2) involves finding $(N+1)$ unknown weights λ_k's and $(N+1)$ unknown nodes x_k's leading to computing

$(2N+2)$ unknowns. To compute these unknowns, the method (22.2) is made exact for polynomial of degree less than or equal to $(2N+1)$, for example, by considering $f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{2N+1} x^{2N+1}$.

For example, when $n = 2$, then $\int_{-1}^{1} f(x)\,dx = w_1\ f(x_1) + w_2\ f(x_2)$.

When the nodes x_k are known, the corresponding methods are called Newton-Cotes methods where the nodes are also to be determined, then the methods are called the quadrature methods.

The interval of integration $[a,b]$ is always transformed to $[-1,1]$ using the transformation $x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$. Depending on the weight function $w(x)$ a variety of methods are developed. We discuss here the Gauss-Legendre integration method for which the weight function $w(x) = 1$.

22.2 Gauss-Legendre Integration Methods

Consider evaluating the integral

$$I = \int_{-1}^{1} f(x)\,dx = \sum_{k=0}^{N} \lambda_k\ f(x_k)$$

where x_k are the nodes and λ_k are the weights.

(I) One Point formula $n = 0$: The formula is $\int_{-1}^{1} f(x)\,dx = \lambda_0\ f(x_0)$

In the above λ_0, x_0 are unknowns, these are obtained by making this integration method exact for $f(x) = 1, x$.

i.e., (a) $\int_{-1}^{1} 1 \cdot dx = \lambda_0 f(x_0) = \lambda_0 = 2$

(b) $\int_{-1}^{1} x \cdot dx = \lambda_0 x_0 \Rightarrow \lambda_0 x_0 = 0 \Rightarrow x_0 = 0 \; (\because \lambda_0 = 2)$.

$\therefore \int_{-1}^{1} f(x)dx = 2 \cdot f(0)$.

(II) The two point formula $n = 1$: The formula is given by

$$\int_{-1}^{1} f(x)dx = \lambda_0 \; f(x_0) + \lambda_1 \; f(x_1)$$

The unknowns are $\lambda_0, \lambda_1, x_0, x_1$. These unknowns are determined by making this method exact for $f(x) = 1, x, x^2, x^3$; we get

$f(x) = 1 \Rightarrow \lambda_0 + \lambda_1 = 2$.

$f(x) = x \Rightarrow \lambda_0 x_0 + \lambda_1 x_1 = 0$.

$f(x) = x^2 \Rightarrow \lambda_0 x_0^2 + \lambda_1 x_1^2 = \frac{2}{3}$.

$f(x) = x^3 \Rightarrow \lambda_0 x_0^3 + \lambda_1 x_0^3 = 0$.

Solving these non-linear equations we obtain

$$x_0 = \pm\frac{1}{\sqrt{3}}, x_1 = \mp\frac{1}{\sqrt{3}}, \;\; \lambda_0 = \lambda_1 = 1.$$

And the two point Gauss-Legendre method is given by

$$\int_{-1}^{1} f(x)dx = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}).$$

Exercise: Show that the three point Gauss-Legendre method is given by

$$\int_{-1}^{1} f(x)dx = \frac{1}{9}[5f(-\sqrt{\frac{3}{5}})+8f(0)+5f(\sqrt{\frac{3}{5}})].$$

Example 1: Evaluate the integral $I = \int_{1}^{2} \frac{2x}{1+x^4}dx$

using the Gauss-Legendre 2-point quadrature rule.

Solution:

The general quadrature formula is written in $[-1,1]$.

So define $x = (\frac{b-a}{2})t + (\frac{b+a}{2})t \Rightarrow x = \frac{1}{2}t + \frac{3}{2},\ dx = \frac{1}{2}dt$

The integral transforms to $\int_{-1}^{1} \frac{8(t+3)}{\left[16+(t+3)^4\right]}dt.$

Using then 3-point rule, we get

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[5(0.4393)+8(0.2474)+5(0.1379)\right]$$

$$= 0.5406$$

We can directly integrate $\int_{1}^{2} \frac{2x}{1+x^4}dx$ and its integral is $\tan^{-1}(4) - \frac{\pi}{4} = 0.5404$.

Example 2: Evaluate the integral $\int_{0}^{1} \frac{1}{1+x}dx$ using the Gauss-Legendre two point formula.

Solution:

Define $x = \frac{1}{2}t + \frac{1}{2} \Rightarrow dx = \frac{1}{2}dt$.

$$\therefore \int_{-1}^{1} \frac{dt}{t+3} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$= 0.69231$

Exercises: Evaluate (a) $\int_0^1 \frac{1}{1+x}dx$ (b) $\int_1^2 \frac{1}{x}dx$ (c) $\int_1^2 \frac{1}{1+x^4}dx$

using Gauss-Legendre (i) 2-point (ii) 3-point quadrature methods.

Keywords: Gaussian Quadrature, One Point formula, Two point formula.

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Lesson 23

Difference Equations

23.1 Introduction

Difference equations arise in problems in which sequential relations exist at various discrete values of the independent variables say $\{t_0, t_1, t_2, ..., t_N\}$. These equations are commonly seen in control engineering, radar tracking etc.

Definition: A difference equation is a relation between the differences such as of an unknown function at one or more general values of the independent variable.

A general difference equation in terms of k unknown function values is written as

$$F(y_n, y_{n+1}, ..., y_{n+k}) = 0 \qquad (23.1)$$

For Example, $\Delta^2 y_{n+1} + \Delta y_n = 0$ (23.2)

is a difference equation written using the forward differences. This can be rewritten as

$$\Delta y_{n+2} - \Delta y_{n+1} + \Delta y_n = 0$$

or $\quad (y_{n+3} - y_{n+2}) - (y_{n+2} - y_{n+1}) + (y_{n+1} - y_n) = 0$

or $\quad y_{n+3} - 2y_{n+2} + 2y_{n+1} - y_n = 0$ (23.3)

If the function F is non-linear in any one of these unknowns, then it is a non-linear difference equation. If F is a linear function in all these unknown function values, then it is a linear difference equation.

Definition 2: The order of a difference equation is the difference between the largest and the smallest argument occurring in the difference equation divided by the unit of increment the equation (3).

The order of the difference equation (3) is $\dfrac{(n+3)-n}{1}=\dfrac{3}{1}=3$.

Example 1: Find the order of the difference equation derived from $\Delta y_{n+1}+2y_n=3$.

Solution:

The difference equation corresponding to $\Delta y_{n+1}+2y_n=3$ is

$$y_{n+2}-y_{n+1}+2y_n-3=0 \qquad (23.4)$$

So the order of this equation is: $\dfrac{(n+2)-n}{1}=2$.

23.2 Formation of Difference Equations:

We now illustrate the formation of difference equations from the given family of curves.

Example 2: Form the difference equation corresponding to the two parameter family of curves given by $y=at+bt^2$.

Solution:

We have $\Delta y=a\Delta t+b\Delta t^2$

$=a(t+1-t)+b[(t+1)^2-t^2]$

$=a+b(2t+1)$

and $\Delta^2 y = 2b[(t+1)-t] = 2b$.

Eliminating the arbitrary constants a and b from Δy and $\Delta^2 y$, we get

$b = \frac{1}{2}\Delta^2 y$ and $a = \Delta y - \frac{1}{2}\Delta^2 y(2t+1)$.

Hence the given family of curve become

$$y = [\Delta y - \frac{1}{2}\Delta^2 y(2t+1)]t + \frac{1}{2}\Delta^2 y t^2$$

or $(t^2+t)\Delta^2 y - 2t\cdot\Delta t + 2y = 0$.

Equivalently, the difference equation is

$$(t^2+t)y_{t+2} - (2t^2+4t)y_{t+1} + \left(t^2+3t+2\right)y = 0 \tag{23.5}$$

Exercise 4: Form the difference equation from

(i) $y_t = at + b2^t$ (ii) $y_t = a2^t + b3^t$.

23.3 Linear Difference Equations:

Consider the linear difference equation

$$a_0 y_{n+k} + a_1 y_{n+k-1} + a_2 y_{n+k-2} + \ldots.. + a_k y_n = g(n) \tag{23.6}$$

where $a_0, a_1, \ldots.., a_k$ are constants.

If $g \equiv 0$ then equation (6) is a homogeneous equation otherwise it is non-homogenous equation. The solution of a difference equation is an expression for y_n which satisfies the given difference equation.

Definition: The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation, i.e. , $Y_n = c_1 y_1(n) + c_2 y_2(n) + ... + c_r y_r(n)$ (23.7)

with $c_1, c_2, ..., c_r$ are arbitrary constants.

Definition: A particular solution is that solution which is derived from the general solution by fixing the arbitrary constants. This is done using the initial conditions on the unknown function at the nodal points.

If V_n is a particular solution of (6), then the complete solution of (6) is

$$y_n = Y_n + V_n \tag{23.8}$$

Y_n is also called as the complementary function.

23.4 Homogenous Equations:

For finding the complementary solution of the equation (23.6), we assume the solution of the form $y_n = A\xi^n$ where A is a constant which is non-zero. Substituting this in (23.6), we get

$$A(a_0\xi^k + a_1\xi^{k-1} + ... + a_k)\xi^n = 0$$

or $a_0\xi^k + a_1\xi^{k-1} + ... + a_k = 0$ (23.9)

which is called the characteristic equation of the difference equation (23.6). Let the roots of the equation be $\xi_1, \xi_2, ..., \xi_k$ are real and distinct. These roots are (i) real, distinct (ii) real, repeating (iii) complex roots.

Let us see how the solution looks like in each case.

Case (i): Here $\xi_1, \xi_2, ..., \xi_k$ are real and distinct, then

$$Y_n = \alpha_1\xi_1^n + \alpha_2\xi_2^n + ... + \alpha_k\xi_k^n \tag{23.10}$$

where $\alpha_1, \alpha_2, ..., \alpha_k$ are arbitrary constants.

Case (ii): Let $\xi_1 = \xi_2 = \xi_3$ and $\xi_4, \xi_5, ..., \xi_k$ are all real, then Y_n is written as

$$Y_n = \left(\alpha_1 + n\alpha_2 + n^2\alpha_3\right)\xi_1^n + \left(\alpha_4\xi_4^n + ... + \alpha_k\xi_k^n\right) \qquad (23.11)$$

As a special case when ξ_1 is a real root with multiplicity k, then

$$Y_n = \left(\alpha_1 + n\alpha_2 + ... + n^k\alpha_k\right)\xi_1^n \qquad (23.12)$$

Case (iii): For the case where two of these roots are complex and rest of them are real distinct: say $\xi_1 = \alpha + i\beta = r\,e^{i\theta}$ and $\xi_2 = \alpha - i\beta = r\,e^{-i\theta}$

$\xi_3, \xi_4, ..., \xi_k$ are real and distinct. Then

$$Y_n = (\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)|\xi_1|^n + c_3\xi_3^n + ... + c_k\xi_k^n$$

with $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\dfrac{\beta}{\alpha}\right)$.

In the same way one can write the solution of the homogeneous difference equation when it has several pairs of complex roots.

Example 3: Solve the difference equations

(i) $y_{n+3} - 2y_{n+2} - 5y_{n+1} + 6y_n = 0$

(ii) $y_{n+2} - y_{n+1} + \dfrac{1}{4}y_n = 0$

Solution:

Note that the above are homogenous difference equations.

(i) By the replace y_n by $A\xi^n$, we obtain the characteristic equation as

$\xi^3 - 2\xi^2 - 5\xi + 6 = 0$, the roots of this are $\xi = 1, -2, 3$ (real, distinct). Hence the complete solution is

$y_n = \alpha_1 (1)^n + \alpha_2 (-2)^n + \alpha_3 (3)^n$.

(ii) The characteristic equation is: $\xi^2 - \xi + \frac{1}{4} = 0$ and its roots are $\xi = \frac{1}{2}, \frac{1}{2}$ (real, repeating). Hence the complete solution is: $y_n = (\alpha_1 + n\alpha_2)\left(\frac{1}{2}\right)^n$.

Example 4: (For complex roots): Find the complete solution of the difference equation $y_{n+2} - 4y_{n+1} + 5y_n = 0$.

Solution:

The characteristic polynomial is $\xi^2 - 4\xi + 5 = 0$ and its roots are $\xi_1 = 2 + i$ and $\xi_2 = 2 - i$. So $r = \sqrt{4+1} = \sqrt{5}$ and $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, and hence $Y_n = (\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)\left(\sqrt{5}\right)^n$.

Exercises:

1: Write the difference equation $\Delta^3 y_n + \Delta^2 y_n + \Delta y_n = 0$ in the subscript form.

2: Write the difference equation $\Delta y_{n+1} + \Delta^2 y_{n-1} = 5$ in the subscript form.

3: Find the order of the difference equation

$y_{n+2} - 2y_n + y_{n-1} = 1$.

5: Find the general solution of $\Delta^2 y_{n+1} - \frac{1}{3}\Delta^2 y_n = 0$.

6: Solve the following difference equations

(i) $y_{n+3} + 16y_{n-1} = 0$

(ii) $y_{n+2} - 6y_{n+1} + 9y_n = 0$

(iii) $y_{n+3} - 3y_{n+1} + 2y_n = 0$

Keywords: Complete solution, Difference Equations, Homogenous difference equations, Linear difference equation,

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Lesson 24

Non Homogeneous Difference Equation

24.1 Introduction

In this lesson we learn how to find the solution of the non-homogeneous difference equation. The solution corresponding to the non-homogeneous term is called the Particular integral of the difference equation:

Consider the non-homogeneous difference equation in the form

$$y_{n+k} + \alpha_1 y_{n+k-1} + \ldots.. + \alpha_k y_k = f(n) \tag{24.1}$$

Note that equation (24.1) is equivalent to equation (23.6).

Using the shift operator E, the above can be put in the operator form as

$$\varphi(E)y_n = f_n \tag{24.2}$$

where $\varphi(E) = E^k + \alpha_1 E^{k-1} + \ldots.. + \alpha_k$.

Then the Particular integral is written as:

$$V_n = \frac{1}{\varphi(E)} f(n)$$

24.2 Finding the Particular Integral:

$\varphi(E)$ is an operator involving E, $\frac{1}{\varphi(E)}$ is its inverse operator [assuming its existence]. The particular solution is obtained for different forms of non-homogeneous function $f(n)$ as given below. We consider the forms for $f(n)$ as a^n, $\sin pn$, $\cos pn$ and $n^p a^n \cdot G(n)$ where $G(n)$ is a polynomial in n.

(A) When $f(n) = a^n$, the particular integral $(P.I.)$ is

$$\frac{1}{\varphi(E)}a^n = \frac{1}{\varphi(a)}a^n$$ provided $\varphi(a) \neq 0$. If for $\varphi(a) = 0$, a is a simple root,

then $\frac{1}{E-a}a^n = na^{n-1}$ and if 'a' is a root with multiplicity $m\ (m < n)$ then

$(E-a)^m = 0$ and the particular integral is

$$\frac{1}{(E-a)^m} = \frac{n(n-1)....(n-m)}{m!}a^{n-m}.$$

This way one can find the P.I. V_n for the given non-homogeneous equation when $f(n) = a^n$.

Example 1: Find the particular integral of $y_{n+2} - 4y_{n+1} + 3y_n = 2^n + 5^n$.

Solution:

$$P.I. = \frac{1}{(E^2 - 4E + 3)}(2^n + 5^n)$$

$$= \frac{1}{(E^2 - 4E + 3)}2^n + \frac{1}{(E^2 - 4E + 3)}5^n$$

Clearly, 2 and 5 are not the roots of the auxiliary equation of $E^2 - 4E + 3$.

$$P.I. = \frac{1}{2^2 - 4\cdot 2 + 3}2^n + \frac{1}{5^2 - 4\cdot 5 + 3}5^n$$

$$= -2^n + \frac{1}{8}5^n$$

Example 2: Solve the difference equation $y_{n+2} - 6y_{n+1} + 9y_n = 3^n + (-1)^n$.

Solution:

The characteristic equation is $\xi^2 - 6\xi + 9 = 0$.

Roots are $\xi = 3,3$.

The complementary function is: $y_{C.F.} = (\alpha_1 + h\alpha_2)(3^n)$.

Note that 3 is a root of the characteristic equation, with multiplicity 2, the particular integral is written as $\dfrac{n(n-1)}{2!}\cdot 3^{n-2} + \dfrac{1}{15}(-1)^n$.

Hence the complete solution is $y_n = y_{C.F} + y_{P.I.} = Y_n + V_n$

$$= (\alpha_1 + \alpha_2 n)3^n + \frac{n(n-1)}{2!}\cdot 3^{n-2} + \frac{1}{15}(-1)^n.$$

(B) When $f(n) = \sin pn$ or $\cos pn$:

$$P.I. = \frac{1}{\varphi(E)}\sin pn = \frac{1}{\varphi(E)}\left(\frac{e^{ipn} - e^{-ipn}}{2i}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{\varphi(E)}a^n - \frac{1}{\varphi(E)}b^n\right); \begin{cases} a^n = e^{ipn} \\ b^n = e^{-ipn} \end{cases}$$

This is in the form discussed for $f(n) = a^n$.

Similarly, $\dfrac{1}{\varphi(E)}\cos pn = \dfrac{1}{2}\left(\dfrac{1}{\varphi(E)}a^n + \dfrac{1}{\varphi(E)}b^n\right)$.

(C) When $f(n) = n^p$, then

$$P.I. = \frac{1}{\varphi(E)}n^p = [\varphi(E)]^{-1} n^p.$$

Recall $E = 1 + \Delta$.

$\therefore P.I. = [\varphi(E)]^{-1} n^p$.

Expanding $[\varphi(1+\Delta)]^{-1}$ in increasing powers of Δ (using binomial theorem) and operating it over n^p, we get the particular integral.

(D) When $f(n) = a^n a(n)$, where $a(n)$ is a polynomial in n.

$$P.I. = \frac{1}{\varphi(E)} a^n a(n) = a^n \frac{1}{\varphi(aE)} a(n).$$

This can be solved using the procedure given in (C).

Example 3: Solve $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = \cos n$.

Solution:

It can be readily seen that the characteristic equation as

$\xi^2 - 2\cos\alpha\,\xi + 1 = 0$ and its roots are $\cos\alpha \pm i\sin\alpha$.

So the C.F. is: $(\alpha_1 \cos\alpha n \pm \alpha_2 \sin\alpha n)(1)^n$.

$$P.I. = \frac{1}{E^2 - 2E\cos\alpha + 1}\cos\alpha = \frac{1}{E^2 - 2E\left(e^{i\alpha} + e^{-i\alpha}\right) + 1}\frac{\left(e^{in} + e^{-in}\right)}{2}$$

$$= \frac{1}{2}\left[\frac{1}{E - e^{i\alpha}} \cdot \frac{1}{\left(e^{i\alpha} - e^{-i\alpha}\right)} \cdot e^{in} + \frac{1}{E - e^{-i\alpha}} \cdot \frac{1}{\left(e^{-i\alpha} - e^{i\alpha}\right)} \cdot e^{-in}\right]$$

$$= \frac{1}{4i\sin\alpha}\left[\frac{1}{E - e^{i\alpha}} \cdot e^{in} - \frac{1}{E - e^{-i\alpha}} \cdot e^{-in}\right]$$

$$= \frac{1}{4i\sin\alpha}\left[\frac{1}{e^{i} - e^{i\alpha}} \cdot e^{in} - \frac{1}{e^{-i} - e^{-i\alpha}} \cdot e^{-in}\right]$$

$\therefore y_n = C.F. + P.I.$

Example 4: Find the particular integral of $y_{n+2} - 2y_{n+1} + y_n = n \cdot 3^n$.

Solution:

$$P.I. = \frac{1}{(E-1)^2} \cdot n \cdot 3^n$$

$$= 3^n \cdot \frac{1}{(3E-1)^2} \cdot n$$

$$= 3^n \cdot \frac{1}{2^2\left(1+\frac{3}{2}\Delta\right)^2} \cdot n$$

$$= \frac{3^n}{2^2}\left(1+\frac{3}{2}\Delta\right)^{-2} \cdot n$$

$$= \frac{3^n}{2^2}(1-3\Delta+....)[n]$$

$$= \frac{3^n}{2^2}\{[n]-3\cdot 1\} = \frac{3^n}{2^2}(n-3).$$

Exercises:

1. Solve $y_{n+2} - 4y_n = n-1$.

2. Solve the following difference equations:

   (i) $\Delta^2 y_n - 5\Delta y_n + 4y_n = n + 2^n$

   (ii) $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$

   (iii) $y_{n+2} - \left(2\cos\frac{1}{2}\right)y_{n+1} + y_n = \sin\frac{n}{2}$.

Keyword: Particular integral, Non-homogeneous difference equation,

Lesson 25

Numerical Solutions of Ordinary Differential Equations

25.1 Introduction

Many problems in engineering and science are modelled as ordinary differential equations which are either linear or non-linear equations satisfying certain given conditions. Only a few of these equations can be solved using the standard analytical methods whereas the other alternative is to find their numerical solution. Here, we see some of the numerical methods to solve a class of problem known as the initial value problems (I.V.P). An initial value problem is one where the differential equation is solved subjected to the required number of initial of initial conditions.

A general first order initial value problem is given by $\frac{dy}{dt} = f(y, t)$ (25.1)

$t \in I = [t_0, b]$; subjected to $y(t_0) = y_0$ (25.2)

The solution of (25.1) - (25.2) can be found as a series for y in terms of power of the independent variable 't', from which the value of y can be obtained by direct substitutionor as a set of tabled values of t and y.

The methods due to Picard and Taylor (Series) find the solution of the IVP (25.1) -(25.2) as the dependent function y as a function of the independent variable 't'. The other methods such as the Euler and Runge-Kutta methods give the solution y at some discrete data set for t in the interval $I = [t_0, b]$.

For ann^{th} order differential equation, the general solution has n arbitrary constants and in order to compute the numerical solution of such an equation, we need n initial conditions each on the function and its derivatives upto the $(n-1)^{th}$ order at the initial value I.V.P can also be found in the similar manner as that of the solution of (25.1)-(25.2), but by using the vector treatment. Let us now discuss the Picard's method of successive approximations.

25.2 Picard's Method

In this method, a sequence of approximations is constructed by stating with an initial approximation to the solution. The limit of this sequence (if exists) will be the approximation for the solution of the I.V.P given by equations $\frac{dy}{dt} = f(t,y),\ t \in [t_0, b]$

$$(25.3)$$

subjected to $(t_0) = y_0$ (25.4)

Integrating (1) between the limits,

$$\int_{y_0}^{y} dy = \int_{t_0}^{t} f(t,y)dt$$

$$\text{or } y = y_0 + \int_{t_0}^{t} f(t,y)dt \qquad (25.5)$$

Notice that the unknown that is to be found isalso seen inside the integral, such an equation is called an integral equation. Such an equation can be solved by the method of successive approximations. To start with the procedure, assume an approximation for the unknown function $y(t)$ as y_0 inside the integral. This makes the equation (25.3) as

$$y = y_0 + \int_{t_0}^{t} f(t, y)dt \tag{25.6}$$

The r.h.s. of (25.4) can be evaluated and call it as $y'(t)$

i.e., $y'(t) = y_0 + \int_{t_0}^{t} f(t, y_0)dt$.

Here $y'(t)$ is the first approximation to the solution. The second approximation to the solution is obtained by using $y'(t)$ in the integral as

$$y^2(t) = y_0 + \int_{t_0}^{t} f(t, y'(t))dt.$$

Repeating this process, we obtain

$$y^n(t) = y_0 + \int_{t_0}^{t} f(t, y^{n-1}(t))dt$$

with $y^{(0)}(t) = y_0,\ n = 1, 2, 3, \ldots$

By this way we generated a sequence of approximating functions $\{y^n(t)\}_{n=1}^{\infty}$.

It can be proved that if the function $f(t, y)$ is bounded in some region about the point (t_0, y_0) and if $f(t, y)$ satisfies the condition $|f(t, y) - f(t, y^*)| \le k|y - y^*|$ for the same positive constant k, then the sequence $\{y^n(t)\}_{n=1}^{\infty}$ converges to the solution of the I.V.P. given by (25.1)-(25.2) .

Example 1: Find the first three approximate analytical solutions to the I.V.P. $\frac{dy}{dt} = 3t + y^2$ subjected to $y(0) = 1$.

Solution:

Integrating $\frac{dy}{dt} = 3t + y^2$ in the domain, we get

$$\int_1^y dy = \int_0^t (3t + y^2)dt$$

$$\Rightarrow y(t) - y(0) = \int_0^t (3t + y^2)dt$$

or $y(t) = 1 + \int_0^t (3t + {y_0}^2)dt$

or $y'(t) = 1 + \int_0^t (3t + 1)dt$

$$= \frac{3t^2}{2} + t + 1$$

is the first approximation. The second approximation is

$$y^2(t) = 1 + \int_0^t \left[3t + \left(y'(t)\right)^2\right]dt$$

$$= 1 + \int_0^t \left[3t + \left(\frac{3t^2}{2} + t + 1\right)^2\right]dt$$

$$= \frac{9}{20}t^5 + \frac{3}{4}t^4 + \frac{4}{3}t^3 + \frac{5}{2}t^2 + t + 1$$

Likewise, the third approximation can be found as

$$y^3(t) = \frac{81}{4400}x^{11} + \frac{27}{400}x^{10} + \frac{47}{240}x^9 + \frac{17}{32}x^8 + \frac{1157}{1260}x^7 + \frac{68}{45}x^6$$

$$+ \frac{25}{12}x^5 + \frac{23}{12}x^4 + 2x^3 + \frac{5}{2}x^2 + x + 1.$$

The advantage of this method is that we can compute the solution of the given I.V.P. at every point of the domain. The disadvantage of this method is, as also seen in the earlier example, that the integration procedure is laborious and at times, integration might not be possible for some functions $f(t, y)$. The limited utility of this method demands for the search of more elegant methods for solving I.V.Ps.

Example 2: Solve the equation $\frac{dy}{dt} = t + y^2$ subject to the condition $y(t=0)=1$.

Solution:

Start with $y^{(0)} = y_0 = 1$.

This generates $y^{(1)}(t) = 1 + \int_0^t (t+1)\,dt$

$$= 1 + t + \frac{t^2}{2}$$

and $y^{(2)}(t) = 1 + \int_0^t \left[t + \left(1 + t + \frac{t^2}{2}\right)^2 \right] dt$

$$= 1 + t + \frac{3}{2}t^2 + \frac{2}{3}t^3 + \frac{1}{4}t^4 + \frac{1}{20}t^5$$

and so on.

Note that finding $y^{(3)}(t)$ itself involves squaring a 5th degree polynomial and integrating it, thus making this method more tedious.

In the next lesson, we learn one other analytical method that gives an approximation for the solution in a function form.

Exercise:

1. Use Picard's method to find the solution y at $t = 0.2, 0.4$ and 1.0 correct to three decimal places for the I.V.P. $\frac{dy}{dt} = \frac{t^2}{1+y^2}$ subject to $y(0) = 0$ [Hint: obtain $y^{(2)}(t)$ which results in a 9^{th} degree polynomial in t as the approximate solution, which gives the solution correct to 3 decimal places].

2. Use Picard's method successive approximations to solve the following I.V.Ps:

a) $\frac{dy}{dx} = 1 + xy$, $y(0) = 1$.

b) $\frac{dy}{dt} = t - y$, $y(0) = 1$.

c) $\frac{dy}{dt} = t + t^4 y$, $y(0) = 3$.

d) $\frac{dy}{dx} = x + y^2$, $y(0) = 0$.

e) $\frac{dy}{dx} = x + y$, $y(0) = 1$.

Keywords: Approximate analytical solutions, Initial value problems, Picard's Method,

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Lesson 26

Taylor Series Method

26.1 Taylor Series Method

Let $y = y(t)$ be a continuously differentiable function in the interval $[t_0, b]$. Expanding $y(t)$ around $t = t_0$ in Taylor Series, we obtain

$$y(t) = y(t_0) + \frac{(t-t_0)}{1!} y'(t_0) + \frac{(t-t_0)^2}{2!} y''(t_0) + \frac{(t-t_0)^3}{3!} y'''(t_0) + \dots \quad (26.1)$$

Taking $t = t_1$ and $t_1 - t_0 = h$, a small increment, i.e., $t_0 + h = t_1, t_1 \in [t_0, b]$ we get

$$y(t_1) = y(t_0) + \frac{h}{1!} y'(t_0) + \frac{h^2}{2!} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \dots \quad (26.2)$$

Now consider the I.V.P. $\frac{dy}{dt} = f(t, y), t \in [t_0, b]$ (26.4)

subject to $y(t_0) = y_0$ (26.5)

The Taylor Series solution of the given I.V.P. (3)-(4) is to find an approximate function $y(t)$ as given in (1) which involves the derivatives of the unknown function $y(t)$ at the initial point $t = t_0$, that satisfies the I.V.P. (3)-(4).

Given $\frac{dy}{dt} = f(t, y)$.

At $t = t_0$; $\frac{dy}{dt}(t = t_0) = f(t_0, y(t_0))$

$= f(t_0, y_0)$

$= f_0$ (say)

$$y''(t) = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(f(t,y)\right) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\left(f(t,y)\right).$$

$$y''(t=t_0) = \frac{\partial f}{\partial t}(t=t_0, y(t_0)) + \frac{\partial f}{\partial y}(t=t_0, y(t_0)) \cdot f\left(t=t_0, y(t_0)\right).$$

In the same manner, we calculate the higher order derivatives of $y(t)$ depending on the necessity. Before we discuss the order of approximation of the Taylor Series method and error associated with a particular order method, let us see its utility for finding the solution of the given I.V.P. through a few examples.

Example 1: Find $y(0.1)$ from the I.V.P. $\frac{dy}{dt} = 3t + y^2$ subject to $y(0) = 1$.

Solution:

Given $y' = 3t + y^2$

$$\Rightarrow y'(0) = 0 + \left[y(0)\right]^2 = 1$$

$$y'' = 3 + 2y \cdot y' \Rightarrow y''(0) = 3 + 2y(0) \cdot y'(0)$$

$$= 3 + 2 \cdot 1 \cdot 1 = 5$$

$$y''' = 2y \cdot y'' + 2y'^2 \Rightarrow y'''(0) = 2y(0) \cdot y''(0) + 2\left[y'(0)\right]^2$$

$$= 2 \cdot 1 \cdot 5 + 2 \cdot 1 = 12$$

$y^{iv} = 2y' \cdot y'' + 2y \cdot y''' + 4y' \cdot y'' = 2 \cdot 1 \cdot 5 + 2 \cdot 1 \cdot 12 + 4 \cdot 1 \cdot 5 = 54$ etc.

Now the Taylor Series solution is written as

$$y(t) = y(t_0) + (t - t_0)\,y'(t_0) + \frac{(t-t_0)^2}{2!}y''(t_0) + \frac{(t-t_0)^3}{3!}y'''(t_0) + \frac{(t-t_0)^4}{4!}y^{iv}(t_0) + \ldots$$

Taking $t = 0.1, t_0 = 0$ (given),

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2!}y''(0) + \frac{(0.1)^3}{3!}y'''(0) + \frac{(0.1)^4}{4!}y^{iv}(0) + \ldots$$

$$= 1 + 0.1 + \frac{5}{2}(0.1)^2 + \frac{12}{3!}(0.1)^3 + \frac{54}{4!}(0.1)^4 + \ldots$$

$\therefore y(0) = 1.12722$.

Example 2: Obtain the Taylor Series solution for the I.V.P. given by $\frac{dy}{dt} = 2y + 3e^t$, $y(0) = 0$. Compare the solution at $t = 0.2$ with the exact solution given by $y(t) = 3\left(e^{2t} - e^t\right)$.

Solution:

Given $y' = 2y + 3e^t \Rightarrow y'(0) = 3$

$y'' = 2y' + 3e^t \Rightarrow y''(0) = 9$

$y''' = 2y'' + 3e^t \Rightarrow y'''(0) = 21$

$y^{iv} = 2y''' + 3e^t \Rightarrow y^{iv}(0) = 45$

The solution is $y(t) = 3t + \frac{9}{2}t^2 + \frac{21}{6}t^3 + \frac{45}{24}t^4 + \ldots$.

At $t = 0.2$, $y(0.2) = 0.811$.

From the exact solution, $y(0.2) = 0.8112$.

So the error in the numerical solution is $|0.8112 - 0.811| = 0.001$.

26.2 Local Truncation Error and Order of the Method

The Taylor series expansion of $y(t)$ about any point t_j is written as

$$y(t) = y(t_9) + (t - t_9)\, y'(t_9) + \frac{(t - t_9)^2}{2!} y''(t_9) + \ldots$$

$$\ldots + \frac{1}{p!}(t - t_0)^p\, y^{(p)}(t_9) + \frac{1}{(p+1)!}(t - t_9)^{(p+1)}\, y^{(p+1)}(t_9 + \theta h) \qquad (26.6)$$

where $y^{(p)}(t)$ is the p^{th} of the $y(t)$ and $0 < \theta < 1$, a real number. The last term in the expansion is called the remainder term. In general, the local truncation error at any location $t = t_{j+1}$ of the method is given by

$$T_{j+1} = \frac{1}{(p+1)!} h^{(p+1)} y^{(p+1)}(t_j + \theta h)$$

where $h = t_{j+1} - t_j$.

The order of a method is the largest integer p for which $\left|\frac{1}{h} T_{j+1}\right| = O(h^p)$.

The notation $O(h^p)$ denoting that all terms of the order p onwards are grouped to a single term representation.

The method given by equation (26.1) is called the Taylor Series method of order p.

Example 3: Determine the first three non-zero terms in the Taylor Series for $y(t)$ from the I.V.P. $y' = t^2 + y^2$, $y(0) = 0$.

Solution:

$y'(0) = 0;$

$y'' = 2t + 2yy' \Rightarrow y''(0) = 0$

$y''' = 2 + 2\left(y'\right)^2 + 2yy'' \Rightarrow y'''(0) = 2$

$y^{iv} = 6y'y'' + 2yy''' \Rightarrow y^{iv}(0) = 0$

$y^{v} = 2yy^{v} + 8y'y''' + 6\left(y''\right)^2 \Rightarrow y^{v}(0) = 0$

$y^{vi} = 2yy^{v} + 10y'y^{iv} + 20y''y''' \Rightarrow y^{vi}(0) = 0$

Similarly $y^{vii}(0) = 80$, $y^{viii}(0) = 0 = y^{ix}(0) = y^{x}(0)$, $y^{xi}(0) = 38400$.

Thus the three term Taylor Series solution for the given I.V.P. is

$$y(t) = \frac{1}{3}t^3 + \frac{1}{63}t^7 + \frac{2}{2079}t^{11}.$$

Example 4: Given the I.V.P.: $y' = 2t + 3y$; $y(0) = 1$ whose exact solution is

$$y(t) = \frac{11}{9}e^{3t} - \frac{2}{9}\left(3t+1\right).$$

a) Use 2nd order Taylor Series method to get $y(0.2)$ with step length $h = 0.1$ (note $h = t_{j+1} - t_j$).

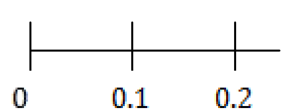
b) Find 't', if the error in $y(t)$ obtained from the first four terms of the Taylor series, is to be less than 5×10^{-5}, after rounding.

c) Determine the number of terms in the Taylor Series required to obtain the result correct to 5×10^{-6} for $t \le 0.4$.

Solution:

a) The second order Taylor Series method is given by $y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + O\left(h^3\right)$.

$n = 0,1; h = 0.1$.



We have $y' = 2t + 3y$

or, $y'(t_n) = y'_n = 2t_n + 3y_n$

or, $y''(t_n) = y''_n = 2 + 3y'_n = 2 + 6t_n + 9y_n$.

Take $n = 0; h = 0.1; y_0 = 1; y'_0 = 3; y''_0 = 11$.

$\therefore\ y(0.1) = 1 + (0.1)\cdot 3 + \dfrac{(0.1)^2}{2}\cdot 11 = 1.355$.

Taking $n = 1; h = 0.1; y_1 = 1.355; y'_1 = 4.265; y''_1 = 14.795$

$\Rightarrow y(0.2) = 1.355 + (0.1)\cdot(4.265) + \dfrac{(0.1)^2}{2!}\cdot(14.795)$

$= 1.8555$.

b) Given $y(0) = 1, y'(0) = 3$.

Compute $y''(0) = 11, y'''(0) = 33$, $y^{iv}(0) = 99$.

So the Four term Taylor Series solution is

$y(t) = 1 + 3t + \dfrac{11}{2}t^2 + \dfrac{11}{2}t^3$.

The remainder term is given by

$R_4 = \dfrac{t^4}{4!}y^{iv}(\xi)$ gives the error in the approximation.

We require $|R_4| \le 5\times 10^{-5}$.

Given the exact solution, use it to find $y^{iv}(\xi)$;

$y(t) = \dfrac{11}{9}e^{3t} - \dfrac{2}{9}(3t+1)$.

$y^{iv}(\xi) = 99e^{3t}$.

$\therefore |R_4| = \left|\dfrac{99}{24}e^{3t}t^4\right| < 5\times 10^{-5}$.

Simplifying and solving this non-linear algebraic equation for finding t, we get

$t^4e^{3t} < 0.000012$

$\Rightarrow t < 0.056$

Note: In the event, if the exact solution is not known, write one more non-vanishing term in the Taylor Series than is required and then differentiate this series p times: here p is 4.

Now to determine the number of terms required in the Taylor series solution to obtain the solution correct to 5×10^{-4} for $t\le 0.4$, we estimate it as:

$$\max_{0\le t\le 0.4}\left|\frac{t^p}{p!}\right|\cdot \max_{\xi t[0,0.4]}\left|y^{(p)}(\xi)\right|\le 5\times10^{-6}$$

Again using the analytical solution, we find the p^{th} derivative of $y(t)$ at $t=\xi$ as:

$$y^p(\xi)=(11)3^{p-2}\cdot e^{1.2}$$

or $\dfrac{(0.4)^p}{p!}(11)3^{p-2}\cdot e^{1.2}\le 5\times10^{-6}$

Solving this non-linear algebraic equation using the Newton-Raphson method or otherwise, we get a lower bound for p as

$$p\ge 10$$

This indicates that a minimum of 10^{th} order Taylor Series method gives the solution which will be accurate upto the 6^{th} decimal place for all values of $t\in[0,0.4]$.

Exercises:

1. Compute an approximation to $y(0.1)$ up to five decimal places from

   $\dfrac{dy}{dt}=t^2y-1, y(0)=1$.

2. Solve $y'=y^2+t$, $y(0)=1$ using Taylor Series method and compute $y(0.2)$, $h=0.1$.

3. Evaluate $y(0.1)$ correct to six decimal places $\left(5\times10^{-6}\right)$ by Taylor Series method if

   $y(t)$ satisfies $\dfrac{dy}{dt}=1+yt, y(0)=1$.

Keywords: Taylor Series method, Taylor series solution.

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Lesson 27

Single Step Methods

27.1 Introduction

Picard's and Taylor Series methods give solution of the given I.V.P. in the form of a power series. We now describe some numerical methods which give the solution in the form of a discrete data at equally spaced points of the interval. The discretization of the interval is considered as described in the lesson 1 and 2 is not repeated here.

Single Step methods:

Consider the I.V.P. $$\left.\begin{array}{l}\dfrac{dy}{dt}=f(t,y), t\in[t_0,b]\\ \text{subject to } y(t_0)=y_0\end{array}\right\} \qquad (27.1)$$

Consider the partition of the interval as

$$t_0<t_1<t_2<\ldots<t_{j-1}<t_j<t_{j+1}<\ldots<t_N=b \qquad (27.2)$$

such that $t_{j+1}-t_j=h; \forall j=0,1,2,\ldots,N-1$ where h is a constant is the step size. Also denote $y_j=y(t_j)$ and $y_{j+1}=y(t_{j+1})$. $t_0,t_1,\ldots,t_N$ are called the nodal points. A general single step method may be written as

$$y_{j+1}=y_j+h\varphi\left(t_{j+1},t_j,y_{j+1},y_j,h\right) \qquad (27.3)$$

Where φ is $\varphi(t,y,h)$ is called the increment function. Note that in (3), we see the dependence of the unknown function on the two nodes t_j and t_{j+1}. In single step methods, the solution at only the previous point.

Also, if y_{j+1} can be obtained by evaluating the right hand side of (27.3), then the method is called an explicit method.

Then equation (27.3) is written as $y_{j+1} = y_j + h\varphi(t_j, y_j, h)$ (27.4)

The method is called an implicit method if increment function φ depends on y_{j+1} also, as seen in equation (27.3). In such a situation, we cannot get the solution y_{j+1} explicitly, we need to solve the equation (3) to get the solution.

27.2 The Local Truncation Error (L.T.E)

Denote $y(t_j)$ as the exact solution and y_j as the numerical solution of I.V.P. (27.1) at $t = t_j$. The exact solution $y(t_j)$ satisfies the equation

$$y(t_{j+1}) = y(t_j) + h\varphi\left(t_{j+1}, t_j, y\left(t_{j+1}\right), y\left(t_j\right), h\right) + T_{j+1} \quad (27.5)$$

where T_{j+1} is called the L.T.E of the method.

Thus the L.T.E $T_{j+1} = y\left(t_{j+1}\right) - y\left(t_j\right) - h\varphi\left(t_{j+1}, t_j, y\left(t_{j+1}\right), y\left(t_j\right), h\right)$ (27.6)

By definition, the order of the single step method is $\left|\frac{1}{h}T_{j+1}\right|$ (27.7)

27.3 Forward Euler Method for the I.V.P. $\frac{dy}{dt} = f(x, y)$; $y(t_0) = y_0$:

Let t_j be any point in the interval $[t_0, b]$. Then the slope of $y(t)$ at $t = t_j$ is given by

$$\left.\frac{dy}{dt}\right|_{t=t_j} = f(t_j, y_j) \quad (27.8)$$

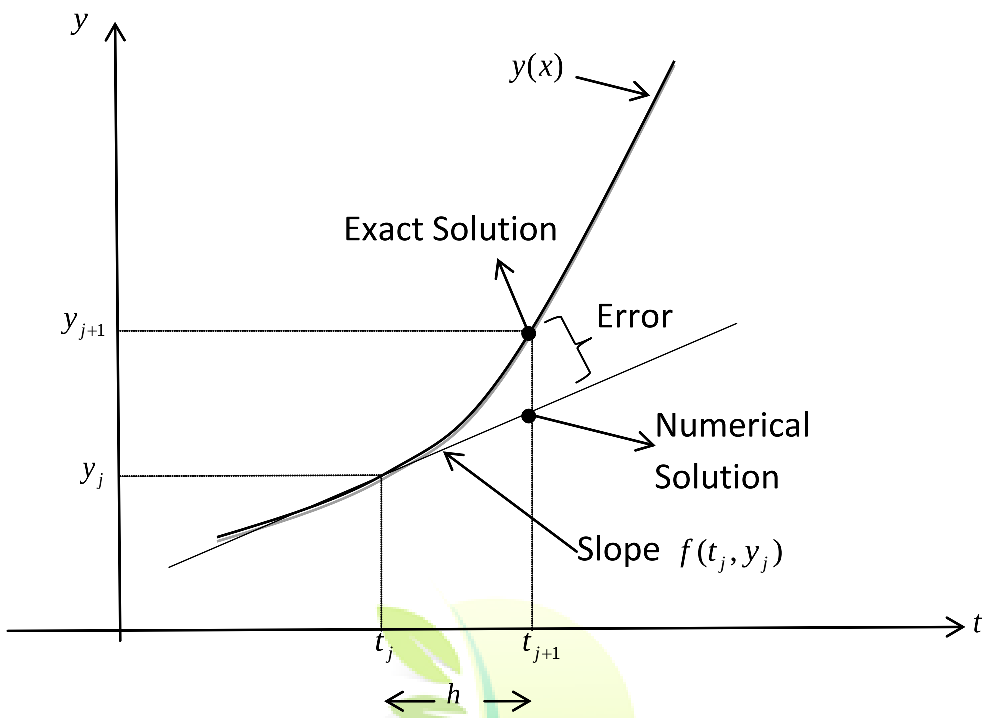
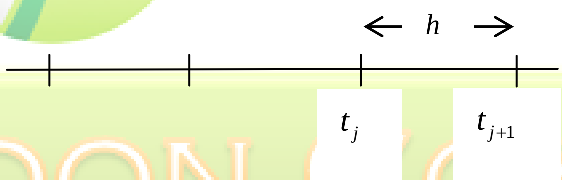


Fig.1: Explicit Euler method



Replacing $\frac{dy}{dt}$ at $t = t_j$ by the first order forward difference at t_j, we get

$\frac{y_{j+1} - y_j}{h} = f(t_j, y_j)$ or $y_{j+1} = y_j + h \cdot f(t_j, y_j),\ j = 0, 1, 2, \ldots, N-1$ at every point of t_j as given in (2). By this, we mean

$y_1 = y_0 + h \cdot f(t_0, y_0)$,

$y_2 = y_1 + h \cdot f(t_1, y_1)$,

… … …

$y_N = y_{N-1} + h \cdot f(t_{N-1}, y_{N-1})$.

For a chosen h and/with the initial condition y_0, one can find the solution explicitly at all the nodal points of the given interval. The local truncation error in the method is given by

$$T_{j+1} = y\left(t_{j+1}\right) - y_{j+1}$$

$$= y\left(t_{j+1}\right) - \left[y\left(t_j\right) + h \cdot f(t_j, y_j)\right]$$

Now expanding $y\left(t_{j+1}\right)$ in the Taylor Series about $t = t_j$ and simplifying, we get

$$T_{j+1} = \frac{h^2}{2} y''(\xi)$$

where $t_j < \xi < t_{j+1}$. Let the maximum value of $y''(t)$ in $[t_0, b]$ be M^*, then

$$\max_{[t_0,b]} \left|T_{j+1}\right| = T \text{ (say) } = \frac{h^2}{2} \max_{[t_0,b]} \left|y''(\xi)\right|$$

$$= \frac{h^2}{2} M^*.$$

Thus $T \le \frac{h^2}{2} M^*$

i.e., the L.T.E is of $O(h^2)$ and by definition, the order of this forward Euler method is one since $\left|\frac{1}{h} T_{j+1}\right| = \left|\frac{1}{h}\frac{h^2}{2} M^*\right| = O(h^1)$.

Example 1: Use Forward Euler method to solve $y' = -y$ with the initial condition $y(0) = 1$ in $[0, 0.04]$ by taking $h = 0.01$.

Solution:

Take $t_0 = 0, t_1 = 0.01, t_2 = 0.02, t_3 = 0.03, t_4 = 0.04$; $f(t, y) = -y$.

Now $y(t_1) = y(t_0) + h \cdot f(t_0, y_0)$

$= 1 + (0.1)(-1) \quad = 0.99$,

$y(t_2) = y(t_1) + h \cdot f(t_1, y_1)$

$= 0.99 + (0.01)(-0.99) \ = 0.9801$,

$y(t_3) = y(t_2) + h \cdot f(t_2, y_2)$

$= 0.9801 + (0.01)(-0.9801) \ = 0.9703$,

$y(t_4) = 0.9703 + (0.01)(-0.9703)$

$= 0.9606$.

We know the exact solution $y' = -y, y(0) = 1$ is $y(t) = e^{-t}$ and gives $y(0.4) = 0.9608$. The error in the Euler method for the solution at $t = 0.4$ is given as $|0.9606 - 0.9608| = 0.0002$.

27.4 Backward Euler Method

We can also replace the slope of $y(t)$ at $t = t_j$ by the first order backward difference approximation which is given by

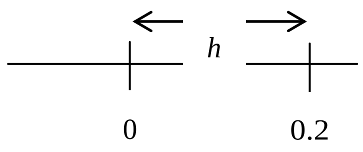
$$\frac{y_j - y_{j-1}}{h} = f(t_j, y_j)$$

or $y_j = y_{j-1} + h \cdot f(t_j, y_j)$

or equivalently, $y_{j+1} = y_j + h \cdot f(t_{j+1}, y_{j+1})$, $j = 0, 1, 2, ..., N-1$.

Evidently, this is an implicit method. It can be shown (left as an exercise!) that the L.T.E. of this method is $-\frac{h^2}{2} y''(\xi), t_j < \xi < t_{j+1}$ and the order of the method is also one. Let us now illustrate this method with an example.

Example 2: Solve the I.V.P. $y' = -2ty^2, y(0) = 1$ in $[0, 0.2]$ with $h = 0.2$.



Solution:

Backward Euler method is

$y_{j+1} = y_j + h \cdot f(t_{j+1}, y_{j+1})$.

Here $t_0 = 0, t_1 = 0.2, h = 0.2, f = -2ty^2$ for $j = 0$.

$y_1 = y_0 + h \cdot f(t_1, y_1)$

$\Rightarrow y(0.2) = y(0) + h\left[-2 \cdot t_1 \cdot y(0.2)^2\right]$

or $y_1 = 1 - 2(0.2)(0.2)y_1^{\,2}$

can be written as a quadratic in y_1 as

$0.08y_1^{\,2} + y_1 - 1 = 0$

whose solution is $y_1 = 0.9307$.

Exercises:

1. Continue this to compute the solution at $t = 0.4$ if the above I.V.P. is solved in the interval $[0, 0.4]$ with $h = 0.2$.

2. Solve the I.V.P. $y' = t + y^2, y(0) = 1, h = 0.1, [0, 0.4]$ using the forward Euler method.

3. Solve the I.V.P. $y' = -2ty^2, 0 \le t \le 0.5, h = 0.1, y(0) = 1$ using the forward Euler method.

Keywords: Backward Euler method, Local truncation error, Forward Euler method.

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Lesson 28

Modified Euler Method

28.1 Introduction

The first order Explicit Euler method is improved to achieve better accuracy in the numerical solution for an initial value problem $y' = f(t,y), y(t_0) = y_0$.

In the modified Euler method, the slope of $y(t)$ at $t = t_j$ is approximated by the average of the slopes at $t = t_j$ and $t = t_{j+1}$.

i.e., $y_{j+1} = y_j + h \cdot y'_j$

$$= y_j + \frac{h}{2} \cdot \left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}\right) \right]$$

or $y_{j+1} = y_j + \frac{h}{2} \cdot \left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}\right) \right]$ (28.1)

This implicit method is used by setting an iterative procedure as follows:

$$y_{j+1}^{(s+1)} = y_j + \frac{h}{2} \cdot \left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}^{(s)}\right) \right], \; s = 0,1,2,... \qquad (28.2)$$

Now the initial approximation for $y_{j+1}^{(0)}$ is considered as the solution y_{j+1} of Euler method. The above iteration process is terminated at each step if the condition

$\left| y_j^{(s+1)} - y_j^{(s)} \right| < \varepsilon$,is satisfied,

where ε is a reassigned error tolerance.

Exercise: Show that the modified Euler method has L.T.E. as $O(h^3)$ and the order of the method is $O(h^2)$.

The exponent of h in $O(h^p)$ is the order of accuracy of the method. It is a measure of accuracy of any numerical scheme. It gives an indication of how rapidly the accuracy can be improved with refinement of the grid spacing h in any given interval. For example, in a first order method such as

$y_{j+1} = y_j + h \cdot f\left(t_j, y_j\right) + O(h)$

If we reduce the mesh size h by $\frac{h}{2}$, the error is reduced by approximately a factor $\frac{1}{2}$. Similarly in a second order method such as

$y_{j+1} = y_j + \frac{h}{2} \cdot \left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}\right) \right] + O(h^2)$.

If we refine the mesh size by a factor of 2, we expect the error to reduce by a factor 2^2 i.e., 4, which gives a rapid decrease in the error. Thus higher order numerical schemes are preferred.

Example 1: Determine the value of $y(0.1)$ from the I.V.P. $y' = y + t^2$, $y(0) = 1, h = 0.05$.

Solution:

$t_0 = 0, t_1 = t_0 + h = 0.05, t_2 = t_0 + 2h = t_1 + h = 0.1$, $y_0 = 0$, $f(t, y) = t^2 + y$; $y_1 = y(0.05)$,

$y_2 = y(0.1)$.

$y_1^{(s+1)} = y_0 + \frac{h}{2} \cdot \left[f\left(t_0, y_0\right) + f\left(t_1, y_1^{(s)}\right) \right]$, $s = 0, 1, 2, \ldots$

Compute $y_1^{(0)}$ using the Euler method.

$y_1^{(0)} = y_0 + \frac{h}{2} f\left(t_0, y_0\right)$

$= 1 + (0.05)(1.0)$

$= 1.05$.

Use modified Euler method now as follows:

Compute $y(0.05)$:

Take $s=0$: calculate $f\left(t_1, y_1^{(0)}\right)=1.0262$

$$y_1^{(1)} = y_0 + \frac{h}{2}\cdot\left[f\left(t_0, y_0\right) + f\left(t_1, y_1^{(0)}\right)\right]$$

$=1.0513$

Take $s=1$; $y_1^{(2)}$ is computed as:

$$y_1^{(2)} = y_0 + \frac{h}{2}\cdot\left[f\left(t_0, y_0\right) + f\left(t_1, y_1^{(1)}\right)\right]$$

$=1.0513$

Note: $\left|y_1^{(2)} - y_1^{(1)}\right| = 0$, so we can stop the iteration process and conclude that

$y(0.05)=1.0513$.

Compute $y(1.0): t_1 = 0.05, y_1 = 1.0513, h = 0.05$

Euler method gives $y_2^{(0)} = y_1 + hf\left(t_1, y_1\right)$

$\Rightarrow y_2^{(0)} = 1.104$.

Now use the modified Euler method:

$$s=0 \Rightarrow y_2^{(1)} = y_1 + \frac{h}{2}\cdot\left[f\left(t_1, y_1\right) + f\left(t_2, y_2^{(0)}\right)\right]$$

$=1.1055$

$$s=1 \Rightarrow y_2^{(2)} = y_1 + \frac{h}{2}\cdot\left[f\left(t_1, y_1\right) + f\left(t_2, y_2^{(1)}\right)\right]$$

$=1.1055$

Take the solution $y(1.0)=1.1055$.

Example 2: Given the I.V.P. $\frac{dy}{dt} = 2ty, y(1) = 1$ find $y(1.4)$ using the modified Euler method by taking $h=0.1$. Compare this solution with the exact solution $y(t)=e^{t^2-1}$. Calculate the percentage relative error.

Solution:

We have the data: $t_0 = 1, t_1 = 1.1, t_2 = 1.2, t_3 = 1.3, t_4 = 1.4, y_0 = 1, h = 0.1$.

Modified Euler method is:

$$y_{j+1} = y_j + \frac{h}{2}\cdot\left[f\left(t_j, y_j\right) + f\left(t_{j+1}, y_{j+1}\right)\right] \text{for } j = 0,1,2,3.$$

Take $j = 0$: We have the iterative method is written as

$$y_1^{(s+1)} = y_0 + \frac{h}{2}\cdot\left[f\left(t_0, y_0\right) + f\left(t_1, y_1^{(s)}\right)\right],\ s = 0,1,2,...$$

Compute $y_1^{(0)}$ using the Euler method:

$$y_1^{(0)} = y_0 + (0.1)\cdot 2\cdot(0.1)\cdot 1 = 1.2.$$

Now $y_1^{(1)} = y_0 + \frac{(0.1)}{2}\left[2t_0y_0 + 2t_1y_1^{(0)}\right] = 1.232$,

$$y_1^{(2)} = y_0 + \frac{(0.1)}{2}\left[2t_0y_0 + 2t_1y_1^{(1)}\right] = 1.232.$$

$j = 1$ to $j = 3$ are calculated (left as an exercise) and are tabulated below. The absolute error and percentage relative errors are calculated as:

$Absolute\ error = |Exact\ value - Numerical\ Value|$ and

$ntage\ relative\ error = \frac{|error|}{Exact\ value}$.

Table 28.1

j	t_n	y_n	Exact value $y = e^{t_n^2-1}$	Absolute error	Percentage relative error
0	1	1	1	0	0
1	1.1	1.232	1.2337	0.0017	0.14
2	1.2	1.5479	1.5527	0.0048	0.31
3	1.3	1.9832	1.9937	0.0106	0.53
4	1.4	1.5908	2.6117	0.0209	0.80

Exercises:

1. Solve the I.V.Ps. using modified Euler method

(i) $\frac{dy}{dt} = -2y, y(0) = 1, h = 0.2$, $t \in [0, 0.6]$.

(ii) $\frac{dy}{dt} = \frac{y-t}{y+t}, y(0) = 1, h = 0.1, t \in [0, 0.2]$.

(iii) $\frac{dy}{dt} = 2 + \sqrt{|ty|}, y(1) = 1, h = 0.5, t \in [1, 2]$.

(iv) $y' = t(1+t^3 y), y(0) = 3, h = 0.1, t \in [0, 0.4]$.

Keywords: Absolute error, Explicit Euler method, Percentage relative errors, Modified Euler method,

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Lesson 29

Runge-Kutta Methods

29.1 Introduction

In single step explicit method, the approximate solution y_{j+1} is computed from the known solution at the point (t_j, y_j) using

$$y_{j+1} = y_j + h \cdot f(t_j, y_j) \tag{29.1}$$

or $y_{j+1} = y_j + h$ (slope of $y(t)$ at $t = t_j$) (29.2)

In equation (1), we used the slope at $t = t_j$ only. Similarly, in the modified Euler method

$$y_{j+1} = y_j + h\left[f(t_j, y_j) + f(t_{j+1}, y_{j+1})\right] \tag{29.3}$$

The slope is replaced by the average of slopes at the end points (t_j, y_j) and (t_{j+1}, y_{j+1}).

29.2 Runge-Kutta Methods

Runge-Kutta methods use a weighted average of slopes on the given interval $[t_j, t_{j+1}]$, instead of a single slope. Thus the general Runge-Kutta method may be defined as

$y_{j+1} = y_j + h$ [Weighted average of slopes at n points on the given interval] (29.4)

This way one can derive N-explicit methods by taking $n = 1, 2, \ldots, N$. Also, n in (29.4) indicates the order of this Runge-Kutta method. The general N^{th} order

Runge-Kutta method is written as

$$y_{j+1} = y_j + \left(w_1 k_1 + w_2 k_2 + \ldots + w_N k_N\right)$$

where w_i are the weights and each k_i is defined as

$$k_1 = h \cdot f\left(t_j, y_j\right)$$

$$k_2 = h \cdot f\left(t_j + c_2 h, y_j + a_{21} k_1\right)$$

$$k_3 = h \cdot f\left(t_j + c_3 h, y_j + a_{31} k_1 + a_{32} k_2\right)$$

$$k_N = h \cdot f\left(t_j + c_N h, y_j + a_{N1} k_1 + a_{N2} k_2 + \ldots + a_{N,N-1} k_{N-1}\right) \qquad (29.5)$$

All w_i's , c_i's, a_i's are the parameters which are determined by forcing the method (29.4) to be of N^{th}order. Deriving a general N^{th} order method is out of purview of this material, but we demonstrate the derivation of the 2^{nd} order Runge-Kutta method below.

29.3 Second Order Runge-Kutta Method

Consider the general form of the (2^{nd} order) Runge-Kutta method with 2 slopes,

$$y_{j+1} = y_j + w_1 k_1 + w_2 k_2 \qquad (29.6)$$

where $k_1 = h \cdot f\left(t_j, y_j\right)$

$$k_2 = h \cdot f\left(t_j + c_2 h, y_j + a_{21} k_1\right)$$

The parameters c_2, a_{21}, w_1, w_2 are chosen to make y_{j+1} closer to the exact solution $y(t_{j+1})$ upto the 2^{nd} order.

Now writing $y(t_{j+1})$ in Taylor series about $t = t_j$,

$$y(t_{j+1}) = y(t_j) + h \cdot y'(t_j) + \frac{h^2}{2!} \cdot y''(t_j) + \ldots$$

$$= y(t_j) + h \cdot f\left(t_j, y_j\right) + \frac{h^2}{2!} \cdot \left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right)\bigg|_{t=t_j} + \ldots \qquad (29.7)$$

Also, $k_1 = h \cdot f_j$

$$k_2 = h \cdot f\left(t_j + c_2 h, y_j + a_{21} h \cdot f_j\right)$$

Expanding f about $\left(t_j, y_j\right)$, we get

$$k_2 = h\left[f\left(t_j\right) + h\left(c_2\frac{\partial f}{\partial t} + a_{21} f\frac{\partial f}{\partial y}\right)\bigg|_{t=t_j} + \frac{h^2}{2!} \cdot \left(c_2^2\frac{\partial^2 f}{\partial t^2} + 2c_2 a_{21} f\frac{\partial^2 f}{\partial y \partial t} + a_{21} f\frac{\partial^2 f}{\partial y^2}\right)\bigg|_{t=t_j} + \ldots\right]$$

Substituting the expressions for k_1 and k_2 in (1),we get

$$y_{j+1} = y_j + \left(w_1 + w_2\right) h \cdot f_j + h^2\left(w_2 c_2\frac{\partial f}{\partial t} + w_2 a_{21} f\frac{\partial f}{\partial y}\right)\bigg|_{t=t_j} + \ldots \qquad (29.8)$$

Comparing the coefficients of h and h^2 in (27.7) and (27.8) we obtain

$$w_1 + w_2 = 1;\ w_2 c_2 = \frac{1}{2}; w_2 a_{21} = \frac{1}{2}.$$

The solution of this may be written as

$$a_{21} = c_2; w_2 = \frac{1}{2c_2}; w_1 = 1 - \frac{1}{2c_2},$$

c_2 is arbitrary non-zero constant.

With the choice of $c_2 = \frac{1}{2}$, we derive the 2nd order Runge-Kutta method.

$a_{21} = \frac{1}{2}; w_2 = 1; w_1 = 0$

we get $k_1 = h \cdot f\left(t_j, y_j\right)$

$k_2 = h \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{1}{2}k_1\right)$

And $y_{j+1} = y_j + k_2$ (29.9)

With the choice of $c_2 = 1$, we get

$w_2 = \frac{1}{2}, w_1 = \frac{1}{2}$ and $a_{21} = 1$

and $y_{j+1} = y_j + \frac{1}{2}(k_1 + k_2)$

where $k_1 = h \cdot f\left(t_j, y_j\right)$

$k_2 = h \cdot f\left(t_j + h, y_j + k_1\right)$.

This method is also a second order method which is known as the Euler-Cauchy method. Clearly, with different choices for c_2 , we get a different second order Runge-Kutta method. Let us demonstrate its utility for solving the initial value problems.

Example1: Compute $y(0.4)$ from the I.V.P. $y' = -2ty^2, y(0) = 1, h = 0.2$

Solution:

$y_{j+1} = y_j + k_2$

where $k_1 = h \cdot f\left(t_j, y_j\right) = (0.2)\left[-2t_j y_j^{\ 2}\right] = -0.4t_j y_j^{\ 2}$

$k_2 = h \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{1}{2}k_1\right)$

$$= -0.4\left(t_j + 0.1\right)\left(y_j + \frac{1}{2}k_1\right)^2$$

Taking $j = 0$; given that $t_0 = 0, y_0 = 1$,

$\Rightarrow k_1 = 0, k_2 = -0.04$.

$\therefore y(0.2) = y(0) + k_2 = 1 - 0.04 = 0.96$.

For $j = 1$, $t_1 = 0.2, y_1 = 0.96$

$\Rightarrow k_1 = -0.073728, k_2 = -0.10226$

and $y(0.4) = y(0.2) + k_2 = 0.96 - 0.10226 = 0.85774$

Keywords: Weighted average of slopes,

References

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Lesson 30

4^{th} Order Runge-Kutta Method

30.1 3^{rd} Order Runge-Kutta Method

The third order Runge-Kutta method is given by

$$y_{j+1} = y_j + \frac{1}{8}(2k_1 + 3k_2 + 3k_3) \qquad (30.1)$$

where $k_1 = h \cdot f(t_j, y_j)$

$$k_2 = h \cdot f\left(t_j + 2\frac{h}{3}, y_j + \frac{2}{3}k_1\right)$$

and $k_3 = h \cdot f\left(t_j + 2\frac{h}{3}, y_j + \frac{2}{3}k_2\right)$.

Derivation of this method involves evaluation of eight unknowns in eight non-linear algebraic equations, which is very tedious. Similarly the 4^{th} order Runge-Kutta method is also. The interested is referred to a standard test book on Numerical analysis for the detailed derivation. The Fourth order Runge-Kutta method is given as:

$$y_{j+1} = y_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (30.2)$$

where $k_1 = h \cdot f(t_j, y_j)$

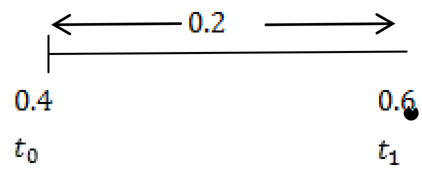
$$k_2 = h \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{k_1}{2}\right)$$

$$k_3 = h \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{k_2}{2}\right)$$

and $k_4 = h \cdot f(t_j + h, y_j + k_3)$

This method is also known as the Classical Runge-Kutta method. The 4^{th}order R-T method is an efficient method which can be used very easily. Let us now illustrate its use for finding the solution of given I.V.P.

Example 1: Use the Classical Runge-Kutta method to find the numerical solution at $t = 0.6$ for $\frac{dy}{dt} = \sqrt{t+y}, y(0.4) = 0.41, h = 0.2$.



Solution:

Given $t_0 = 0.4, y_0 = 0.41;\ f(t,y) = \sqrt{t+y}$.

First let us evaluate k_i's.

$$k_1 = h \cdot f(t_0, y_0)$$

$$= (0.2)[0.4 + 0.41]^{\frac{1}{2}} = 0.18$$

$$k_2 = h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.2)[(0.4+0.1)+(0.41+0.09)]^{\frac{1}{2}} = 0.2$$

$$k_3 = h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.2)[(0.4+0.1)+(0.41+0.01)]^{\frac{1}{2}} = 0.20099$$

$$k_4 = h \cdot f(t_0 + h, y_0 + k_3)$$

$$= (0.2)[0.6 + (0.41 + 0.20099)]^{\frac{1}{2}}$$

$$= 0.22009.$$

Now $y(0.6) = y(0.4) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 0.41 + 0.20035$$

$= 0.61035$.

Example 2: Find $y(0.1)$ form $\frac{dy}{dt} = y - t, y(0) = 2$ by taking $h = 0.1$.

Solution:

Given $f(t, y) = y - t, t_0 = 0, y_0 = 2, h = 0.1$

$k_1 = h \cdot f(t_0, y_0)$

$= (0.1)[2 - 0] = 0.2$

$k_2 = h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= (0.1)[2.1 - 0.05] = 0.205$

$k_3 = h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.1)[2.1025 - 0.05] = 0.20525$

$k_4 = h \cdot f(t_0 + h, y_0 + k_3)$

$= (0.1)[2.20525 - 0.1] = 0.21053$

Hence $y(0.1) = y(0) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 2 + 0.2056$

$\therefore y(0.1) = 2.2056$.

Example 3: Given $\frac{dy}{dt} = 1 + y^2, y(0.2) = 0.2027, h = 0.2$ compute $y(0.4)$ using the 4^{th} order R-K method.

Solution:

Given $f(t, y) = 1 + y^2, t_0 = 0.2, y_0 = 0.2027, h = 0.2$

Evaluating k_i's, we get

$k_1 = 0.2082, k_2 = 0.2188,$

$k_3 = 0.2195, k_4 = 0.2356$.

and hence $y(0.4) = y(0.2) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.4228$.

Exercises:

1. Use 3rd order Runge-Kutta method to find

   a) $y(0.1)$ given $\frac{dy}{dt} = 3e^t + 2y, y(0) = 0, h = 0.1$.

   b) $y(0.8)$ given $\frac{dy}{dt} = \sqrt{t+y}, y(0.4) = 0.41, h = 0.2$.

   c) $y(0.2)$ given $\frac{dy}{dt} = \frac{y-t}{y+t}, y(0) = 1, h = 0.1$.

   d) $y(0.2)$ given $\frac{dy}{dt} = 3t + \frac{1}{2}y, y(0) = 1, h = 0.1$.

2. Use 4thorder Runge-Kutta method to solve the problems 1(a)-1(d) and make a comparison table.

3. Solve the non-linear I.V.P.

$\frac{dy}{dt} = \frac{y^2 - 2t}{y^2 + 2t}$ subjected to $y(0) = 1$ in the interval $[0,1]$ by taking $h = 0.2$.

4. Use the Classical Runge-Kutta method to find $y(1.4)$ insteps of 0.2 given that

$\frac{dy}{dt} = t^2 + y^2, y(1) = 1.5$.

Keywords: 3rd order Runge - Kutta method, 4th order Runge - Kutta method,

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Lesson 31

Methods for Solving Higher Order Initial Value Problems

31.1 Introduction

From the Theory of ordinary differential equations, it is evident that an n^{th}order ordinary differential equation, be a linear or a non-linear one, can be reduced to a system of n-first order equations. To see this, consider the second order o. d. e.

$$y'' - y' + 4t^2 y = 0$$

subject to the initial conditions $y(1) = 1$, $y'(1) = 2$.

Let $u = y(t)$ and $v = y'(t)$,

then $v' = y''$

and we have $u' = v$ and $v' = v - 4t^2 u$.

Thus the 2-first order equations are

$$u' = v\,;\ u(1) = 1$$

$$v' = v - 4t^2 u\,;\ v(1) = 2$$

This is known as the initial value problem in the first order system corresponding to the given 2^{nd} order initial value problem.

In general an n^{th}order differential equation

$$y^{(n)} = F(t, y, y', y'', \ldots, y^{n-1}) \tag{31.1}$$

is written in the first order system as follows set $u_1 = y(t)$.

$$\left.\begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ &\dots \\ u_{n-1}' &= u_n \\ u_n' &= F(t, u_1, u_2, \dots, u_{n-1}) \end{aligned}\right\} \qquad (31.2)$$

With the transformed initial conditions

$$u_1(t_0) = \eta_0, u_2(t_0) = \eta_1 \dots, u_n(t_0) = \eta_{n-1} \qquad (31.3)$$

This system is written in the vector form as

$$\left.\begin{aligned} \underline{u}' &= \underline{f}(t, \underline{u}) \\ \underline{u}(t_0) &= \underline{\eta} \end{aligned}\right\}$$

(31.4)

where $\underline{u} = [u_1, u_2, \dots, u_n]^T$,

$\underline{f} = [u_2, u_3, \dots, u_n, F]^T$,

$\underline{\eta} = [\eta_0, \eta_1, \dots, \eta_{n-1}]^T$.

Thus the methods of solution of the first order I.V.P

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0$$

can be used to solve the above system of first order I.V.Ps.

31.2 Taylor Series Method

In what follows is shown the utility of Taylor series method to the system of I.V.Ps. through two examples.

1. The vector form of the Taylor Series method is written as

$$\underline{y}_{j+1} = \underline{y}_j + h\underline{y}'_j + \frac{h^2}{2!}\underline{y}''_j + \ldots + \frac{h^p}{p!}\underline{y}_j^{(p)}$$

Here $j = 0,1,2,..,N-1$ denote the nodal point

and $\underline{y}_j^{(k)} = \begin{bmatrix} y_{1,j}^{(k)} \\ y_{2,j}^{(k)} \\ \ldots \\ y_{n,j}^{(k)} \end{bmatrix} = \begin{bmatrix} \frac{d}{dt^{k-1}} f_1(t_j, y_{1,j}, y_{2,j}, \ldots, y_{n,j}) \\ \ldots \\ \frac{d^{k-1}}{dt^{k-1}} f_n(t_j, y_{1,j}, y_{2,j}, \ldots, y_{n,j}) \end{bmatrix}$.

31.3 Euler Method

The vector form of Euler method can be written as:

$$\underline{y}_{j+1} = \underline{y}_j + h\underline{y}'_j, \; j = 0,1,2,\ldots,N-1.$$

Example: 1. Reduce the 3rd order I.V.P. into a system of first order I.V.P.:

$$y''' + 2y'' + y' - y = \cos t, t \in [0,1]$$

subject to $y(0) = 0, y'(0) = 1, y''(0) = 2$.

Solution:

Set $y = u_1, u_1' = u_2, u_2' = u_3$.

Now the system of 3 first order equations is

$$u_1' = u_2$$

$$u_2' = u_3$$

$$u_3' = \cos t - 2u_3 - u_2 + u_1$$

The initial conditions are: $u_1(0) = 0, u_3(0) = 1, u_3(0) = 2$.

Example 2. Use the 2^{nd} order Taylor Series method to compute $y(1), y'(1)$ and $y''(1)$ by taking $h = 1.0$ in the above example.

Solution:

The Second order Taylor Series method is

$$\underline{u}(t_0 + h) = \underline{u}(t_0) + h\underline{u}'(t_0) + \frac{h^2}{2}\underline{u}''(t_0)$$

Given $h = 1$

$$\therefore \underline{u}(1) = \underline{u}(0) + \underline{u}'(0) + \frac{1}{2}\underline{u}''(0).$$

The system of I.V.P. is:

$$\underline{u}' = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}' = \begin{bmatrix} u_1 \\ u_3 \\ \cos t - 2u_3 - u_2 + u_1 \end{bmatrix}$$

subject to $\underline{u}(0) = \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

We now require to compute $\underline{u}'(0)$ and $\underline{u}''(0)$:

$$\underline{u}'(0) = \begin{bmatrix} u_2(0) \\ u_3(0) \\ 1 - 2u_3(0) - u_2(0) + u_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} u_1'(0) \\ u_2'(0) \\ u_3'(0) \end{bmatrix}$$

Also $\underline{u}''(0) = \begin{bmatrix} u_2'(0) \\ u_3'(0) \\ -2u_3'(0) - u_2'(0) + u_1'(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$

$$\therefore \underline{u}(1) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} y(1) \\ y'(1) \\ y''(1) \end{bmatrix}.$$

Exercises:

1. Solve the system equations $u' = -3u + 2v, u(0) = 0$ and $v' = 3u - 4v, v(0) = \frac{1}{2}$ using

(i) Forward Euler method and

(ii) 2nd order Taylor Series method by taking $h = 0.2$ on the interval $[0, 0.6]$.

Keywords: Euler Method, Higher order initial value problems, Taylor Series method.

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Lesson 32

System of I.V.Ps.-4^{th} Order R-K Method

32.1 Introduction

We now present the vector form of the 2^{nd} order and 4^{th} order Runge-Kutta methods.

Given the I.V.P. : $$\left.\begin{aligned} \frac{dy}{dt} &= f(t,y,z);\, y(t_0)=y_0 \\ \frac{dz}{dt} &= g(t,y,z);\, z(t_0)=z_0 \end{aligned}\right\} \qquad (32.1)$$

The Euler-Cauchy method (which belong to the class of 2^{nd} order Runge-Kutta method) when applied to the above system of I.V.P. is written in the vector form as

$$\underline{u}_{j+1} = \underline{u}_j + \frac{1}{2}(\underline{k}_1 + \underline{k}_2)$$

where $\underline{u} = [y,z]^T$; $\underline{k}_1 = [k_{11}, k_{21}]^T$; $\underline{k}_2 = [k_{12}, k_{22}]^T$

with $k_{11} = h \cdot f(t_j, y_j, z_j)$

$k_{21} = h \cdot g(t_j, y_j, z_j)$

and $k_{12} = h \cdot f(t_j + h, y_j + k_{11}, z_j + k_{21})$

$k_{22} = h \cdot f(t_j + h, y_j + k_{11}, z_j + k_{21})$

Example 1

Find $y(0.2)$ and $z(0.2)$ from the system of I.V.P. :

$y' = -3y + 2z, y(0) = 0$

$z' = 3y - 4z, z(0) = 0.5$

by taking $h = 0.2$ using the Euler-Cauchy method.

Solution:

Given $t_0 = 0, y_0 = 0, z_0 = 0.5,$

$f(t,y,z) = -3y + 2z, g(t,y,z) = 3y - 4z$.

$k_{11} = 0.2, k_{21} = -0.4$

$k_{12} = -0.08, k_{22} = 0.04$

$\therefore y(0.2) = y(0) + \frac{1}{2}(k_{11} + k_{12}) = 0.06,$

$z(0.2) = z(0) + \frac{1}{2}(k_{21} + k_{22}) = 0.32$.

The 4^{th} order Runge-Kutta method for the system of equations as given in (1) is written as

$$y_{j+1} = y_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_{j+1} = z_j + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4), \quad j = 0,1,2,..,N-1$$

where $k_1 = h \cdot f\left(t_j, y_j, z_j\right)$

$$l_1 = g \cdot f\left(t_j, y_j, z_j\right)$$

$$k_2 = h \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{k_1}{2}, z_j + \frac{l_1}{2}\right)$$

$$l_2 = g \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{k_1}{2}, z_j + \frac{l_1}{2}\right)$$

$$k_3 = h \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{k_2}{2}, z_j + \frac{l_2}{2}\right)$$

$$l_3 = g \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{k_2}{2}, z_j + \frac{l_2}{2}\right)$$

$$k_4 = h \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{k_3}{2}, z_j + \frac{l_3}{2}\right)$$

$$l_4 = g \cdot f\left(t_j + \frac{h}{2}, y_j + \frac{k_3}{2}, z_j + \frac{l_3}{2}\right)$$

In the similar manner, the method can be extended to 3 or more first order equations.

Example 2: Solve $\left.\begin{array}{l} y'' = xy'^2 - y^2 \\ y(0) = 1, y'(0) = 0 \end{array}\right\}$ to compute $y(0.2)$.

Solution:

The given second order equation with the initial conditions can be written as the system of two first order equations as:

$$\frac{dy}{dt} = z = f(t, y, z), \text{ say}$$

$$\frac{dz}{dt} = tz^2 - y^2 = g(t, y, z), \text{ say.}$$

Given $t_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$. Compute $k_1, l_1, k_2, l_2, k_3, l_3$ and k_4, l_4 in this order, we see

$$k_1 = h \cdot f\left(t_0, y_0, z_0\right) = 0.2 \times 0 = 0$$

$$l_1 = g \cdot f\left(t_0, y_0, z_0\right) = 0.2(-1) = -0.2$$

$$k_2 = h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = -0.02$$

$$l_2 = g \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = -0.1998$$

$$k_3 = h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = -0.02$$

$$l_3 = g \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = -0.1958$$

$$k_4 = h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}, z_0 + \frac{l_3}{2}\right) = -0.0392$$

$$l_4 = g \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}, z_0 + \frac{l_3}{2}\right) = -0.1905$$

Thus $y(0.2) = y(0) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 - 0.0199$

$= 0.9801$

and $y'(0.2) = z(0.2) = z(0) + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= 0 - 0.1970$

$= 0.197$.

Exercises

1. Find $y(0.2)$ and $z(0.2)$ using the **4th order Runge-Kutta method** to solve

   $y' = -3y + 2z, y(0) = 0$

   $z' = 3y - 4z, z(0) = 0.5$

   by taking $h = 0.1$.

2. Solve $y'' = y + ty'$, $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$ using the 4th order R-K method. Take $h = 0.1$.

3. Find $y'' = t^3 y' + t^3 y, y(0) = 1$, $y'(0) = \frac{1}{2}$ in $[0,1]$ by taking $h = 0.2$.

Keywords: System of I.V.Ps.,

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Lesson 33

Introduction

In this lesson we will discuss the idea of integral transform, in general, and Laplace transform in particular. Integral transforms turn out to be a very efficient method to solve certain ordinary and partial differential equations. In particular, the transform can take a differential equation and turn it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution. The idea of solving differential equations is given in Figure 33.1.

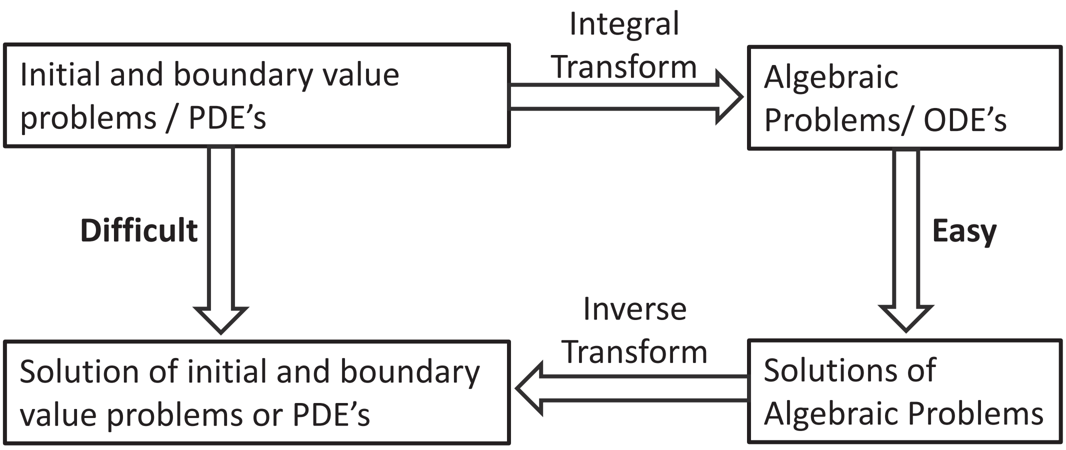

Figure 33.1: Idea of Solving Differential/Integral Equations

33.1 Concept of Transformations

An integral of the form

$$\int_a^b K(s,t)f(t)\,\mathrm{d}t$$

is called integral transform of $f(t)$. The function $K(s,t)$ is called kernel of the transform. The parameter s belongs to some domain on the real line or in the complex plane. Choosing different kernels and different values of a and b, we get different integral transforms. Examples include Laplace, Fourier, Hankel and Mellin transforms. For $K(s,t) = e^{-st},\ a = 0,\ b = \infty$, the improper integral

$$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

is called Laplace transform of $f(t)$. If we set $K(s,t) = e^{-ist},\ a = -\infty,\ b = \infty$, then

$$\int_{-\infty}^\infty e^{ist} f(t)\,\mathrm{d}t$$

where $i = \sqrt{-1}$ is called the Fourier transform of $f(t)$. A common property of integral transforms is linearity, i.e.,

$$\text{I.T. } [\alpha\, f(t) + \beta\, g(t)] = \int_a^b K(s,t)\,[\alpha\, f(t) + \beta\, g(t)]\ \mathrm{d}t = \alpha\,\text{I.T.}(f(t)) + \beta\,\text{I.T. }(g(t))$$

The symbol I.T. stands for integral transforms.

33.2 Laplace Transform

The Laplace transform of a function f is defined as

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

provided the improper integral converges for some s.

Remark 1: *The integral $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ is said to be convergent (absolutely convergent) if*

$$\lim_{R\to\infty} \int_0^R e^{-st} f(t)\,\mathrm{d}t \quad \left(\lim_{R\to\infty} \int_0^R \mid e^{-st} f(t) \mid\ \mathrm{d}t\right)$$

exists as a finite number.

33.3 Laplace Transform of Some Elementary Functions

We now give Laplace transform of some elementary functions. Laplace transform of these elementary functions together with properties of Laplace transform will be used to evaluate Laplace transform of more complicated functions.

33.4 Example Problems

33.4.1 Problem 1

Evaluate Laplace transform of $f(t) = 1,\ t \geq 0.$

Solution: Using definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st}\,\mathrm{dt} = \frac{e^{-st}}{-s}\Big|_0^\infty$$

Assuming that s is real and positive, therefore

$$L[f(t)] = \frac{1}{s},\ \text{since}\ \lim_{R\to\infty} e^{-sR} = 0$$

What will happen if we take s to be a complex number, i.e., $s = x + iy$. Since $e^{-iyR} = \cos yR - i\sin yR$, and therefore $\mid e^{-iyR} \mid = 1$, then, we find

$$\lim_{R\to\infty} \mid e^{xR} \mid\mid e^{-iyR} \mid = 0\ \text{for}\ \mathrm{Re}(s) = x > 0$$

Thus, we have

$$L[f(t)] = L[1] = \frac{1}{s},\ \mathrm{Re}(s) > 0.$$

33.4.2 Problem 2

Find the Laplace transform of the functions $e^{at},\ e^{iat},\ e^{-iat}$.

Solution: Using the definition of Laplace transform

$$L[e^{at}] = \int_0^\infty e^{-st}e^{at}\,\mathrm{dt} = \int_0^\infty e^{-(s-a)t}\,\mathrm{dt} = \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty$$
$$= \frac{1}{s-a},\ \text{provided}\ \mathrm{Re}(s) > a\ (\text{or}\ s > a)$$

Similarly, we can evaluate

$$L[e^{iat}] = \int_0^\infty e^{-(s-ia)t}\,\mathrm{dt} = \frac{e^{-(s-ia)t}}{-(s-ia)}\Big|_0^\infty$$
$$= \frac{1}{s-ia},\ \text{provided}\ \mathrm{Re}(s) > 0.$$

Here we have used the fact that, for $s = x + iy$, we have

$$\lim_{R\to\infty}\left|\frac{e^{-(s-ia)R}}{-(s-ia)}\right| = -\frac{1}{s-ia}\lim_{R\to\infty}\left|e^{-xR}e^{-i(y-a)R}\right| = 0$$

Similarly, we get

$$L[e^{-iat}] = \frac{1}{s+ia}.$$

33.4.3 Problem 3

Fins the Laplace transform of the unit step function (commonly known as the Heaviside function). This function is given as

$$u(t-a) = \begin{cases} 0 & \textit{if } t < a, \\ 1 & \textit{if } t \geq a. \end{cases}$$

Solution: Let us find the Laplace transform of $u(t-a)$, where $a \geq 0$ is some constant. That is, the function that is 0 for $t < a$ and 1 for $t \geq a$.

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_{t=a}^\infty = \frac{e^{-as}}{s},$$

where of course $s > 0$ and $a \geq 0$.

33.4.4 Problem 4

Find the Laplace transform of $t^n, \ n = 1, 2, 3, ...$

Solution: Using definition of Laplace transform we get

$$\begin{aligned} L[t^n] &= \int_0^\infty e^{-st}t^n\,\mathrm{d}t = \left[t^n\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}nt^{n-1}\,\mathrm{d}t \\ &= 0 + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,\mathrm{d}t = \frac{n}{s}L[t^{n-1}] \end{aligned}$$

Putting $n = 1$:

$$L[t] = \frac{1}{s}L[1] = \frac{1}{s^2} = \frac{1!}{s^2}$$

Putting $n = 2$:

$$L[t^2] = \frac{2}{s^3} = \frac{2!}{s^3}$$

If we assume $L[t^n] = \frac{n!}{s^{n+1}}$, then

$$L[t^{n+1}] = \frac{n+1}{s} L[t^n] = \frac{(n+1)!}{s^{n+2}} \quad \Rightarrow \quad L[t^n] = \frac{n!}{s^{n+1}}, \quad \text{Re}(s) > 0.$$

One can also extend this result for non-integer values of n.

33.4.5 Problem 5

Find $L[t^\gamma]$ for non-integer values of γ.

Solution: Using the definition of Laplace transform we get

$$L[t^\gamma] = \int_0^\infty e^{-st} t^\gamma \, \mathrm{d}t, \quad (\gamma > -1)$$

Note that the above integral is convergent only for $\gamma > -1$. We substitute $u = st \ \Rightarrow \mathrm{d}u = s\mathrm{d}t$ where $s > 0$. Thus we get

$$L[t^\gamma] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^\gamma \frac{1}{s} \, \mathrm{d}u = \frac{1}{s^{\gamma+1}} \int_0^\infty e^{-u} u^\gamma \, \mathrm{d}u$$

We know

$$\Gamma(p) = \int_0^\infty u^{p-1} e^{-u} \, \mathrm{d}u \ (p > 0)$$

Then,

$$L[t^\gamma] = \frac{\Gamma(\gamma+1)}{s^{\gamma+1}}, \quad \gamma > -1, s > 0$$

Note that for $\gamma = 1, 2, 3, ...$, the above formula reduces to the formula we got in previous example for integer values, i.e., $L[t^\gamma] = \dfrac{\gamma!}{s^{\gamma+1}}$.

33.4.6 Problem 6

Let $f(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$. Find $L[f(t)]$.

Solution: Applying the definition of Laplace transform we obtain

$$L[f(t)] = L\left[\sum_{k=0}^{n} a_k t^k\right]$$

Using the linearity of the transform we get

$$L[f(t)] = \sum_{k=0}^{n} L[t^k] = \sum_{k=0}^{n} a_k \frac{k!}{s^{k+1}}.$$

Remark 2: *For an infinite series $\sum_{n=0}^{\infty} a_n t^n$, it is not possible, in general, to obtain Laplace transform of the series by taking the transform term by term.*

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Lesson 34

Laplace Transform of Some Elementary Functions

In this lesson we compute the Laplace transform of some elementary functions, before discussing the restriction that have to be imposed on $f(t)$ so that it has a Laplace transform. With the help of Laplace transform of elementary function we can get Laplace transform of complicated function using properties of the transform that will be discussed later. Another important aspect of the finding Laplace transform of elementary function relies on using them for getting inverse Laplace transform.

34.1 Example Problems

34.1.1 Problem 1

Find Laplace transform of (i) $\cosh \omega t$, *(ii)* $\cos \omega t$, *(iii)* $\sinh \omega t$ *(iv)* $\sin \omega t$.

Solution: (i) Using the definition of Laplace transform we get

$$L[\cosh \omega t] = L\left[\frac{e^{\omega t} - e^{-\omega t}}{2}\right]$$

Using linearity of the transform we obtain

$$L[\cosh \omega t] = \frac{1}{2}\Big(L\left[e^{\omega t}\right] - L\left[e^{-\omega t}\right]\Big)$$

Applying the Laplace transform of exponential function we obtain

$$L[\cosh \omega t] = \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] = \frac{s}{s^2+\omega^2}$$

(ii) Following similar steps we obtain

$$L[\cos \omega t] = L\left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right]$$

Using linearity, we obtain

$$L[\cos \omega t] = \frac{1}{2}L\left[e^{i\omega t}\right] + \frac{1}{2}L\left[e^{-i\omega t}\right]$$

We know the Laplace transform of exponential functions which can be used now to get

$$L[\cos\omega t] = \frac{1}{2}\left\{\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right\} = \frac{1}{2}\frac{2s}{s^2+\omega^2}$$

Thus we have

$$L[\cos\omega t] = \frac{s}{s^2+\omega^2}$$

Similarly we get the last two cases (iii) and (iv) as

$$L[\sinh\omega t] = \frac{\omega}{s^2-\omega^2} \quad \text{and} \quad L[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$

34.1.2 Problem 2

Find the Laplace transform of $(3+e^{6t})^2$.

Solution: We determine the Laplace transform as follows

$$L(3+e^{6t})^2 = L(3+e^{6t})(3+e^{6t}) = L(9+6e^{6t}+e^{12t})$$

Using linearity we get

$$\begin{aligned} L(3+e^{6t})^2 &= L(9) + L(6e^{6t}) + L(e^{12t}) \\ &= 9L(1) + 6L(e^{6t}) + L(e^{12t}) \end{aligned}$$

Using the Laplace transform of elementary functions appearing above we obtain

$$L(3+e^{6t})^2 = \frac{9}{s} + \frac{6}{s-6} + \frac{1}{s-12}$$

34.1.3 Problem 3

Find the Laplace transform of $\sin^3 2t$.

Solution: We know that

$$\sin 3t = 3\sin t - 4\sin^3 t$$

This implies that we can write

$$\sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$$

Applying Laplace transform and using its linearity property we get

$$L[\sin^3 2t] = \frac{1}{4}\left(3L[\sin 2t] - L[\sin 6t]\right)$$

Using the Laplace transforms of $\sin at$ we obtain

$$L[\sin^3 2t] = \frac{3}{4}\frac{2}{s^2+4} - \frac{1}{4}\frac{6}{s^2+36}$$

Thus we get

$$L[\sin^3 2t] = \frac{48}{(s^2+4)(s^2+36)}$$

34.1.4 Problem 4

Find Laplace transform of the function $f(t) = 2^t$.

Solution: First we rewrite the given function as

$$f(t) = 2^t = e^{\ln 2^t} = e^{t\ln 2}$$

Now $f(t)$ is function of the form e^{at} and therefore

$$L[f(t)] = \frac{1}{s-\ln 2}, \text{ for } s > \ln 2$$

34.1.5 Problem 5

Find (a) $L[t^3 - 4t + 5 + 3\sin 2t]$ *and (b)* $L[H(t-a) - H(t-b)]$.

Solution: (a) Using linearity of the transform we get

$$L[t^3 - 4t + 5 + 3\sin 2t] = L[t^3] - 4L[t] + L[5] + 3L[\sin 2t]$$

Using Laplace transform evaluated in previous previous examples, we have

$$L[t^3 - 4t + 5 + 3\sin 2t] = \frac{6}{s^4} - \frac{4}{s^2} + \frac{5}{s} + \frac{6}{(s^2+4)}$$

On simplification we find

$$L[t^3 - 4t + 5 + 3\sin 2t] = \frac{(5s^5 + 2s^4 + 20s^3 10s^2 + 24)}{[s^4(s^2+4)]}$$

(b) Using Linearity property we get

$$L[H(t-a)-H(t-b)] = L[H(t-a)] - L[H(t-b)]$$

Applying the definition of Laplace transform we obtain

$$\begin{aligned} L[H(t-a)-H(t-b)] &= \int_0^\infty H(t-a)e^{-st}\,\mathrm{d}t - \int_0^\infty H(t-b)e^{-st}\,\mathrm{d}t \\ &= \int_a^\infty H(t-a)e^{-st}\,\mathrm{d}t - \int_b^\infty H(t-b)e^{-st}\,\mathrm{d}t \end{aligned}$$

Integration gives

$$L[H(t-a)-H(t-b)] = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$$

This implies

$$L[H(t-a)-H(t-b)] = \frac{e^{-as}-e^{-bs}}{s}$$

34.1.6 Problem 6

Find Laplace transform of the following function

$$f(t) = \begin{cases} t/c, & \text{if } 0 < t < c\,; \\ 1, & \text{if } t > c. \end{cases}$$

Here c is some constant.

Solution: Using the definition of Laplace transform we have

$$L[f(t)] = \int_0^c e^{-st}\left(\frac{t}{c}\right)\,\mathrm{d}t + \int_c^\infty e^{-st}\,\mathrm{d}t$$

Integrating by parts we find

$$L[f(t)] = \left[\frac{t}{c}\left(-\frac{e^{-st}}{s}\right) - \frac{1}{c}\left(-\frac{e^{-st}}{s^2}\right)\right]_0^c + \left[-\frac{e^{-st}}{s}\right]_c^\infty$$

On simplifications we obtain

$$L[f(t)] = \frac{1-e^{sc}}{cs^2}$$

34.1.7 Problem 7

Find Laplace transform of the function $f(t)$ *given by*

$$f(t) = \begin{cases} 0, & \text{if } 0 < t < 1\,; \\ t, & \text{if } 1 < t < 2; \\ 0, & \text{if } t > 2. \end{cases}$$

Solution: By the definition of Laplace transform we have

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_1^2 e^{-st} t\,\mathrm{d}t$$

Integrating by parts we obtain

$$\begin{aligned} L[f(t)] &= \left[t\left(-\frac{e^{-st}}{s}\right)\right]_1^2 + \int_1^2 \frac{e^{-st}}{s}\,\mathrm{d}t \\ &= -\frac{2e^{-2s} - e^{-s}}{s} - \frac{e^{-2s} - e^{-s}}{s^2} \end{aligned}$$

34.1.8 Problem 8

Find Laplace transform of $\sin\sqrt{t}$.

Solution: We have

$$\sin\sqrt{t} = t^{1/2} - \frac{1}{3!}t^{3/2} + \frac{1}{5!}t^{5/2} - \frac{1}{7!}t^{7/2} + \ldots$$

Then, taking the Laplace transform of each term in the series we get

$$\begin{aligned} L[\sin\sqrt{t}] &= L[t^{1/2}] - \frac{1}{3!}L[t^{3/2}] + \frac{1}{5!}L[t^{5/2}] - \frac{1}{7!}L[t^{7/2}] + \ldots \\ &= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \frac{1}{7!}\frac{\Gamma(9/2)}{s^{9/2}} + \ldots \end{aligned}$$

Further simplifications leads to

$$\begin{aligned} L[\sin\sqrt{t}] &= \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}}\left[1 - \frac{1}{3!}\frac{3}{2}\frac{1}{s} + \frac{1}{5!}\frac{5}{2}\frac{3}{2}\frac{1}{s^2} - \frac{1}{7!}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{s^3} + \ldots\right] \\ &= \frac{1}{2s}\sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{2^2 s} + \frac{1}{2!}\frac{1}{(2^2 s)^2} - \frac{1}{3!}\frac{1}{(2^2 s)^3} + \ldots\right] \end{aligned}$$

Thus, we have

$$L[\sin\sqrt{t}] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}e^{-\frac{1}{4s}}.$$

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Lesson 35

Existence of Laplace Transform

In this lesson we shall discuss existence theorem on Laplace transform. Since every Laplace integral is not convergent, it is very important to know for which functions Laplace transform exists.

Consider the function $f(t) = e^{t^2}$ and try to evaluate its Laplace integral. In this case we realize that

$$\lim_{R\to\infty} \int_0^R e^{t^2-st}\,\mathrm{d}t = \infty, \text{ for any choice of } s$$

Naturally question arises in mind that for which class of functions, the Laplace integral converges? So before answering this question we go through some definition.

35.1 Piecewise Continuity

A function f is called piecewise continuous on $[a, b]$ if there are finite number of points $a < t_1 < t_2 < \ldots < t_n < b$ such that f is continuous on each open subinterval $(a, t_1), (t_1, t_2), \ldots, (t_n, b)$ and all the following limits exists

$$\lim_{t\to a+} f(t), \quad \lim_{t\to b-} f(t), \quad \lim_{t\to t_j+} f(t), \text{ and } \lim_{t\to t_j-} f(t), \ \forall j.$$

Note: A function f is said to be piecewise continuous on $[0, \infty)$ if it is piecewise continuous on every finite interval $[0, b]$, $b \in \mathbb{R}_+$.

35.1.1 Example 1

The function defined by

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1; \\ 3 - t, & 1 < t \le 2; \\ t + 1, & 2 < t \le 3; \end{cases}$$

is piecewise continuous on $[0, 3]$.

35.1.2 Example 2

The function defined by

$$f(t) = \begin{cases} \frac{1}{2-t}, & 0 \leq t < 2; \\ t+1, & 2 \leq t \leq 3; \end{cases}$$

is not piecewise continuous on $[0,3]$.

35.2 Example Problems

35.2.1 Problem 1

Discuss the piecewise continuity of

$$f(t) = \frac{1}{t-1}$$

.

Solution: $f(t)$ is not piecewise continuous in any interval containing 1 since

$$\lim_{t \to 1\pm} f(t)$$

do not exists.

35.2.2 Problem 2

Check whether the function

$$f(t) = \begin{cases} \frac{1-e^{-t}}{t}, & t \neq 0; \\ 0, & \text{otherwise} \end{cases}$$

is piecewise continuous or not.

Solution: The given function is continuous everywhere other than at 0. So we need to check limits at this point. Since both the left and right limits

$$\lim_{t \to 0-} f(t) = 1 \quad \text{and} \quad \lim_{t \to 0+} f(t) = 1$$

exists, the given function is piecewise continuous.

35.3 Functions of Exponential Orders

A function f is said to be of exponential order α if there exist constant M and α such that for some $t_0 \geq 0$

$$|f(t)| \leq Me^{\alpha t} \quad \text{for all } t \geq t_0$$

Equivalently, a function $f(t)$ is said to be of exponential order α if

$$\lim_{t\to\infty} e^{-\alpha t}|f(t)| = \text{ a finite quantity}$$

Geometrically, it means that the graph of the function f on the interval (t_0, ∞) does not grow faster than the graph of exponential function $Me^{\alpha t}$

35.4 Example Problems

35.4.1 Problem 1

Show that the function $f(t) = t^n$ has exponential order α for any value of $\alpha > 0$ and any natural number n.

Solution: We check the limit

$$\lim_{t\to\infty} e^{-\alpha t}t^n$$

Repeated application of L'hospital rule gives

$$\lim_{t\to\infty} e^{-\alpha t}t^n = \lim_{t\to\infty} \frac{n!}{\alpha^n e^{\alpha t}} = 0$$

Hence the function is of exponential order.

35.4.2 Problem 2

Show that the function $f(t) = e^{t^2}$ is not of exponential order.

Solution: For given function we have

$$\lim_{t\to\infty} e^{-\alpha t}e^{t^2} = \lim_{t\to\infty} e^{t(t-\alpha)} = \infty$$

for all values of α. Hence the given function is not of exponential order.

35.4.3 Theorem (Sufficient Conditions for Laplace Transform)

If f is piecewise continuous on $[0, \infty)$ and of exponential order α then the Laplace transform exists for $Re(s) > \alpha$. Moreover, under these conditions Laplace integral converges absolutely.

Proof: Since f is of exponential order α, then

$$|f(t)| \leq M_1 e^{\alpha t}, \quad t \geq t_0 \tag{35.1}$$

Also, f is piecewise continuous on $[0, \infty)$ then

$$|f(t)| \leq M_2, \quad 0 < t < t_0 \tag{35.2}$$

From equation (35.1) and (35.2) we have

$$|f(t)| \leq M e^{\alpha t}, \quad t \geq 0$$

Then

$$\int_0^R |e^{-st} f(t)| dt \leq \int_0^R |e^{-(x+iy)t} M e^{\alpha t}| dt$$

Here we have assumed s to be a complex number so that $s = x + iy$. Noting that $|e^{-iy}| = 1$ we find

$$\int_0^R |e^{-st} f(t)| dt \leq M \int_0^R e^{-(x-\alpha)t} dt$$

On integration we obtain

$$\int_0^R |e^{-st} f(t)| dt \leq \frac{M}{x-\alpha} - \frac{M}{x-\alpha} e^{-(x-\alpha)R}$$

Letting $R \to \infty$ and noting $Re(s) = x > \alpha$, we get

$$\int_0^\infty |e^{-st} f(t)| dt \leq \frac{M}{x-\alpha}$$

Hence the Laplace integral converges absolutely and thus converges. This implies the existence of Laplace transform. For piecewise continuous functions of exponential order, the Laplace transform always exists. Note that it is a sufficient condition, that means if a function is not of exponential order or piecewise continuous then the Laplace transform may or may not exist. ■

Remark 1: *We have observed in the proof of existence theorem that*

$$\left|\int_0^\infty e^{-st}f(t)dt\right| \leq \int_0^\infty |e^{-st}f(t)|dt \leq \frac{M}{Re(s)-\alpha} \quad \text{for } Re(s) > \alpha$$

We now deduce two important conclusions with this observation:

- $L[f(t)] = \displaystyle\int_0^\infty e^{-st}f(t)dt = F(s) \to 0$ *as* $Re(s) \to \infty$

- *if* $L[f(t)] \not\to 0$ *as* $s \to \infty$ *(or* $Re(s) \to \infty$*) then* $f(t)$ *cannot be piecewise continuous function of exponential order. For example functions such as* $F_1(s) = 1$ *and* $F_2(s) = s/(s+1)$ *are not Laplace transforms of piecewise continuous functions of exponential order, since* $F_1(s) \not\to 0$ *and* $F_2(s) \not\to 0$ *as* $s \to \infty$*.*

Remark 2: *It should be noted that the conditions stated in existence theorem are sufficient rather than necessary conditions. If these conditions are satisfied then the Laplace transform must exist. If these conditions are not satisfied then Laplace transform may or may not exist. We can observe this fact in the following examples:*

- *Consider, for example,*

  $$f(t) = 2te^{t^2}\cos(e^{t^2})$$

  Note that $f(t)$ *is continuous on* $[0,\infty)$ *but not of exponential order, however the Laplace transform of* $f(t)$ *exists, since*

  $$L[f(t)] = \int_0^\infty e^{-st}2te^{t^2}\cos(e^{t^2})dt$$

  Integration by parts leads to

  $$L[f(t)] = e^{-st}\sin(e^{t^2})\Big|_0^\infty + s\int_0^\infty e^{-st}\sin(e^{t^2})dt$$

  Using the definition of Laplace transform we obtain

  $$L[f(t)] = -\sin(1) + sL[\sin(e^{t^2})]$$

  Note that $L[\sin(e^{t^2})]$ *exists because the function* $\sin(e^{t^2})$ *satisfies both the conditions of existence theorem. This example shows that Laplace transform of a function which is not of exponential order exists.*

- *Consider another example of the function*

$$f(t) = \frac{1}{\sqrt{t}},$$

which is not piecewise continuous since $f(t) \to \infty$ *as* $t \to 0$. *But we know that*

$$L[f(t)] = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad s > 0.$$

This example shows that Laplace transform of a function which is not piecewise continuous exists. These two examples clearly shows that the conditions given in existence theorem are sufficient but not necessary.

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Lesson 36

Properties of Laplace Transform

In this lesson we discuss some properties of Laplace transform. There are several useful properties of Laplace transform which can extend its applicability. In this lesson we mainly present shifting and translation properties.

36.1 First Shifting Property

If $L[f(t)] = F(s)$ then $L\left[e^{at}f(t)\right] = F(s-a)$, where a is any real or complex constant.

Proof: By the definition of Laplace transform we find

$$\begin{aligned} L\left[e^{at}f(t)\right] &= \int_0^\infty e^{at}f(t)e^{-st}\,\mathrm{d}t \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,\mathrm{d}t \end{aligned}$$

Again by the definition of Laplace transform we get

$$L\left[e^{at}f(t)\right] = F(s-a).$$

36.2 Example Problems

36.2.1 Problem 1

Find the Laplace transform of $e^{-t}\sin^2 t$.

Solution: First we get the Laplace transform of $\sin^2 t$ as

$$\begin{aligned} L\left[\sin^2 t\right] &= L\left[\frac{1-\cos 2t}{2}\right] \\ &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)} = F(s). \end{aligned}$$

Now using the first shifting property we obtain

$$L\left[e^{-t}\sin^2 t\right] = F(s+1) = \frac{2}{(s+1)(s^2+2s+5)}$$

36.2.2 Problem 2

Find $L[e^{-2t}\sin 6t]$.

Solution: Setting $f(t) = \sin 6t$ we find

$$L[f(t)] = F(s) = \frac{6}{s^2+36}$$

Now using the first shifting property we get

$$L[e^{-2t}\sin 6t] = \frac{6}{(s+2)^2+36}$$

36.2.3 Problem 3

Evaluate $L[e^{2t}(t+3)^2]$.

Solution: By the definition and linearity of Laplace transform we have

$$\begin{aligned} L[(t+3)^2] =& L[t^2+6t+9] = L[t^2] + 6L[t] + 9L[1] \\ =& \frac{2!}{s^3} + \frac{6}{s^2} + \frac{9}{s} \end{aligned}$$

Further simplifications lead to

$$L[(t+3)^2] = \frac{2+6s+9s^2}{s^3} = F(s)$$

Using the first shifting property we get

$$\begin{aligned} L[e^{2t}(t+3)^2] = F(s-2) =& \frac{2+6(s-2)+9(s-2)^2}{(s-2)^3} \\ =& \frac{9s^2-30s+26}{(s-2)^3} \end{aligned}$$

36.2.4 Problem 4

Using shifting property evaluate $L[\sinh 2t\cos 2t]$ *and* $L[\sinh 2t\sin 2t]$

Solution: We know that

$$L[\sinh 2t] = \frac{2}{s^2-4} = F(s)$$

Using shifting property we can get

$$L[e^{2it}\sinh 2t] = F(s-2i)$$

This implies

$$L[e^{2it}\sinh 2t] = \frac{2}{(s-2i)^2-4} = \frac{2}{(s^2-8)-4is}$$

Multiplying numerator and denominator by $(s^2-8)+4is$, we find

$$L[e^{2it}\sinh 2t] = \frac{2(s^2-8)+8is}{(s^2-8)^2+16s^2} = \frac{2(s^2-8)+8is}{(s^4+64)}$$

Replacing e^{2it} by $\cos 2t + i\sin 2t$ and using linearity of the transform we obtain

$$L[\cos 2t\sinh 2t] + iL[\cos 2t\sinh 2t] = \frac{2(s^2-8)}{(s^4+64)} + i\frac{8s}{(s^4+64)}$$

Equating real and imaginary parts we have

$$L[\cos 2t\sinh 2t] = \frac{2(s^2-8)}{(s^4+64)} \quad \text{and} \quad L[\cos 2t\sinh 2t] = \frac{8s}{(s^4+64)}$$

36.3 Second Shifting Property

If $L[f(t)] = F(s)$ and $g(t) = \begin{cases} f(t-a) & \text{when } t > a \\ 0 & \text{when } 0 < t < a \end{cases}$

then

$$L[g(t)] = e^{-as}F(s).$$

Proof: By the definition of Laplace transform we have

$$\begin{aligned} L[g(t)] &= \int_0^\infty e^{-st} g(t)\,\mathrm{d}t \\ &= \int_a^\infty e^{-st} f(t-a)\,\mathrm{d}t \end{aligned}$$

Substituting $t-a=u$ so that $\mathrm{d}t = \mathrm{d}u$, we find

$$\begin{aligned} L[g(t)] &= \int_0^\infty e^{-s(u+a)} f(u)\,\mathrm{d}u \\ &= e^{-sa}\int_0^\infty e^{-su} f(u)\,\mathrm{d}u \end{aligned}$$

Again using the definition of Laplace transform we get

$$L[g(t)] = e^{-as}F(s).$$

Alternative form: It is sometimes useful to present this property in the following compact form.

If $L[f(t)] = F(s)$ then

$$L\left[f(t-a)H(t-a)\right] = e^{-as}F(s)$$

where

$$H(t) = \begin{cases} 1 & \text{when } t > 0 \\ 0 & \text{when } t < 0 \end{cases}$$

Note that $f(t-a)H(t-a)$ is same as the function $g(t)$ given above.

36.4 Example Problems

36.4.1 Problem 1

Find $L[g(t)]$ where $g(t) = \begin{cases} 0 & \text{when } 0 \le t < 1 \\ (t-1)^2 & \text{when } t \ge 1 \end{cases}$

Solution: On comparison with the function $g(t)$ given in second shifting theorem we get

$$f(f) = t^2 \quad \Rightarrow \quad L[f(t)] = \frac{2}{s^3}$$

Using the second shifting property we find

$$L[g(t)] = e^{-s}\left(\frac{2}{s^3}\right).$$

36.4.2 Problem 2

Find the Laplace transform of the function $g(t)$, where

$$g(t) = \begin{cases} \cos(t-\pi/3), & t > \pi/3; \\ 0, & t > \pi/3. \end{cases}$$

Solution: Comparing with the notations used in the second shifting theorem we have $f(t) = \cos t$. Thus, we find

$$L[f(t)] = F(s) = \frac{s}{s^2+1}.$$

Hence by the second shifting theorem we obtain

$$L[g(t)] = e^{-\pi/3}F(s) = e^{-\pi/3}\frac{s}{s^2+1}.$$

36.5 Change of Scale Property

If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof: By definition, we have

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\,\mathrm{d}t.$$

Substituting $at = u$ so that $a\,\mathrm{d}t = \mathrm{d}u$ we find

$$L[f(at)] = \int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\frac{1}{a}\,\mathrm{d}u.$$

Using definition of the Laplace transform we get

$$L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right).$$

36.5.1 Example

If

$$L[f(t)] = \frac{s^2 - s + 1}{(2s+1)^2(s-1)}$$

then find $L[f(2t)]$.

Solution: Direct application of the second shifting theorem we obtain

$$L[f(2t)] = \frac{1}{2}\frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2} + 1}{\left(2\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}$$

On simplifications, we get

$$L[f(2t)] = \frac{1}{4}\frac{s^2 - 2s + 4}{(s+1)^2(s-2)}.$$

Suggested Readings

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Lesson 37

Properties of Laplace Transform (Cont.)

In this lesson we continues discussing various properties of Laplace transform. In particular we shall discuss Laplace transform of derivatives and integrals. These two properties are very important for solving differential and integral equations.

37.1 Laplace Transform of Derivatives

Before we state the derivative theorem, it should be noted that this results is the key aspect for its application of solving differential equations.

37.1.1 Derivative Theorem

Suppose f is continuous on $[0,\infty)$ and is of exponential order α and that $f^{'}$ is piecewise continuous on $[0,\infty)$. Then

$$L[f^{'}(t)] = sL[f(t)] - f(0), \quad \text{Re(s)} > \alpha.$$

Proof: By the definition of Laplace transform, we have

$$L[f^{'}(t)] = \int_0^{\infty} f^{'}(t)e^{-st}\,\mathrm{d}t$$

Integrating by parts, we get

$$L[f^{'}(t)] = f(t)e^{-st}\Big|_0^{\infty} - \int_0^{\infty} f(t)e^{-st}(-s)\,\mathrm{d}t$$

Using the definition of Laplace transform we obtain

$$L[f^{'}(t)] = -f(0) + sL[f(t)], \qquad \text{Re(s)} > \alpha.$$

This completes the proof. ∎

Remark 1: *Suppose $f(t)$ is not continuous at $t = 0$, then the results of the above theorem takes the following form*

$$L[f^{'}(t)] = -f(0+0) + sL[f(t)]$$

Remark 2: *An interesting feature of the derivative theorem is that $L[f^{'}(t)]$ exists without the requirement of $f^{'}$ to be of exponential order. Recall the existence of Laplace transform of $f(t) = 2te^{t^2}\cos\left(e^{t^2}\right)$ which is obvious now by the derivative theorem because*

$$f(t) = \left(\sin\left(e^{t^2}\right)\right)^{'}.$$

Remark 3: *The derivative theorem can be generalized as*

$$\begin{aligned} L[f^{''}(t)] &= -f^{'}(0) + sL[f^{'}(t)] \\ &= -f^{'}(0) + s\left\{-f(0) + sL[f(t)]\right\} = s^2L[f(t)] - sf(0) - f^{'}(0). \end{aligned}$$

In general, for nth derivative we have

$$L[f^n(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f^{'}(0) - ... - f^{n-1}(0).$$

37.2 Example Problems

37.2.1 Problem 1

Determine $L[\sin^2\omega t]$.

Solution: Let us assume that

$$f(t) = \sin^2\omega t$$

Now we compute the derivative of f as

$$f^{'}(t) = 2\sin\omega t\cos\omega t\omega = \omega\sin 2\omega t.$$

Using the derivative theorem we have

$$L[f^{'}(t)] = -f(0) + sL[f(t)]$$

Substituting the function $f(t)$ and its derivative we find

$$L[\omega \sin 2\omega t] = sL[\sin^2 \omega t] - 0$$

Therefore, we have

$$L[\sin^2 \omega t] = \frac{\omega}{s}\left(\frac{2\omega}{s^2 + 4\omega^2}\right)$$

37.2.2 Problem 2

Using derivative theorem, find $L[t^n]$.

Solution: Let

$$f(t) = t^n.$$

Then

$$f^{'}(t) = nt^{n-1}, \quad f^{''}(t) = n(n-1)t^{n-2}, \ldots, \quad f^n(t) = n!.$$

From derivative theorem we have

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f^{'}(0) - \ldots - f^{n-1}(0).$$

Therefore, we find

$$L[n!] = s^n L[t^n] \Rightarrow L[t^n] = \frac{n!}{s^{n+1}}.$$

37.2.3 Problem 3

Using derivative theorem, find $L[\sin kt]$.

Solution: Let $f(t) = \sin kt$ and therefore we have

$$f'(t) = k \cos kt \quad \text{and} \quad f''(t) = -k^2 \sin kt$$

Substituting in the derivative theorem

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

yields

$$L[-k^2 \sin kt] = s^2 L[\sin kt] - 0 - k$$

On simplifications we get

$$L[\sin kt] = \frac{k}{s^2 + k^2}$$

37.2.4 Problem 4

Using $L[t^2] = 2/s^3$ *and derivative theorem, find* $L[t^5]$.

Solution: Let $f(t) = t^5$ so that $f'(t) = 5t^4, \quad f''(t) = 20t^3 \quad f'''(t) = 60t^2$. The derivative theorem for third derivative reads as

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

This implies

$$L[60t^2] = s^3 L[f(t)] \quad \Rightarrow \quad L[f(t)] = \frac{120}{s^6}.$$

37.2.5 Problem 5

Using the Laplace transform of $L\left[\sin\sqrt{t}\right]$ *and applying the derivative theorem, find the Laplace transform of the function*

$$\frac{\cos\sqrt{t}}{\sqrt{t}}$$

Solution: We know that

$$L\left[\sin\sqrt{t}\right] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

Let $f(t) = \sin\sqrt{t}$, then we have

$$f(0) = 0 \text{ and } f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

Substitution of $f(t)$ in the derivative theorem

$$L[f'(t)] = sL[f(t)] - f(0)$$

yields

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = s\frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

Thus, we get

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

37.3 Laplace Transform of Integrals

37.3.1 Theorem

Suppose $f(t)$ is piecewise continuous on $[0,\infty)$ and the function

$$g(t) = \int_0^t f(u)\,\mathrm{d}u$$

is of exponential order. Then

$$L[g(t)] = \frac{1}{s}F(s).$$

Proof: Clearly $g(0) = 0$ and $g^{'}(t) = f(t)$. Note that $g(t)$ is piecewise continuous and is of exponential order as well as $g^{'}(t) = f(t)$ is piecewise continuous. Then, we get using the derivative theorem

$$L[g^{'}(t)] = sL[g(t)] - g(0)$$

Since $g(0) = 0$ we obtain the desired result as

$$L[g(t)] = \frac{1}{s}L[f(t)]$$

This completes the proof. ■

37.4 Example Problems

37.4.1 Problem 1

Given that

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{1+s^2}\,\mathrm{d}s.$$

Find the Laplace transform of the integral

$$\int_0^t \frac{\sin u}{u}\,\mathrm{d}u.$$

Solution: Direct application of the above result gives

$$L\left[\int_0^t \frac{\sin u}{u}\,\mathrm{d}u\right] = \frac{1}{s}L\left[\frac{\sin t}{t}\right]$$
$$= \frac{1}{s}\int_s^\infty \frac{1}{1+s^2}\,\mathrm{d}s = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1} s\right]$$

Thus, we have

$$L\left[\int_0^t \frac{\sin u}{u}\,\mathrm{d}u\right] = \frac{1}{s}\cot^{-1} s$$

37.4.2 Problem 2

Find Laplace transform of the following integral

$$\int_0^t u^n e^{-au} du$$

Solution: With the application of the first shifting theorem we know that

$$L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}$$

It follows from the above result on Laplace transform of integrals

$$L\left[\int_0^t u^n e^{-au} du\right] = \frac{1}{s}L[t^n e^{-at}] = \frac{n!}{s(s+a)^{n+1}}.$$

Suggested Readings

Debnath, L. and Bhatta, D. (2007). Integral Transforms and Their Applications. Second Edition. Chapman and Hall/CRC (Taylor and Francis Group). New York.

Dyke, P.P.G. (2001). An Introduction to Laplace Transforms and Fourier Series. Springer-Verlag London Ltd.

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Lesson 38

Properties of Laplace Transform (Cont.)

In this lesson we further continue discussing properties of Laplace transform. In particular, this lesson is devoted to Laplace transform of functions which are multiplied and divide by t.

38.1 Multiplication by t^n

38.1.1 Theorem

If $F(s)$ is the Laplace transform of $f(t)$, i.e., $L[f(t)] = F(s)$ then,

$$L[tf(t)] = -\frac{d}{ds}F(s)$$

and in general the following result holds

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s).$$

Proof: By definition we know

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

Using Leibnitz rule for differentiation under integral sign we obtain

$$\frac{dF(s)}{ds} = \int_0^\infty (-t)e^{-st} f(t)\,\mathrm{d}t$$

Thus we get

$$\frac{dF(s)}{ds} = -L[tf(t)]$$

Repeated differentiation under integral sign gives the general rule. ∎

Applicability of the above result will now be demonstrated by some examples.

38.2 Example Problems

38.2.1 Problem 1

Find Laplace transform of the function $t^2 \cos at$.

Solution: We know from Laplace transform of elementary functions that

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

Direct application of the above rule gives

$$L\left[t^2 \cos at\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2 + a^2}\right) = \frac{d}{ds}\left(\frac{s^2 + a^2 - 2s^2}{\left(s^2 + a^2\right)^2}\right) = \frac{d}{ds}\left(\frac{a^2 - s^2}{\left(s^2 + a^2\right)^2}\right)$$

On simplifications we find

$$L\left[t^2 \cos at\right] = \frac{2s\left(s^2 - 3a^2\right)}{\left(s^2 + a^2\right)^3}$$

38.2.2 Problem 2

Evaluate (i) $L[te^{-t}]$ *(ii)* $L[t^2 e^{-t}]$ *(iii)* $L[t^k e^{-t}]$

Solution: (i) We know that

$$L[e^{-t}] = \frac{1}{s+1}$$

Using the above mentioned rule we find

$$L[te^{-t}] = -\frac{d}{ds}\frac{1}{s+1} = \frac{1}{(s+1)^2}$$

(ii) Applying the same idea once again, we obtain

$$L[t^2 e^{-t}] = -\frac{d}{ds}\frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

(iii) Similarly, we can further generalize this result as

$$L[t^k e^{-t}] = \frac{k!}{(s+1)^{k+1}}$$

38.2.3 Problem 3

Find the Laplace transform of $f(t) = (t^2 - 3t + 2)\sin t$

Solution: Using linearity of the Laplace transform we have

$$L[f(t)] = L[t^2 \sin t] - 3L[t \sin t] + 2L[\sin t] \tag{38.1}$$

Since we know

$$L[\sin t] = \frac{1}{1+s^2}$$

then

$$L[t \sin t] = -\frac{d}{ds}\frac{1}{1+s^2} = \frac{2s}{(1+s^2)^2}$$

and

$$L[t^2 \sin t] = -\frac{d}{ds}\frac{2s}{(1+s^2)^2} = \frac{2(1+s^2)^2 - 8s^2(1+s^2)}{(1+s^2)^4} = \frac{6s^2-2}{(1+s^2)^3}$$

Substituting the above values in the equation (38.1), we find

$$L[f(t)] = \frac{6s^2-2}{(1+s^2)^3} - \frac{6s}{(1+s^2)^2} + \frac{2}{1+s^2}$$

Further simplifications lead to

$$L[f(t)] = \frac{6s^2 - 2 - 6s(1+s^2) + 2(1+s^2)^2}{(1+s^2)^3}$$

Finally, we obtain

$$L[f(t)] = \frac{(2s^4 - 6s^3 + 10s^2 - 6s)}{(s^6 + 3s^4 + 3s^2 + 1)}$$

38.3 Division by t

38.3.1 Theorem

If f *is piecewise continuous on* $[0,\infty)$ *and is of exponential order* α *such that*

$$\lim_{t\to 0+}\frac{f(t)}{t}$$

exists, then,

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,\mathrm{d}u, \quad [s>\alpha]$$

Proof: This can easily be proved by letting $g(t) = \dfrac{f(t)}{t}$ so that $f(t) = tg(t)$.

Hence,

$$F(s) = L[f(t) = L[tg(t)] = \frac{d}{ds}L[g(t)]$$

Integrating with respect to s we get,

$$-L[g(t)]\Big|_s^\infty = \int_s^\infty F(s)\,\mathrm{d}s.$$

Since $g(t)$ is piecewise continuous and of exponential order, it follows that $\lim\limits_{s\to\infty} L[g(t)] \to 0$. Thus we have

$$L[g(t)] = \int_0^\infty F(s)\,\mathrm{d}s.$$

This completes the proof. ■

Remark: *It should be noted that the condition* $\lim\limits_{t\to 0+}[f(t)/t]$ *is very important because without this condition the function* $g(t)$ *may not be piecewise continuous on* $[0,\infty)$. *Thus without this condition we can not use the fact* $\lim\limits_{s\to\infty} L[g(t)] \to 0$.

38.3.2 Corollary

If $L[f(t)] = F(s)$ *then* $\displaystyle\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty f(s)\,ds$*, provided that the integrals converge.*

Proof: We know that

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,\mathrm{d}u$$

Using the definition of Laplace transform we get

$$\int_0^\infty e^{-st}\frac{f(t)}{t}\,\mathrm{d}t = \int_s^\infty F(u)\,\mathrm{d}u$$

Taking limit $s \to 0$ in above two integrals we obtain

$$\int_0^\infty \frac{f(t)}{t}\,\mathrm{d}t = \int_0^\infty F(u)\,\mathrm{d}u$$

This completes the proof. ■

38.4 Example Problems

38.4.1 Problem 1

Find the Laplace transform of the function

$$f(t) = \frac{\sin at}{t}$$

Solution: We know,

$$L[\sin at] = \frac{a}{s^2 + a^2} \quad \text{and} \quad L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\, \mathrm{d}u$$

On integrating we get,

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2 + a^2}\, \mathrm{d}s = \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

Thus we have

$$L\left[\frac{\sin at}{t}\right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

38.4.2 Problem 2

Find the Laplace transform of the function

$$f(t) = \frac{2 \sin t \sinh t}{t}$$

Solution: Using Division by t property of the Laplace transform we get

$$L[f(t)] = \int_s^\infty L\left[\sin t\left(e^t - e^{-t}\right)\right]\, \mathrm{d}s \tag{38.2}$$

Now we evaluate $L\left[\sin t\left(e^t - e^{-t}\right)\right]$ using linearity of the Laplace transform as

$$L\left[\sin t\left(e^t - e^{-t}\right)\right] = L\left[e^t \sin t\right] - L\left[e^{-t} \sin t\right]$$

Applying the first shifting theorem we obtain

$$L\left[\sin t\left(e^t - e^{-t}\right)\right] = \frac{1}{1 + (s-1)^2} - \frac{1}{1 + (s+1)^2}$$

Substituting this value in the equation (38.2) we find

$$L[f(t)] = \int_s^\infty \left[\frac{1}{1+(s-1)^2} - \frac{1}{1+(s+1)^2}\right] \mathrm{d}s$$

On integrating, we have

$$\begin{aligned} L[f(t)] &= \tan^{-1}(s-1)\Big|_s^\infty - \tan^{-1}(s+1)\Big|_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1) \end{aligned}$$

On cancellation of $\pi/2$ we get

$$L[f(t)] = \tan^{-1}(s+1) - \tan^{-1}(s-1)$$

This can be further simplified to obtain

$$L[f(t)] = \tan^{-1}\left(\frac{2}{s^2}\right)$$

Suggested Readings

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Lesson 39

Properties of Laplace Transform (Cont.)

In this lesson we evaluate Laplace transform of periodic functions. Periodic functions frequently occur in various engineering problems. We shall now show that with the help of a simple integral, we can evaluate Laplace transform of periodic functions. We shall further continue the discussion for stating initial and final value theorems of Laplace transforms and their applications with the help of simple examples.

39.1 Laplace Transform of a Periodic Function

Let f be a periodic function with period T so that $f(t) = f(t+T)$ then,

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\,\mathrm{d}t.$$

Proof: By definition we have,

$$L[f(t)] = \int_0^\infty e^{-sT} f(t)\,\mathrm{d}t$$

We break the integral into two integrals as

$$L[f(t)] = \int_0^T e^{-sT} f(t)\,\mathrm{d}t + \int_T^\infty e^{-sT} f(t)\,\mathrm{d}t$$

Substituting $t = \tau + T$ in the second integral

$$L[f(t)] = \int_0^T e^{-sT} f(t)\,\mathrm{d}t + \int_0^\infty e^{-(\tau+T)} f(\tau+T)\,\mathrm{d}\tau$$

Noting $f(\tau+T) = f(\tau)$ we find

$$L[f(t)] = \int_0^T e^{-sT} f(t)\,\mathrm{d}t + e^{-sT} L[f(t)],$$

On simplifications, we obtain

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t.$$

This completes the proof. ■

Remark 1: *Just to remind that if a function f is periodic with period $T > 0$ then $f(t) = f(t+T)$, $-\infty < t < \infty$. The smallest of T, for which the equality $f(t) = f(t+T)$ is true, is called fundamental period of $f(t)$. However, if T is the period of a function f then nT, n is any natural number, is also a period of f. Some familiar periodic functions are $\sin x$, $\cos x$, $\tan x$ etc.*

39.2 Example Problems

39.2.1 Problem 1

Find Laplace transform for

$$f(t) = \begin{cases} 1 & \text{when } 0 < t \leq 1 \\ 0 & \text{when } 1 < t < 2 \end{cases}$$

with $f(t+2) = f(t),\ t > 0.$

Solution: Using the above result on periodic function, we have,

$$L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-2s}} \int_0^1 e^{-st}\,\mathrm{d}t$$

On integration we obtain

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left(\frac{1}{-s}\right) \left[e^{-s} - 1\right] = \frac{1}{s(1+e^{-s})}$$

39.2.2 Problem 2

Find Laplace transform for

$$f(t) = \begin{cases} \sin t & \text{when } 0 < t < \pi \\ 0 & \text{when } \pi < t < 2\pi \end{cases}$$

with $f(t+2\pi) = f(t),\ t > 0.$

Solution: Since $f(t)$ is periodic with period 2π we have

$$L[f(t)] = \frac{1}{1-e^{-2s\pi}} \int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t$$

We now evaluate the above integral as

$$\int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t = \int_0^{\pi} e^{-st} f(t)\,\mathrm{d}t + \int_{\pi}^{2\pi} e^{-st} f(t)\,\mathrm{d}t$$

Substituting the given value of $f(t)$ we obtain

$$\int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t = \int_0^{\pi} e^{-st} \sin t\,\mathrm{d}t + 0 = \frac{1+e^{-s\pi}}{1+s^2}$$

This implies

$$L[f(t)] = \frac{1}{1-e^{-2s\pi}}\frac{1+e^{-s\pi}}{1+s^2} = \frac{1}{(1+s^2)(1-e^{-s\pi})}$$

39.2.3 Problem 3

Find the Laplace transform of the square wave with period T:

$$f(t) = \begin{cases} h & \text{when } 0 < t < T/2 \\ -h & \text{when } T/2 < t < T \end{cases}$$

Solution: Using Laplace transform of periodic function we find

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^{T} e^{-st} f(t)\,\mathrm{d}t$$

Substituting $f(t)$ we obtain

$$L[f(t)] = \frac{1}{1-e^{-sT}}\left(\int_0^{T/2} he^{-st}\,\mathrm{d}t - \int_{T/2}^{T} he^{-st}\,\mathrm{d}t\right)$$

Evaluating integrals we get

$$L[f(t)] = \frac{1}{(1-e^{-sT})}\frac{h}{s}\left(1-2e^{sT/2}+e^{-sT}\right) = \frac{h(1-e^{sT/2})}{s(1-e^{sT/2})}$$

39.3 Limiting Theorems

These theorems allow the limiting behavior of the function to be directly calculated by taking a limit of the transformed function.

39.3.1 Theorem (Initial Value Theorem)

Suppose that f is continuous on $[0,\infty)$ and of exponential order α and f' is piecewise continuous on $[0,\infty)$ and of exponential order. Let

$$F(s) = L[f(t)],$$

then

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s), \quad \text{[assuming } s \text{ is real]}$$

Proof: By the derivative theorem,

$$L[f'(t)] = sL[f(t)] - f(0+)$$

Note that $\lim_{s\to\infty} L[f'(t)] = 0$, since f' is piecewise continuous on $[0,\infty)$ and of exponential order. Therefore we have

$$0 = \lim_{s\to\infty} sF(s) - f(0+)$$

Hence we get

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$

This completes the proof. ■

39.3.2 Theorem (Final Value Theorem)

Suppose that f is continuous on $[0,\infty)$ and is of exponential order α and f' is piecewise continuous on $[0,\infty)$ and furthermore $\lim_{t\to\infty} f(t)$ exists then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sL[f(t)] = \lim_{s\to 0} sF(s)$$

Proof: Note that f has exponential order $\alpha = 0$ since it is bounded, since $\lim_{t\to 0+} f(t)$ and $\lim_{t\to\infty} f(t)$ exist and $f(t)$ is continuous in $[0,\infty)$. By the derivative theorem, we have

$$L[f'(t)] = sF(s) - f(0+), \quad s > 0,$$

Taking limit as $s \to 0$, we obtain

$$\lim_{s\to 0}\int_0^\infty e^{-st} f^{'}(t)\,\mathrm{d}t = \lim_{s\to 0} sF(s) - f(0)$$

Taking the limit inside the integral

$$\int_0^\infty f^{'}(t)\,\mathrm{d}t = \lim_{s\to 0} sF(s) - f(0)$$

On integrating we obtain

$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

Cancellation of $f(0)$ gives the desired results. ■

Remark 2: *In the final value theorem, existence of* $\lim_{t\to\infty} f(t)$ *is very important. Consider* $f(t) = \sin t$*. Then* $\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{1+s^2} = 0$*. But* $\lim_{t\to\infty} f(t)$ *does not exist. Thus we may say that if* $\lim_{s\to 0} sF(s) = L$ *exists then either* $\lim_{t\to\infty} f(t) = L$ *or this limit does not exist.*

39.3.3 Example

Without determining $f(t)$ *and assuming that* $f(t)$ *satisfies the hypothesis of the limiting theorems, compute*

$$\lim_{t\to 0+} f(t) \text{ and } \lim_{t\to\infty} f(t) \quad \text{if} \quad L[f(t)] = \frac{1}{s} + \tan^{-1}\left(\frac{a}{s}\right).$$

Solution: By initial value theorem, we get

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[1 + s\tan^{-1}\left(\frac{a}{s}\right)\right]$$

Application of L'hospital rule gives

$$\lim_{t\to 0+} f(t) = 1 + \lim_{s\to\infty} \frac{\frac{s^2}{s^2+a^2}\left(\frac{-a}{s^2}\right)}{-\frac{1}{s^2}} = 1 + a$$

Using the final value theorem we find

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1 + s\tan^{-1}\frac{a}{s}\right] = 1.$$

Remark 3: *Final value theorem says* $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$, *if* $\lim_{t\to\infty} f(t)$ *exists. If* $F(s)$ *is finite as* $s \to 0$ *then trivially* $\lim_{t\to\infty} f(t) = 0$. *However, there are several functions whose Laplace transform is not finite as* $s \to 0$, *for example,* $f(t) = 1$ *and its Laplace transform* $F(s)$ *is equal to* $\frac{1}{s}$, $s > 0$. *In this case we have* $\lim_{s\to 0} sF(s) = \lim_{s\to 0} 1 = 1 = \lim_{t\to\infty} f(t)$.

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Lesson 40

Inverse Laplace Transform

In this lesson we introduce the concept of inverse Laplace transform and discuss some of its important properties that will be helpful to evaluate inverse Transform of some complicated functions. As mention in the beginning of this module that the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the transformed domain we will like to get back to the time domain and therefore we need to introduce the concept of inverse Laplace transform.

40.1 Inverse Laplace Transform

If $F(s) = L[f(t)]$ for some function $f(t)$. We define the *inverse Laplace transform* as

$$L^{-1}[F(s)] = f(t).$$

There is an integral formula for the inverse, but it is not as simple as the transform itself as it requires complex numbers and path integrals. The easiest way of computing the inverse is using table of Laplace transform. For example,

$$L[\sin wt] = \frac{w}{s^2 + w^2}$$

This implies

$$L^{-1}\left[\frac{w}{s^2 + w^2}\right] = \sin wt, \ t \geq 0$$

and similarly

$$L[\cos wt] = \frac{s}{s^2 + w^2} \quad \Rightarrow \quad L^{-1}\left[\frac{s}{s^2 + w^2}\right] = \cos wt, \ t \geq 0$$

40.2 Uniqueness of Inverse Laplace Transform

If we have a function $F(s)$, to be able to find $f(t)$ such that $L[f(t)] = F(s)$, we need to first know if such a function is unique.

Consider

$$g(t) = \begin{cases} 1 & \text{when } t = 1 \\ \sin(t) & \text{when otherwise} \end{cases}$$

$$L[g(t)] = \frac{1}{s^2+1} = L[\sin t]$$

Thus we have two different functions $g(t)$ and $\sin t$ whose Laplace transform are same. However note that the given two functions are different at a point of discontinuity. Thanks to the following theorem where we have uniqueness for continuous functions:

40.2.1 Theorem (Lerch's Theorem)

If f and g are continuous and are of exponential order, and if $F(s) = G(s)$ for all $s > s_0$ then $f(t) = g(t)$ for all $t > 0$.

Proof: If $F(s) = G(s)$ for all $s > s_0$ then,

$$\int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^\infty e^{-st} g(t)\mathrm{d}t, \ \ \forall s > s_0$$

$$\Rightarrow \int_0^\infty e^{-st}[f(t) - g(t)]\mathrm{d}t = 0, \ \ \forall s > s_0$$

$$\Rightarrow f(t) - g(t) \equiv 0, \ \ \forall t > t_0.$$

$$\Rightarrow f(t) = g(t), \ \ \forall t > t_0.$$

This completes the proof. ■

Remark: *The uniqueness theorem holds for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points where it has jump discontinuities like the Heaviside function or the function $g(t)$ defined above. Since the Laplace integral however does not "see" values at the discontinuities. So in this case we can only conclude that $f(t) = g(t)$ outside of discontinuities.*

We now state some important properties of the inverse Laplace transform. Though, these properties are the same as we have listed for the Laplace transform, we repeat them without proof for the sake of completeness and apply them to evaluate inverse Laplace transform of some functions.

40.3 Linearity of Inverse Laplace Transform

If $F_1(s)$ and $F_2(s)$ are the Laplace transforms of the function $f_1(t)$ and $f_2(t)$ respectively, then

$$L^{-1}[a_1F_1(s) + a_2F_2(s)] = a_1L^{-1}[F_1(s)] + L^{-1}[F_2(s)] = a_1f_1(t) + a_2f_2(t)$$

where a_1 and a_2 are constants.

40.4 Example Problems

40.4.1 Problem 1

Find the inverse Laplace transform of

$$F(s) = \frac{6}{2s-3} + \frac{8-6s}{16s^2+9}$$

Solution: Using linearity of the inverse Laplace transform we have

$$f(t) = 6L^{-1}\left[\frac{1}{2s-3}\right] + 8L^{-1}\left[\frac{1}{16s^2+9}\right] - 6L^{-1}\left[\frac{s}{16s^2+9}\right]$$

Rewriting the above expression as

$$f(t) = 3L^{-1}\left[\frac{1}{s-(3/2)}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2+(9/16)}\right] - \frac{3}{8}L^{-1}\left[\frac{s}{s^2+(9/16)}\right]$$

Using the result

$$L\left[\frac{1}{s-a}\right] = e^{as}$$

and taking the inverse transform we obtain

$$f(t) = 3e^{3t/2} + \frac{2}{3}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}.$$

40.4.2 Problem 2

Find the inverse Laplace transform of

$$F(s) = \frac{s^2+s+1}{s^3+s}$$

Solution: We use the method of partial fractions to write F in a form where we can use the table of Laplace transform. We factor the denominator as $s(s^2+1)$ and write

$$\frac{s^2+s+1}{s^3+s} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Putting the right hand side over a common denominator and equating the numerators we get $A(s^2+1)+s(Bs+C) = s^2+s+1$. Expanding and equating coefficients we obtain $A+B=1$, $C=1$, $A=1$, and thus $B=0$. In other words,

$$F(s) = \frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}.$$

By linearity of the inverse Laplace transform we get

$$L^{-1}\left[\frac{s^2+s+1}{s^3+s}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] = 1+\sin t.$$

40.5 First Shifting Property of Inverse Laplace Transform

If $L^{-1}[F(s)] = f(t)$, *then* $L^{-1}[F(s-a)] = e^{at}f(t)$

40.6 Example Problems

40.6.1 Problem 1

Evaluate $L^{-1}\left[\frac{1}{(s+1)^2}\right]$

Solution: Rewriting the given expression as

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = L^{-1}\left[\frac{1}{(s-(-1))^2}\right]$$

Applying the first shifting property of the inverse Laplace transform

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$

Thus we obtain

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = te^{-t}.$$

40.6.2 Problem 2

Find $L^{-1}\left[\dfrac{1}{s^2+4s+8}\right]$.

Solution: First we complete the square to make the denominator $(s+2)^2+4$. Next we find

$$L^{-1}\left[\frac{1}{s^2+4}\right]=\frac{1}{2}\sin(2t).$$

Putting it all together with the shifting property, we find

$$L^{-1}\left[\frac{1}{s^2+4s+8}\right]=L^{-1}\left[\frac{1}{(s+2)^2+4}\right]=\frac{1}{2}e^{-2t}\sin(2t).$$

40.7 Second Shifting Property of Inverse Laplace Transform

If $L^{-1}[F(s)]=f(t),$ *then* $L^{-1}\left[e^{-as}f(s)\right]=f(t-a)\,H(t-a)$

40.8 Example Problems

40.8.1 Problem 1

Find the inverse Laplace transform of

$$F(s)=\frac{e^{-s}}{s(s^2+1)}$$

Solution: First we compute the inverse Laplace transform

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]=L^{-1}\left[\frac{1}{s}-\frac{s}{(s^2+1)}\right]$$

Using linearity of the inverse transform we get

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]=L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{s}{(s^2+1)}\right]=1-\cos t$$

We now find

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right]=L^{-1}\left[e^{-s}L[1-\cos t]\right]$$

Using the second shifting theorem we obtain

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = [1-\cos(t-1)]\, H(t-1).$$

40.8.2 Problem 2

Find the inverse Laplace transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+4} + \frac{e^{-2s}}{s^2+4} + \frac{e^{-3s}}{(s+2)^2}$$

Solution: First we find that

$$L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t$$

and using the first shifting property

$$L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}$$

By linearity we have

$$f(t) = L^{-1}\left[\frac{e^{-s}}{s^2+4}\right] + L^{-1}\left[\frac{e^{-2s}}{s^2+4}\right] + L^{-1}\left[\frac{e^{-3s}}{(s+2)^2}\right]$$

Putting it all together and using the second shifting theorem we get

$$f(t) = \frac{1}{2}\sin 2(t-1)\, H(t-1) + \frac{1}{2}\sin 2(t-2)\, H(t-2) + e^{-2(t-3)}\, H(t-3)$$

Suggested Readings

Debnath, L. and Bhatta, D. (2007). Integral Transforms and Their Applications. Second Edition. Chapman and Hall/CRC (Taylor and Francis Group). New York.

Grewal, B.S. (2007). Higher Engineering Mathematics. Fourteenth Edition. Khanna Publishers, New Delhi.

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Lesson 41

Properties of Inverse Laplace Transform

We shall continue discussing various properties of inverse Laplace transform. We mainly cover change of scale property, inverse Laplace transform of integrals and derivatives etc.

41.1 Change of Scale Property

If $L^{-1}[F(s)] = f(t)$ *then* $L^{-1}[F(as)] = \frac{1}{a}F\left(\frac{t}{a}\right)$

41.1.1 Example

If

$$L^{-1}\left[\frac{s}{s^2-16}\right] = \cosh 4t,$$

then find

$$L^{-1}\left[\frac{s}{2s^2-8}\right]$$

Solution: Given that

$$L^{-1}\left[\frac{s}{s^2-16}\right] = \cosh 4t$$

Replacing s by $2s$ and using scaling property we find

$$L^{-1}\left[\frac{2s}{4s^2-16}\right] = \frac{1}{2}\cosh 2t$$

Thus, we obtain

$$L^{-1}\left[\frac{s}{2s^2-8}\right] = \frac{1}{2}\cosh 2t$$

41.2 Inverse Laplace Transform of Derivatives (Derivative Theorem)

If $L^{-1}[F(s)] = f(t)$ *then* $L^{-1}\left[\frac{d^n}{ds^n}f(s)\right] = (-1)^n t^n f(t), \quad n = 1, 2 \dots$

41.2.1 Example

Find the Laplace transform of

$$(i)\ \frac{2as}{(s^2+a^2)^2} \qquad (ii)\ \frac{s^2-a^2}{(s^2+a^2)^2}$$

Solution: Note that

$$\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{-2as}{(s^2+a^2)^2} \text{ and } \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{a^2-s^2}{(s^2+a^2)^2}$$

Direct application of the derivative theorem we obtain

$$(i) \qquad L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right] = (-1)\, t\, L^{-1}\left[-\frac{a}{s^2+a^2}\right] = t \sin at$$

and

$$(ii) \qquad L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = (-1)\, t\, L^{-1}\left[-\frac{s}{s^2+a^2}\right] = t \cos at$$

41.3 Inverse Laplace Transform of Integrals

If $L^{-1}[F(s)] = f(t)$ *then* $L^{-1}\left[\int_s^\infty f(s)\, ds\right] = \frac{f(t)}{t}$

41.3.1 Example

Find the inverse Laplace transform $f(t)$ *of the function*

$$\int_s^\infty \frac{1}{s(s+1)}\, ds$$

Solution: By the method of partial fraction we obtain

$$L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] = 1 - e^{-t}.$$

Using the inverse Laplace transform of integrals we get

$$L^{-1}\left[\int_s^\infty \frac{1}{s(s+1)}\right] = \frac{1-e^{-t}}{t}.$$

41.4 Multiplication by Powers of s

If $L^{-1}[F(s)] = f(t)$ *and* $f(0) = 0$, *then* $L^{-1}[sF(s)] = f'(t)$

41.4.1 Example

Using $L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$, *and with the application of above result compute* $L^{-1}\left[\frac{s}{s^2+1}\right]$.

Solution: Direct application of the above result leads to

$$L^{-1}\left[\frac{s}{s^2+1}\right] = \frac{d}{dt}\sin t = \cos t.$$

41.5 Division by Powers of s

Let $L^{-1}[F(s)] = f(t)$. *if* $f(t)$ *is piecewise continuous and of exponential order* α *such that* $\lim_{t\to 0}\frac{f(t)}{t}$ *exists, then*

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)\,du.$$

41.6 Example Problems

41.6.1 Problem 1

Compute

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]$$

Solution: we could proceed by applying this integration rule.

$$L^{-1}\left[\frac{1}{s}\frac{1}{s^2+1}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+1}\right]\ \mathrm{d}u = \int_0^t \sin\tau\ \mathrm{d}u = 1 - \cos t.$$

41.6.2 Problem 2

Find inverse Laplace transform of $\dfrac{1}{(s^2+1)^2}$

Solution: We know that

$$L^{-1}\left[\frac{s}{(1+s^2)^2}\right]=\frac{1}{2}t\sin t.$$

We now apply the above result as

$$L^{-1}\left[\frac{1}{(1+s^2)^2}\right]=L^{-1}\left[\frac{1}{s}\frac{s}{(1+s^2)^2}\right]=\frac{1}{2}\int_0^t t\sin t\,\mathrm{d}t.$$

Evaluating the above integral we get

$$L^{-1}\left[\frac{1}{(1+s^2)^2}\right]=\frac{1}{2}(-t\cos t+\sin t).$$

41.6.3 Problem 3

Find inverse Laplace transform of $\dfrac{s-1}{s^2(s^2+1)}$.

Solution: It is easy to compute

$$L^{-1}\left[\frac{s-1}{(s^2+1)}\right]=L^{-1}\left[\frac{s}{(s^2+1)}\right]-L^{-1}\left[\frac{1}{(s^2+1)}\right]=\cos t-\sin t.$$

Now repeated application of the above result we get

$$L^{-1}\left[\frac{s-1}{s(s^2+1)}\right]=\int_0^t(\cos t-\sin t)\,\mathrm{d}t=\sin t+\cos t-1.$$

Finally, we obtain the desired transform as

$$L^{-1}\left[\frac{s-1}{s^2(s^2+1)}\right]=\int_0^t(\sin t+\cos t-1)\,\mathrm{d}t=1-t+\sin t-\cos t.$$

41.7 Evaluation of Integrals

With the application of Laplace and inverse Laplace transform we can also compute some complicated integrals.

41.8 Example Problems

41.8.1 Problem 1

Evaluate $\displaystyle\int_0^\infty \frac{\cos tx}{x^2+1}\,\mathrm{d}x, \quad t>0.$

Solution: Let

$$f(t)=\int_0^\infty \frac{\cos tx}{x^2+1}\,\mathrm{d}x.$$

Taking Laplace transform on both sides,

$$\begin{aligned}L[f(t)] &= \int_0^\infty \frac{s}{(x^2+1)(s^2+x^2)}\,\mathrm{d}x\\ &= \frac{s}{s^2+1}\int_0^\infty \left(\frac{1}{x^2+1}-\frac{1}{s^2+x^2}\right)\mathrm{d}x\\ &= \frac{s}{s^2-1}\left[\tan^{-1}x-\frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)\right]_0^\infty\\ &= \frac{s}{s^2-1}\left(\frac{\pi}{2}-\frac{\pi}{2s}\right)=\frac{\pi}{2}\frac{1}{s+1}.\end{aligned}$$

Taking inverse Laplace transform on both sides,

$$f(t)=\frac{\pi}{2}e^{-t}.$$

41.8.2 Problem 2

Evaluate $\displaystyle\int_0^\infty e^{-x^2}\mathrm{d}x.$

Solution: Let

$$g(t)=\int_0^\infty e^{-tx^2}\,\mathrm{d}x$$

Now taking Laplace on both sides,

$$L[g(t)]=\int_0^\infty \frac{1}{s+x^2}\,\mathrm{d}x=\frac{1}{\sqrt{s}}\arctan\left(\frac{x}{\sqrt{s}}\right)\Big|_0^\infty=\frac{1}{\sqrt{s}}\frac{\pi}{2}$$

Taking inverse Laplace transform we obtain

$$g(t)=\frac{\pi}{2}L^{-1}\left[\frac{1}{\sqrt{s}}\right]=\frac{\pi}{2}\frac{1}{\sqrt{\pi}\sqrt{t}}.$$

Hence for $t=1$ we get

$$\int_0^\infty e^{-x^2}\mathrm{d}x=\frac{\sqrt{\pi}}{2}.$$

Remark: *Theoretical results on applicability of Laplace transform for evaluating of integrals and evaluation of some more integrals will be further elaborated in one of the next lessons.*

Suggested Readings

Arfken, G.B., Weber, H.J. and Harris, F.E. (2012). Mathematical Methods for Physicists (A comprehensive guide), Seventh Edition, Elsevier Academic Press, New Delhi.

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Lesson 42

Convolution for Laplace Transform

In this lesson we introduce the convolution property of the Laplace transform. We shall start with the definition of convolution followed by an important theorem on Laplace transform of convolution. Convolution theorem plays an important role for finding inverse Laplace transform of complicated functions and therefore very useful for solving differential equations.

42.1 Convolution

The convolution of two given functions $f(t)$ and $g(t)$ is written as $f * g$ and is defined by the integral

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\, d\tau. \tag{42.1}$$

As you can see, the convolution of two functions of t is another function of t.

42.2 Example Problems

42.2.1 Problem 1

Find the convolution of $f(t) = e^t$ *and* $g(t) = t$ *for* $t \geq 0$.

Solution: By the definition we have

$$(f * g)(t) = \int_0^t e^{\tau}(t-\tau)\, d\tau$$

Integrating by parts, we obtain

$$(f * g)(t) = e^t - t - 1.$$

42.2.2 Problem 2

Find the convolution of $f(t) = \sin(\omega t)$ *and* $g(t) = \cos(\omega t)$ *for* $t \geq 0$.

Solution: By the definition of convolution we have

$$(f * g)(t) = \int_0^t \sin(\omega\tau)\,\cos\big(\omega(t-\tau)\big)\,d\tau.$$

We apply the identity $\cos(\theta)\sin(\psi) = \frac{1}{2}\big(\sin(\theta+\psi) - \sin(\theta-\psi)\big)$ to get

$$(f * g)(t) = \int_0^t \frac{1}{2}\,\big(\sin(\omega t) - \sin(\omega t - 2\omega\tau)\big)\,d\tau$$

On integration we obtain

$$(f * g)(t) = \left[\frac{1}{2}\tau\sin(\omega t) + \frac{1}{4\omega}\,\cos(2\omega\tau - \omega t)\right]_{\tau=0}^{t} = \frac{1}{2}\,t\sin(\omega t).$$

The formula holds only for $t \geq 0$. We assumed that f and g are zero (or simply not defined) for negative t.

42.3 Properties of Convolution

The convolution has many properties that make it behave like a product. Let c be a constant and f, g, and h be functions, then

(i) $f * g = g * f,$ [symmetry]

(ii) $c(f * g) = cf * g = f * cg,$ [c=constant]

(iii) $f * (g * h) = (f * g) * h,$ [associative property]

(iv) $f * (g + h) = f * g + f * h,$ [distributive property]

Proof: We give proof of (i) and all others can be done similarly. By the definition of convolution we have

$$f * g = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$$

Substituting $t - \tau = u \Rightarrow -d\tau = du$ we get

$$f * g = -\int_t^0 f(t-u)g(u)\mathrm{d}u = \int_0^t f(t-u)g(u)\mathrm{d}u = g * f$$

This completes the proof. ■

The most interesting property for us, and the main result of this lesson is the following theorem.

42.4 Convolution Theorem

If f and g are piecewise continuous on $[0,\infty)$ and of exponential order α, then

$$L[(f*g)(t)] = L[f(t)]L[g(t)].$$

Proof: From the definition,

$$L[(f*g)(t)] = \int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\mathrm{d}t,\ \ [\text{Re(s)}> \alpha\ ]$$

Changing the order of integration,

$$L[(f*g)(t)] = \int_0^\infty \int_0^t e^{-st} f(\tau)g(t-\tau)\mathrm{d}t\mathrm{d}\tau,$$

We now put $t-\tau = u \Rightarrow -d\tau = du$ and get,

$$\begin{aligned} L[(f*g)(t)] &= \int_0^\infty \int_0^\infty e^{-s(u+\tau)} f(\tau)g(u)\mathrm{d}u\mathrm{d}\tau \\ &= \int_0^\infty e^{-s\tau)} f(\tau)\mathrm{d}\tau \int_0^\infty e^{-su} g(u)\mathrm{d}u \\ &= L[f(t)]L[g(t)] \end{aligned}$$

This completes the proof. ■

In other words, the Laplace transform of a convolution is the product of the Laplace transforms. The simplest way to use this result is in reverse, i.e., to find inverse Laplace transform.

42.5 Example Problems

42.5.1 Problem 1

Find the inverse Laplace transform of the function of s defined by

$$\frac{1}{(s+1)s^2} = \frac{1}{s+1}\frac{1}{s^2}.$$

Solution: We recognize the two elementary entries

$$L^{-1}\left[\frac{1}{s+1}\right]=e^{-t} \qquad \text{and} \qquad L^{-1}\left[\frac{1}{s^2}\right]=t.$$

Therefore,

$$L^{-1}\left[\frac{1}{s+1}\,\frac{1}{s^2}\right]=\int_0^t \tau e^{-(t-\tau)}\,d\tau$$

On integration by parts we obtain

$$L^{-1}\left[\frac{1}{s+1}\,\frac{1}{s^2}\right]=e^{-t}+t-1.$$

42.5.2 Problem 2

Use the convolution theorem to evaluate

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right].$$

Solution: Note that

$$L[\sin t]=\frac{1}{s^2+1} \qquad \text{and} \qquad L[\cos t]=\frac{s}{s^2+1}$$

Using convolution theorem,

$$L[\sin t*\cos t]=L[\sin t]L[\cos t]=\frac{s}{(s^2+1)^2}.$$

Therefore, we have

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right]=\int_0^t \sin\tau\cos(t-\tau)\mathrm{d}\tau.$$

Using the trigonometric equality $2\sin A\cos B=\sin(A+B)+\sin(A-B)$ we get

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right]=\frac{1}{2}\int_0^t [\sin t+\sin(2\tau-t)]\mathrm{d}\tau.$$

On integration we find

$$\begin{aligned}L^{-1}\left[\frac{s}{(s^2+1)^2}\right]&=\frac{1}{2}t\sin t+\frac{1}{2}\left[-\frac{\cos(2\tau-t)}{2}\right]_0^t\\&=\frac{1}{2}t\sin t\frac{1}{4}[\cos t-\cos t].\end{aligned}$$

Finally we have the following result

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2}t\sin t.$$

42.5.3 Problem 3

Use convolution theorem to evaluate

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right]$$

Solution: We know the following elementary transforms

$$L\left[\frac{1}{\sqrt{t}}\right] = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \Rightarrow L^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{t\pi}}$$

and

$$L^{-1}\left[\frac{1}{s-1}\right] = e^t.$$

Then by the convolution theorem, we find

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = \frac{1}{\sqrt{t\pi}} * e^t = \int_0^t \frac{1}{\sqrt{t\pi}} e^{t-\tau}\mathrm{d}\tau.$$

Substitution $u = \sqrt{\tau} \Rightarrow du = \dfrac{1}{2\sqrt{\tau}}d\tau$ gives

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = \frac{e^t}{\sqrt{\pi}}\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\mathrm{d}\tau = 2\frac{e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\mathrm{d}u.$$

Thus, we have

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = e^t\mathrm{erf}(\sqrt{t}).$$

42.5.4 Problem 4

Use convolution theorem to evaluate

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right].$$

Solution: We know

$$L^{-1}\left[\frac{1}{s^3}\right]=\frac{t^2}{2} \text{ and } L^{-1}\left[\frac{1}{s^2+1}\right]=\sin t.$$

By the convolution theorem we have

$$\begin{aligned}
L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] &= \frac{1}{2}t^2 * \sin t = \frac{1}{2}\int_0^t \sin\tau\,(t-\tau)^2\,\mathrm{d}\tau \\
&= \frac{1}{2}\left[\left(-\cos\tau(t-\tau)^2\right)\Big|_0^t - 2\int_0^t (t-\tau)\cos\tau\mathrm{d}\tau\right] \\
&= \frac{1}{2}\left[t^2 - 2\left((t-\tau)\sin\tau\right)\Big|_0^t + \int_0^t \sin\tau\mathrm{d}\tau\right].
\end{aligned}$$

Finally we get the desired inverse Laplace transform as

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]=\frac{t^2}{2}+\cos t - 1.$$

Suggested Readings

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Lesson 43

Laplace Transform of Some Special Functions

In this lesson we discuss Laplace transform of some special functions like error functions, Dirac delta functions, etc. There functions appears in various applications of science and engineering to some of them we shall encounter while solving differential equations using Laplace transform.

43.1 Error Function

The error appears in probability, statistics and solutions of some partial differential equations. It is defined as

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, \mathrm{d}u$$

Its complement, known as complementary error function, is defined as

$$\text{erfc}(t) = 1 - \text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2}\, \mathrm{d}u$$

We find Laplace transform of different forms of error function in the following examples.

43.2 Example Problems

43.2.1 Problem 1

Find $L[erf(\sqrt{t})]$.

Solution: From definition of the error function and the Laplace transform we have,

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^{\sqrt{t}} e^{-st} e^{-x^2} \mathrm{d}x \mathrm{d}t$$

By changing the order of integration we get,

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_{x=0}^\infty \int_{t=x^2}^\infty e^{-st} e^{-x^2} \mathrm{d}t \mathrm{d}x$$

Evaluating the inner integral we obtain

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_{x=0}^{\infty} e^{-x^2} \frac{e^{-sx^2}}{s} \mathrm{d}x = \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_{x=0}^{\infty} e^{-(1+s)x^2} \mathrm{d}x$$

Substituting $\sqrt{(1+s)}x = u \Rightarrow dx = \dfrac{1}{\sqrt{1+s}} du$

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \frac{1}{s\sqrt{1+s}} \int_{x=0}^{\infty} e^{-u^2} \mathrm{d}u = \frac{1}{s\sqrt{s+1}}$$

Note that we have used the value of Gaussian integral $\displaystyle\int_{x=0}^{\infty} e^{-u^2} \mathrm{d}u = \frac{\sqrt{\pi}}{2}$.

43.2.2 Problem 2

Find $L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right]$. *and show that* $L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right] = \text{erfc}\left(\frac{k}{\sqrt{t}}\right)$

Solution: By the definition of Laplace transform we have

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \int_0^{\infty} e^{-st} \frac{2}{\sqrt{\pi}} \int_0^{\frac{k}{\sqrt{t}}} e^{-u^2} \mathrm{d}u \, \mathrm{d}t$$

Changing the order of integration we get

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_0^{\frac{k^2}{u^2}} e^{-st} e^{-u^2} \mathrm{d}t \, \mathrm{d}u$$

Evaluation of the inner integral leads to

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_0^{\infty} e^{-u^2} \left(1 - e^{-s\frac{k^2}{u^2}}\right) \mathrm{d}u$$

Using the value of Gaussian integral we have

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}} \frac{1}{s} \left[\frac{\sqrt{\pi}}{2} - \int_0^{\infty} \left(e^{-u^2 - s\frac{k^2}{u^2}}\right) \mathrm{d}u\right] \tag{43.1}$$

Let us assume

$$I(s) = \int_0^{\infty} e^{-u^2 - s\frac{k^2}{u^2}} \mathrm{d}u$$

By differentiation under integral sign

$$\frac{dI}{ds} = \int_0^{\infty} e^{-u^2 - s\frac{k^2}{u^2}} \left(-\frac{k^2}{u^2}\right) \mathrm{d}u$$

Substitution $\frac{\sqrt{s}k}{u} = x \Rightarrow -\frac{\sqrt{s}k}{u^2}du = dx$ leads to

$$\frac{dI}{ds} = -\frac{k}{\sqrt{s}}\int_0^\infty e^{-x^2 - s\frac{k^2}{x^2}}\mathrm{d}x = -\frac{k}{\sqrt{s}}I$$

Solving the above differential equation we get

$$\ln I(s) = -2k\sqrt{s} + \ln c \quad \Rightarrow \quad I(s) = ce^{-2k\sqrt{s}}$$

Further note that

$$I(0) = \int_0^\infty e^{-u^2}\mathrm{d}u = \frac{\sqrt{\pi}}{2} \quad \Rightarrow \quad c = \frac{\sqrt{\pi}}{2}$$

Therefore, we get

$$I(s) = \frac{\sqrt{\pi}}{2}e^{-2k\sqrt{s}}$$

Substituting this value in the equation (43.1), we obtain

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{s\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}e^{-2k\sqrt{s}}\right] = \frac{1-e^{-2k\sqrt{s}}}{s}$$

Taking inverse Laplace transform on both sides we get

$$\text{erf}\left(\frac{k}{\sqrt{t}}\right) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right] = 1 - L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right]$$

This leads to the desired result as

$$L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right] = 1 - \text{erf}\left(\frac{k}{\sqrt{t}}\right) = \text{erf}_c\left(\frac{k}{\sqrt{t}}\right)$$

43.3 Dirac-Delta Function

Often in applications we study a physical system by putting in a short pulse and then seeing what the system does. The resulting behaviour is often called *impulse response*. Let us see what we mean by a pulse. The simplest kind of a pulse is a simple rectangular pulse defined by

$$\varphi_\epsilon^a(t) = \begin{cases} 0 & \text{if } t < a, \\ 1/\epsilon & \text{if } a \le t < a+\epsilon, \\ 0 & \text{if } a+\epsilon \le t. \end{cases}$$

Let us take the Laplace transform of a square pulse,

$$L\left[\varphi_\epsilon^a(t)\right] = \int_0^\infty e^{-st}\varphi_\epsilon(t)\,\mathrm{d}t$$

Substituting the value of the function we obatin

$$L\left[\varphi_\epsilon^a(t)\right] = \frac{1}{\epsilon}\int_a^{a+\epsilon} e^{-st}\,\mathrm{d}t$$

On integration we get

$$L\left[\varphi_\epsilon^a(t)\right] = \frac{e^{-sa}}{s\epsilon}\left[1 - e^{-s\epsilon}\right]$$

We generally want ϵ to be very small. That is, we wish to have the pulse be very short and very tall. By letting ϵ go to zero we arrive at the concept of the *Dirac delta function*, $\delta(t-a)$. Thus, the Dirac-Delta can be thought as the limiting case of $\varphi_\epsilon(t)$ as $\epsilon \to 0$

$$\delta(t-a) = \lim_{\epsilon\to 0}\varphi_\epsilon^a(t)$$

So $\delta(t)$ is a "function" with all its "mass" at the single point $t = 0$. In other words, the Dirac-delta function is defined as having the following properties:

(i) $\delta(t-a) = 0, \quad \forall t,\ t \neq a$

(ii) for any interval $[c, d]$

$$\int_c^d \delta(t-a)\,dt = \begin{cases} 1 & \text{if the interval } [c,d] \text{ contains } a, \text{ i.e. } c \le a \le d, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) for any interval $[c, d]$

$$\int_c^d \delta(t-a)f(x)\,dt = \begin{cases} f(a) & \text{if the interval } [c,d] \text{ contains } a, \text{ i.e. } c \le a \le d, \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately there is no such function in the classical sense. You could informally think that $\delta(t)$ is zero for $t \neq 0$ and somehow infinite at $t = 0$.

As we can integrate $\delta(t)$, let us compute its Laplace transform.

$$L\left[\delta(t-a)\right] = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}$$

In particular,

$$L\left[\delta(t)\right] = 1.$$

Remark: *Notice that the Laplace transform of $\delta(t-a)$ looks like the Laplace transform of the derivative of the Heaviside function $u(t-a)$, if we could differentiate the Heaviside function. First notice*

$$\mathcal{L}\big[u(t-a)\big] = \frac{e^{-as}}{s}.$$

To obtain what the Laplace transform of the derivative would be we multiply by s, to obtain e^{-as}, which is the Laplace transform of $\delta(t-a)$. We see the same thing using integration,

$$\int_0^t \delta(s-a)\, ds = u(t-a).$$

So in a certain sense

$$\text{"}\ \frac{d}{dt}\big[u(t-a)\big] = \delta(t-a)\ \text{"}$$

This line of reasoning allows us to talk about derivatives of functions with jump discontinuities. We can think of the derivative of the Heaviside function $u(t-a)$ as being somehow infinite at a, which is precisely our intuitive understanding of the delta function.

43.3.1 Example

Compute $L^{-1}\left[\frac{s+1}{s}\right]$.

Solution: We write,

$$L^{-1}\left[\frac{s+1}{s}\right] = L^{-1}\left[1+\frac{1}{s}\right] = L^{-1}[1] + L^{-1}\left[\frac{1}{s}\right] = \delta(t) + 1.$$

The resulting object is a generalized function which makes sense only when put under an integral.

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Lesson 44

Laplace and Inverse Laplace Transform: Miscellaneous Examples

In this lesson we evaluate Laplace and inverse Laplace transforms of some useful functions. Some important special functions include Bessel's functions and Laguerre polynomial. Additionally, some examples demonstrating potential of Laplace and inverse Laplace transform for evaluating special integrals will be presented.

44.1 Bessel's Functions

The Bessel's functions of order n (of first kind) is defined as

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!}\left(\frac{t}{2}\right)^{n+2r}.$$

This Bessel's function is a solution of the Bessel's equation of order n

$$y^n + \frac{1}{t}y' + \left(1 - \frac{n^2}{t^2}\right)y = 0$$

The Bessel's functions of order 0 and 1 are given as

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \dots$$

and

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^2 4} + \frac{t^5}{2^2 4^2 6} + \dots$$

Note that $J_0'(t) = -J_1(t)$.

44.1.1 Example

Find the Laplace transform of $J_0(t)$ and $J_1(t)$.

Solution: Taking Laplace transform of the $J_0(t)$ we have

$$L[J_0(t)] = L\left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \dots\right]$$

Using linearity of the Laplace transform we get

$$L[J_0(t)] = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2 4^2}\frac{4!}{s^5} - \frac{1}{2^2 4^2 6^2}\frac{6!}{s^7} + \dots$$

This can be rewritten as

$$L[J_0(t)] = \frac{1}{s}\left[1 - \frac{1}{2}\frac{1}{s^2} + \frac{1}{2}\frac{3}{4}\frac{1}{s^4} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{s^6} + \dots\right]$$

With Binomial expansion we can write

$$L[J_0(t)] = \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2} = \frac{1}{\sqrt{1+s^2}}$$

Further note that $L[J_1(t)] = -L[J_0'(t)]$ and therefore using the derivative theorem we find

$$L[J_1(t)] = -sL[J_0(t)] + J_0(0)] = 1 - sL[J_0(t)], \text{ since } J_0(0) = 1$$

Hence, we obtain

$$L[J_1(t)] = 1 - \frac{s}{\sqrt{1+s^2}}$$

44.2 Laguerre Polynomials

Laguerre polynomials are defined as

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right), \; n = 0, 1, 2, \dots$$

The Laguerre polynomials are solutions of Laguerre's differential equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0, \; n = 0, 1, 2, \dots$$

44.2.1 Example

Show that $L[L_n(t)] = \dfrac{(s-1)^n}{s^{n+1}}$

Solution: By definition of the Laplace transform we have

$$\begin{aligned} L[L_n(t)] &= \int_0^\infty e^{-st}\frac{e^t}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right)\,\mathrm{d}t \\ &= \frac{1}{n!}\int_0^\infty e^{-(s-1)t}\frac{d^n}{dt^n}\left(e^{-t}t^n\right)\,\mathrm{d}t \end{aligned}$$

Integrating by parts, we find

$$L[L_n(t)] = \frac{1}{n!}\left[e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\Big|_0^\infty + (s-1)\int_0^\infty e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\,\mathrm{d}t\right]$$

Noting that each term in $\frac{d^{n-1}}{dt^{n-1}}$ contains some integral power of t so that it vanishes as $t \to 0$ and $e^{-(s-1)t}$ vanishes for $t \to \infty$ provided $s > 1$. Thus, we have

$$L[L_n(t)] = \frac{s-1}{n!}\left[\int_0^\infty e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\,\mathrm{d}t\right]$$

Repeated use of integration by parts leads to

$$L[L_n(t)] = \frac{(s-1)^n}{n!}\left[\int_0^\infty e^{-(s-1)t}e^{-t}t^n\,\mathrm{d}t\right] = \frac{(s-1)^n}{n!}L[t^n]$$

Hence, we get

$$L[L_n(t)] = \frac{(s-1)^n}{n!}\frac{n!}{s^{n+1}} = \frac{(s-1)^n}{s^{n+1}}$$

44.3 Miscellaneous Example Problems

44.3.1 Problem 1

Using the convolution theorem prove that

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\mathrm{d}u = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad [m,n>0].$$

Solution: Let $f(t) = t^{m-1},\; g(t) = t^{n-1}$, then

$$(f*g)(t) = \int_0^t \tau^{m-1}(t-\tau)^{n-1}\mathrm{d}\tau,$$

Substituting $\tau = ut$ so that $\mathrm{d}\tau = t\,\mathrm{d}u$ we obtain

$$(f*g)(t) = \int_0^1 t^{m-1}u^{m-1}t^{n-1}(1-u)^{n-1}t\mathrm{d}u$$

We simplify the above expression to get

$$(f*g)(t) = t^{m+n-1}\int_0^1 u^{m-1}(1-u)^{n-1}\mathrm{d}u = t^{m+n-1}B(m,n)$$

Taking Laplace transform and using convolution property, we find

$$L[t^{m+n-1}B(m,n)] = L[f(t)] * L[g(t)] = L[t^{m-1}] * L[t^{n-1}] = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}$$

Taking inverse Laplace transform,

$$t^{m+n-1}B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} t^{m+n-1}$$

Hence, we get the desired result as

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

44.3.2 Problem 2

Show that

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}.$$

Solution: We know

$$L[\sin t] = \frac{1}{s^2+1}$$

Therefore, we get

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s = \frac{\pi}{2} - \tan^{-1} s.$$

Taking limit as $s \to 0$ (see remarks below for details) we find

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2}.$$

Remark 1: *Suppose that f is piecewise continuous on $[0,\infty)$ and $L[f(t)] = F(s)$ exists for all $s > 0$, and $\int_0^\infty f(t)\,\mathrm{d}t$ converges. Then* $\lim_{s\to 0+} F(s) = \lim_{s\to 0+} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\infty f(t)\,\mathrm{d}t.$

Remark 2: *If f is a piecewise continuous function ans $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s)$ converges uniformly for all $s \in E$, then $F(s)$ is a continuous function on E, that is, for* $s \to s_0 \in E$,

$$\lim_{s\to s_0} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s_0) = \int_0^\infty \lim_{s\to s_0} e^{-st} f(t)\,\mathrm{d}t.$$

Remark 3: *Recall that the integral $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ is said to converge uniformly for s in some domain Ω if for any $\epsilon > 0$ there exists some number τ_0 such that if $\tau \geq \tau_0$ then*

$$\left|\int_\tau^\infty e^{-st} f(t)\,\mathrm{d}t\right| < \epsilon$$

for all s in Ω.

44.3.3 Problem 3

Using Laplace transform, evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{x \sin xt}{x^2 + a^2}\,\mathrm{d}x$$

Solution: Let

$$f(t) = \int_0^\infty \frac{x \sin xt}{x^2 + a^2}\,\mathrm{d}x$$

Taking Laplace transform, we get

$$F(s) = \int_0^\infty \frac{x}{x^2 + a^2}\,\frac{x}{x^2 + s^2}\,\mathrm{d}x$$

Using the method of partial fractions we obtain

$$F(s) = \int_0^\infty \frac{1}{x^2 + s^2}\,\mathrm{d}x - \frac{a^2}{s^2 - a^2}\int_0^\infty \left(\frac{1}{x^2 + a^2} - \frac{1}{x^2 + s^2}\right)\mathrm{d}x$$

Evaluating the above integrals we have

$$F(s) = \frac{1}{s}\tan^{-1}\left(\frac{x}{s}\right)\Big|_0^\infty - \frac{a^2}{s^2 - a^2}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{s}\tan^{-1}\left(\frac{x}{s}\right)\right]_0^\infty$$

On simplification we obtain

$$F(s) = \frac{1}{2}\,\frac{\pi}{s + a}$$

Taking inverse Laplace transform we find

$$f(t) = \frac{1}{2}\pi e^{-at}$$

Hence the value of the given integral

$$\int_{-\infty}^{\infty} \frac{x \sin xt}{x^2 + a^2}\,\mathrm{d}x = 2\int_0^\infty \frac{x \sin xt}{x^2 + a^2}\,\mathrm{d}x = \pi e^{-at}.$$

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Lesson 45

Application of Laplace Transform

In previous lessons we have evaluated Laplace transforms and inverse Laplace transform of various functions that will be used in this and following lessons to solve ordinary differential equations. In this lesson we mainly solve initial value problems.

45.1 Solving Differential/Integral Equations

We perform the following steps to obtain the solution of a differential equation.

(i) Take the Laplace transform on both sides of the given differential/integral equations.

(ii) Obtain the equation $L[y] = F(s)$ from the transformed equation.

(iii) Apply the inverse transform to get the solution as $y = L^{-1}[F(s)]$.

In the process we assume that the solution is continuous and is of exponential order so that Laplace transform exists. For linear differential equations with constant coefficients one can easily prove that under certain assumption that the solution is continuous and is of exponential order. But for the ordinary differential equations with variable coefficients we should be more careful. The whole procedure of solving differential equations will be clear with the following examples.

45.2 Example Problems

45.2.1 Problem 1

Solve the following initial value problem

$$\frac{d^2y}{dt^2} + y = 1, \qquad y(0) = y^{'}(0) = 0.$$

Solution: Take the Laplace transform on both sides, we get

$$L[y^{''}] + L[y] = L[1]$$

Using derivative theorems we find

$$s^2 L[y] - sy(0)y^{'}(0) + L[y] = L[1]$$

We plug in the initial conditions now to obtain

$$L[y]\left(1+s^2\right) = \frac{1}{s} = \frac{1}{s\left(1+s^2\right)}$$

Using partial fractions we obtain

$$L[y] = \frac{1}{s} - \frac{s}{1+s^2}$$

Taking inverse Laplace transform we get

$$y(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{s}{1+s^2}\right] = 1 - \cos t$$

45.2.2 Problem 2

Solve the initial value problem

$$x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

Solution: We will take the Laplace transform on both sides. By $X(s)$ we will, as usual, denote the Laplace transform of $x(t)$.

$$L[x''(t) + x(t)] = L[\cos(2t)],$$
$$s^2 X(s) - sx(0) - x'(0) + X(s) = \frac{s}{s^2+4}.$$

Plugging the initial conditions, we obtain

$$s^2 X(s) - 1 + X(s) = \frac{s}{s^2+4}$$

We now solve for $X(s)$ as

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}$$

We use partial fractions to write

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}$$

Now take the inverse Laplace transform to obtain

$$x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t).$$

45.2.3 Problem 3

Solve the following initial value problem

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = t^2e^{3t}, \quad y(0) = 2,\, y^{'}(0) = 6.$$

Solution: Taking the Laplace transform on both sides, we get

$$s^2Y(s) - sy(0) - y^{'}(0) - 6\,[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

Using initial values we obtain

$$s^2Y(s)2s - 6 - 6\,[sY(s) - 2] + 9Y(s) = \frac{2}{(s-3)^3}$$

We solve for $Y(s)$ to get

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2(s-3)}{(s-3)^2}$$

Taking inverse Laplace transform, we find

$$y(t) = \frac{2}{4!}t^4e^{t3t} + 2e^{3t} = \frac{1}{12}t^4e^{t3t} + 2e^{3t}.$$

45.2.4 Problem 4

Solve

$$y'' + y = CH(t-a), \quad y(0) = 0,\, y^{'}(0) = 1.$$

Solution: Taking Laplace transform on both sides, we get

$$s^2Y(s) - sy(0) - y^{'}(0) + Y(s) = C\int_a^{\infty} e^{-st}\,\mathrm{d}t$$

We substitute the given initial values to obtain

$$(s^2+1)Y(s) = 1 + C\frac{-as}{s}$$

Solve for $Y(s)$ as

$$Y(s) = \frac{1}{s^2+1} + C\frac{-as}{s(s^2+1)}$$

Method of partial fractions leads to

$$y(t) = \sin t + CL^{-1}\left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-as}\right]$$

By inverse Laplace transform we obtain

$$y(t) = \sin t + CH(t-a)[1-\cos(t-a)].$$

45.2.5 Problem 5

Solve the following initial value problem

$$x''(t) + x(t) = H(t-1) - H(t-5), \quad x(0) = 0, \quad x'(0) = 0,$$

Solution: We transform the equation and we plug in the initial conditions as before to obtain

$$s^2X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

We solve for $X(s)$ to obtain

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

We can easily show that

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t.$$

In other words $L[1-\cos t] = \dfrac{1}{s(s^2+1)}$. So using the shifting theorem we find

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = L^{-1}\left[e^{-s}L[1-\cos t]\right] = \left[1-\cos(t-1)\right]H(t-1).$$

Similarly, we have

$$L^{-1}\left[\frac{e^{-5s}}{s(s^2+1)}\right] = L^{-1}\left[e^{-5s}L[1-\cos t]\right] = \left[1-\cos(t-5)\right]H(t-5).$$

Hence, the solution is

$$x(t) = \left[1-\cos(t-1)\right]H(t-1) - \left[1-\cos(t-5)\right]H(t-5).$$

45.2.6 Problem 6

Solve the initial value problem

$$x'' + \omega_0^2 x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0.$$

Solution: We first apply the Laplace transform to the equation

$$s^2 X(s) + \omega_0^2 X(s) = 1$$

Solving for $X(s)$ we find

$$X(s) = \frac{1}{s^2 + \omega_0^2}$$

Taking the inverse Laplace transform we obtain

$$x(t) = \frac{\sin(\omega_0 t)}{\omega_0}.$$

45.2.7 Problem 7

Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t-3)H(t-3), \quad x(0) = 0, \quad x'(0) = 0.$$

Solution: Recall the second shifting theorem

$$L\left[f(t-a)H(t-a)\right] = e^{-as}F(s)$$

We now apply the Laplace transform to the differential equation to get

$$s^2Y(s) - sy(0) - y'(0) + 2\left[sY(s) - y(0)\right] + 2Y(s) = e^{-3s}$$

Plugging the initial values we find

$$\left[s^2 + 2s + 2\right] Y(s) = e^{-3s}$$

Solving for $Y(s)$ we get

$$Y(s) = \frac{1}{[(s+1)^2 + 1]} e^{-3s}$$

Taking inverse Laplace transform with the use of first and second shifting properties we obtain

$$y(t) = L^{-1}\left[\frac{1}{[(s+1)^2+1]}e^{-3s}\right] = H(t-3)e^{-(t-3)}\sin(t-3).$$

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Lesson 46

Application of Laplace Transform (Cont.)

In this lesson we continue the application of Laplace transform for solving initial and boundary value problems. In this lesson we will also look for differential equations with variable coefficients and some boundary value problems.

46.1 Example Problems

46.1.1 Problem 1

Find the solution to

$$x'' + \omega_0^2 x = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$.

Solution: We first apply the Laplace transform to the equation. Denoting the transform of $x(t)$ by $X(s)$ and the transform of $f(t)$ by $F(s)$ as usual, we have

$$s^2X(s) + \omega_0^2 X(s) = F(s),$$

or in other words

$$X(s) = F(s)\frac{1}{s^2+\omega_0^2}.$$

We know

$$L^{-1}\left[\frac{1}{s^2+\omega_0^2}\right] = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Therefore, using the convolution theorem, we find

$$x(t) = \int_0^t f(\tau)\frac{\sin\big(\omega_0(t-\tau)\big)}{\omega_0}\, d\tau,$$

or if we reverse the order

$$x(t) = \int_0^t \frac{\sin(\omega_0 t)}{\omega_0} f(t-\tau)\, d\tau.$$

46.1.2 Problem 2

Find the general solution of

$$y'' + y = e^{-t}.$$

Solution: Taking Laplace transform on both sides

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s+1}$$

Denoting $y(0)$ by y_0 and $y'(0)$ by y_1 we find

$$(s^2+1)Y(s) - sy_0 - y_1 = \frac{1}{s+1}$$

Now we solve for $Y(s)$ to obtain

$$Y(s) = \frac{1}{(s+1)(s^2+1)} + \frac{sy_0}{s^2+1} + \frac{y_1}{s^2+1}$$

Method of partial fractions leads to

$$Y(s) = \frac{1}{2}\left[\frac{1}{s+1} - \frac{s-1}{s^2+1}\right] + \frac{sy_0}{s^2+1} + \frac{y_1}{s^2+1}$$

Taking the inverse transform we get

$$y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t + y_0\cos t + y_1\sin t$$

This can be rewritten as

$$y(t) = \frac{1}{2}e^{-t} + \left(y_0 - \frac{1}{2}\right)\cos t) + \left(y_1 + \frac{1}{2}\right)\sin t$$

Note that y_0 and y_1 are arbitrary, so the general solution is given by

$$y(t) = \frac{1}{2}e^{-t} + C_0\cos t + C_1\sin t.$$

46.1.3 Problem 3

Solve the following boundary value problem

$$y'' + y = \cos t, \quad y(0) = 1, y\left(\frac{\pi}{2}\right) = 1.$$

Solution: Taking Laplace transform on both sides we get,

$$s^2Y(s) - sy(0) - y^{'}(0) + Y(s) = \frac{s}{s^2+1}$$

We solve for $Y(s)$ to get

$$Y(s) = \frac{s}{(s^2+1)^2} + \frac{s}{s^2+1} + \frac{y^{'}(0)}{s^2+1}$$

Taking inverse Laplace transform on both sides we get,

$$y(t) = \frac{1}{2}t\sin t + \cos t + y^{'}(0)\sin t.$$

Given $y(\frac{\pi}{2}) = 1$, therefore

$$1 = \frac{1}{2}\frac{\pi}{2} + 0 + y^{'}(0) \Rightarrow y^{'}(0) = \left(1 - \frac{\pi}{4}\right).$$

Hence, we obtain the solution as

$$y(t) = \frac{1}{2}t\sin t + \cos t + \left(1 - \frac{\pi}{4}\right)\sin t.$$

46.1.4 Problem 4

Solve the following fourth order initial value problem

$$\frac{d^4y}{dx^4} = -\delta(x-1),$$

with the initial conditions

$$y(0) = 0, \qquad y''(0) = 0, \qquad y(2) = 0, \qquad y''(2) = 0.$$

Solution: We apply the transform and get

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = -e^{-s}.$$

We notice that $y(0) = 0$ and $y''(0) = 0$. Let us call $C_1 = y'(0)$ and $C_2 = y'''(0)$. We solve for $Y(s)$,

$$Y(s) = \frac{-e^{-s}}{s^4} + \frac{C_1}{s^2} + \frac{C_2}{s^4}.$$

We take the inverse Laplace transform utilizing the second shifting property to take the inverse of the first term.

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) + C_1 x + \frac{C_2}{6} x^3.$$

We still need to apply two of the endpoint conditions. As the conditions are at $x = 2$ we can simply replace $u(x-1) = 1$ when taking the derivatives. Therefore,

$$0 = y(2) = \frac{-(2-1)^3}{6} + C_1(2) + \frac{C_2}{6} 2^3 = \frac{-1}{6} + 2C_1 + \frac{4}{3} C_2,$$

and

$$0 = y''(2) = \frac{-3 \cdot 2 \cdot (2-1)}{6} + \frac{C_2}{6} 3 \cdot 2 \cdot 2 = -1 + 2C_2.$$

Hence $C_2 = \frac{1}{2}$ and solving for C_1 using the first equation we obtain $C_1 = \frac{-1}{4}$. Our solution for the beam deflection is

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) - \frac{x}{4} + \frac{x^3}{12}.$$

We now demonstrate the potential of Laplace transform for solving ordinary differential equations with variable coefficients.

46.1.5 Problem 5

Solve the initial value problem

$$y'' + ty' - 2y = 4; \qquad y(0) = -1,\ y'(0) = 0.$$

Solution: Taking Laplace transform on both sides we get,

$$s^2 Y(s) - sy(0) - y'(0) + \left(-\frac{d}{ds} L[y']\right) - 2Y(s) = 4L[1]$$

Using the given initial values and applying derivative theorem once again, we get

$$s^2 Y(s) + s - \frac{d}{ds}\left(sY(s) - y(0)\right) - 2Y(s) = \frac{4}{s}$$

On simplification we find the following differential equation

$$\frac{dY}{ds} + \left(\frac{3}{s} - s\right) Y(s) = -\frac{4}{s^2} + 1$$

Integrating factor of the above differential equation is given as

$$e^{\int\left(\frac{3}{s}-s\right)\mathrm{d}s} = s^3 e^{-\frac{s^2}{2}}$$

Hence, the solution of the differential equation can be written as

$$Y(s)s^3e^{-\frac{s^2}{2}} = \int\left(-\frac{4}{s^2}-s\right)s^3e^{-\frac{s^2}{2}}\,\mathrm{d}s + c$$

On integration we find

$$Y(s)s^3e^{-\frac{s^2}{2}} = 4e^{-\frac{s^2}{2}} - \left(S^2e^{-\frac{s^2}{2}}\right) + \int 2se^{-\frac{s^2}{2}}\,\mathrm{d}s + c$$

We can simplify the above expression to get

$$Y(s) = \frac{2}{s^3} - \frac{1}{s} + \left(\frac{c}{s^3}\right)e^{\frac{s^2}{2}}$$

Since, $Y(s) \to 0$ as $s \to \infty, c$ must be zero. Putting $c = 0$ and taking inverse Laplace transform we get the desired solution as

$$y(t) = t^2 - 1$$

46.1.6 Problem 6

Solve the initial value problem

$$ty'' + y' + ty = 0;\;\; y(0) = 1,\; y'(0) = 0$$

Solution: Taking Laplace transform on both sides we get,

$$-\frac{d}{ds}L[y''] + L[y'] + \left(-\frac{d}{ds}L[y]\right) = 0$$

Application of derivative theorem leads to

$$-\frac{d}{ds}\left\{s^2Y(s) - sy(0)y'(0)\right\} + \{sY(s) - y(0)\} - \frac{d}{ds}Y(s) = 0$$

Plugging initial values, we find

$$\left(s^2+1\right)Y'(s) + sY(s) = 0$$

On integration we get

$$Y(s) = \frac{c}{\sqrt{1+s^2}}$$

Taking inverse Laplace transform we find

$$y(t) = cJ_0(t)$$

Noting $y(0) = 1,\ J_0(0) = 1$, we find $c = 1$. Thus, the required solution is

$$y(t) = J_0(t).$$

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Lesson 47

Application of Laplace Transform (Cont.)

In this lesson we discuss application of Laplace transform for solving integral equations, integro-differential equations and simultaneous differential equations.

47.1 Integral Equation

An equation of the form

$$f(t) = g(t) + \int_0^t K(t,u)f(u)\,\mathrm{d}u,$$

or

$$g(t) = \int_0^t K(t,u)f(u)\,\mathrm{d}u$$

are known as the integral equations, where $f(t)$ is the unknown function. When the kernel $K(t,u)$ is of the particular form $K(t,u) = K(t-u)$ then the equations can be solved using Laplace transforms. We apply the Laplace transform to the first equation to obtain

$$F(s) = G(s) + K(s)F(s),$$

where $F(s)$, $G(s)$, and $K(s)$ are the Laplace transforms of $f(t)$, $g(t)$, and $K(t)$ respectively. Solving for $F(s)$, we find

$$F(s) = \frac{G(s)}{1-K(s)}.$$

To find $f(t)$ we now need to find the inverse Laplace transform of $F(s)$. Similar steps can be followed to solve the integral equation of second type mentioned above.

47.2 Example Problems

47.2.1 Problem 1

Solve the following integral equation

$$f(t) = e^{-t} + \int_0^t \sin(t-u)f(u)\,\mathrm{d}u.$$

Solution: Applying Laplace transform on both sides and using convolution theorem we get,

$$L[f(t)] = \frac{1}{s+1} + L[\sin t]L[f(t)]$$

On simplifications, we obtain

$$L[f(t)]\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s+1}$$

This further implies

$$L[f(t)] = \frac{s^2+1}{s^2(s+1)}$$

Partial fractions leads to

$$L[f(t)] = \frac{2}{s+1} + \frac{1}{s^2} - \frac{1}{s}$$

Taking inverse Laplace transform we obtain the desired solution as

$$f(t) = 2e^{-t} + t - 1$$

47.2.2 Problem 2

Solve the differential equation

$$x(t) = e^{-t} + \int_0^t \sinh(t-\tau)x(\tau)\, d\tau.$$

Solution: We apply Laplace transform to obtain

$$X(s) = \frac{1}{s+1} + \frac{1}{s^2-1}X(s),$$

or

$$X(s) = \frac{\frac{1}{s+1}}{1-\frac{1}{s^2-1}} = \frac{s-1}{s^2-2} = \frac{s}{s^2-2} - \frac{1}{s^2-2}.$$

It is not difficult to take inverse Laplace transform to find

$$x(t) = \cosh(\sqrt{2}\,t) - \frac{1}{\sqrt{2}}\sinh(\sqrt{2}\,t).$$

47.2.3 Problem 3

Solve the following integral equation for $x(t)$

$$t^2 = \int_0^t e^\tau x(\tau)\, d\tau,$$

Solution: We apply the Laplace transform and the shifting property to get

$$\frac{2}{s^3} = \frac{1}{s} L[e^t x(t)] = \frac{1}{s} X(s-1),$$

where $X(s) = L[x(t)]$. Thus, we have

$$X(s-1) = \frac{2}{s^2} \qquad \text{or} \qquad X(s) = \frac{2}{(s+1)^2}.$$

We use the shifting property again to obtain

$$x(t) = 2e^{-t}t.$$

47.3 Integro-Differential Equations

In addition to the integral we have a differential term in the integro differential equations. The idea of solving ordinary differential equations and integral equations are now combined. We demonstrate the procedure with the help of the following example.

47.3.1 Example

Solve

$$\frac{dy}{dt} + 4y + 13\int_0^t y(u)\,\mathrm{d}u = 3e^{-2t}\sin 3t, \qquad y(0) = 3.$$

Solution: Taking Laplace transform and using its appropriate properties we obtain,

$$sY(s) - y(0) + 4Y(s) + 13\frac{Y(s)}{s} = 3\frac{3}{(s+2)^2+9}.$$

Collecting terms of $Y(s)$ we get

$$\frac{s^2+4s+13}{s}Y(s) = \frac{9}{(s+2)^2+9} + 3$$

On simplification we have

$$Y(s) = \frac{9s}{[(s+2)^2+9]^2} + \frac{3s}{(s+2)^2+9}$$

Taking inverse Laplace transform and using shifting theorem we get

$$y(t) = e^{-2t} L^{-1}\left[\frac{9(s-2)}{(s^2+9)^2} + \frac{3(s-2)}{s^2+9}\right].$$

We now break the functions into the known forms as

$$\begin{aligned} y(t) =& e^{-2t} L^{-1}\left[\frac{9s}{(s^2+9)^2} - \frac{18}{(s^2+9)^2} + \frac{1}{(s^2+9)^2} + \frac{3s}{s^2+9} - \frac{7}{s^2+9}\right] \\ =& e^{-2t} L^{-1}\left[\frac{9s}{(s^2+9)^2} + \frac{s^2-9}{(s^2+9)^2} + \frac{3s}{s^2+9} - \frac{7}{s^2+9}\right] \end{aligned}$$

Using the the following basic inverse transforms

$$L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at, \quad L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right] = t\sin at, \quad L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at.$$

We find the desired solution as

$$y(t) = e^{-2t}\left[\frac{3}{2}t\sin 3t + t\cos 3t + 3\cos 3t - \frac{7}{3}\sin 3t\right]$$

47.4 Simultaneous Differential Equations

At the end we show with the help of an example the application of Laplace transform for solving simultaneous differential equations.

47.4.1 Example

Solve

$$\frac{dx}{dt} = 2x - 3y, \quad \frac{dy}{dt} = y - 2x$$

subject to the initial conditions

$$x(0) = 8, \quad y(0) = 3.$$

Solution: Taking Laplace transform on both sides we get

$$sX(s) - x(0) = 2X(s) - 3Y(s)$$

and

$$sY(s) - y(0) = Y(s) - 2X(s)$$

Collecting terms of $X(s)$ and $Y(s)$ we have the following equations

$$(s-2)X(s) + 3Y(s) = 8 \tag{47.1}$$

$$2X(s) + (s-1)Y(s) = 3 \tag{47.2}$$

Eliminating $Y(s)$ we obtain

$$[(s-1)(s-2) - 6]\, X(s) = 8(s-1) - 9$$

On simplifications we receive

$$X(s) = \frac{8s - 17}{(s-4)(s+1)}$$

Partial fractions lead to

$$X(s) = \frac{5}{s+1} + \frac{3}{s-4},$$

Taking inverse Laplace transform both sides we get

$$x(t) = 5e^{-t} + 3e^{4t}$$

Now we solve the above equations (47.1) and for (47.2) $Y(s)$

$$[6 - (s-1)(s-2)]\, Y(s) = 16 - 3(s-2)$$

On simplifications we get

$$Y(s) = \frac{3s - 22}{s^2 - 3s - 4} = \frac{3s - 22}{(s-4)(s+1)}$$

Using the method of partial fractions we obtain

$$Y(s) = \frac{5}{s+1} - \frac{2}{s-4}$$

Taking inverse transform we get

$$y(t) = 5e^{-t} - 2e^{4t}.$$

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Lesson 48

Application of Laplace Transform (Cont.)

In this last lesson of this module we demonstrate the potential of Laplace transform for solving partial differential equations. If one of the independent variables in partial differential equations ranges from 0 to ∞ then Laplace transform may be used to solve partial differential equations.

48.1 Solving Partial Differential Equations

Working steps are more or less similar to what we had for solving ordinary differential equations. We take the Laplace transform with respect to the variable that ranges from 0 to ∞. This will convert the partial differential equation into an ordinary differential equation. Then, the transformed ordinary differential equation must be solved considering the given conditions. At the end we take the inverse Laplace transform which results the required solution.

Denoting the Laplace transform of unknown variable $u(x,t)$ with respect to t by $U(x,s)$ and using the definition of Laplace transform we have

$$U(x,s) = L[u(x,t)] = \int_0^\infty e^{-st} u(x,t)\,\mathrm{d}t$$

Then, for the first order derivatives, we have

$$(i)\ \ L\left[\frac{\partial u}{\partial x}\right] = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}\,\mathrm{d}t = \frac{d}{dx}\int_0^\infty e^{-st}u(x,t)\,\mathrm{d}t = \frac{dU}{dx}$$

$$(ii)\ \ L\left[\frac{\partial u}{\partial t}\right] = \int_0^\infty e^{-st}\frac{\partial u}{\partial t}\,\mathrm{d}t = e^{-st}u\Big|_0^\infty - \int_0^\infty u(-s)e^{-st}\,\mathrm{d}t$$

$$= -u(x,0) + s\int_0^\infty ue^{-st}\,\mathrm{d}t$$

$$\Rightarrow\ \ L\left[\frac{\partial u}{\partial t}\right] = -u(x,0) + sU(x,s)$$

Similarly for the second order derivatives we find

$$(iii)\ \ L\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2U}{dx^2}$$

$$(iv)\ \ L\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2U(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0)$$

$$(v)\ \ L\left[\frac{\partial^2 u}{\partial x \partial t}\right] = s\frac{d}{dx}U(x,s) - \frac{d}{dx}u(x,0)$$

Remark: *In order to derive the above results, besides the assumptions of piecewise continuity and exponential order of $u(x,t)$ with respect to t, we have also used the following assumptions: (i) The differentiation under integral sign is valid and (ii) The limit of the Laplace transform is the Laplace transform of the limit, i.e.,* $\lim_{x\to x_0} L[u(x,t)] = L[\lim_{x\to x_0} u(x,t)]$.

48.2 Example Problems

48.2.1 Problem 1

Solve the following initial boundary value problem

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \quad u(x,0) = x,\ u(0,t) = t$$

Solution: Taking Laplace transform

$$\frac{d}{dx}U(x,s) = sU(x,s) - u(x,0)$$

Using the initial values we get

$$\frac{d}{dx}U(x,s) - sU(x,s) = -x$$

The integrating factor is

$$I.F. = e^{-\int s\,\mathrm{d}x} = e^{-sx}$$

Hence, the solution can be written as

$$U(x,s)e^{-sx} = -\int xe^{-sx}\,\mathrm{d}x + c$$

On integration by parts we find

$$U(x,s)e^{-sx} = -x\frac{e^{-sx}}{-s} - \int \frac{e^{-sx}}{s}\,\mathrm{d}x + c$$

Simplify, the above expression we have

$$U(x,s) = \frac{x}{s} + \frac{1}{s} + ce^{sx}$$

Using given boundary condition we find

$$\frac{1}{s^2} = \frac{1}{s^2} + ce^{sx} \quad \Rightarrow \quad c = 0$$

With this we obtain

$$U(x,s) = \frac{x}{s} + \frac{1}{s}$$

Taking inverse Laplace transform, we find the desired solution as

$$u(x,t) = x + t$$

48.2.2 Problem 2

Solve the following partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x, \quad x > 0,\ t > 0$$

with the following initial and boundary condition

$$u(x,0) = 0,\ x > 0 \ \textit{and}\ u(0,t) = 0,\ t > 0$$

Solution: Taking Laplace transform with respect to t we have

$$sU(x,s) - u(x,0) + x\frac{d}{dx}U(x,s) = \frac{x}{s}, \quad s > 0$$

Using the given initial value we find

$$\frac{d}{dx}U(x,s) + \frac{s}{x}U(x,s) = \frac{1}{s}$$

Its integrating factor is x^s and therefore the solution can be written as

$$U(x,s)x^s = \int \frac{1}{s} x^s \, \mathrm{d}x + c \quad \Rightarrow \quad U(x,s) = \frac{1}{s(s+1)} x + \frac{c}{x^s}$$

Boundary condition provides

$$u(0,t) = 0 \quad \Rightarrow \quad U(0,s) = 0, \quad \Rightarrow \quad c = 0$$

Thus we have

$$U(x,s) = \frac{x}{s(s+1)} = x \left[\frac{1}{s} - \frac{1}{s+1} \right]$$

Taking inverse Laplace transform we find the desired solution as

$$u(x,t) = x \left[1 - e^{-t} \right]$$

48.2.3 Problem 3

Solve the following heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}, \qquad x > 0, \, t > 0$$

with the initial and boundary conditions

$$u(x,0) = 1, \; u(0,t) = 0, \; \lim_{x \to \infty} = 1$$

Solution: Taking Laplace transform we find

$$sU(x,s) - u(x,0) = \frac{d^2}{dx^2} U(x,s)$$

Using the given initial condition we have

$$\frac{d^2}{dx^2} U(x,s) - sU(x,s) = -1$$

Its solution is given as

$$U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s}$$

The given boundary conditions give

$$\lim_{x \to \infty} U(x,s) = \frac{1}{s} \quad \Rightarrow \quad c_1 = 0$$

and

$$U(0,s)=0 \;\Rightarrow\; c_1+c_2+\frac{1}{s}=0 \;\Rightarrow\; c_2=-\frac{1}{s}$$

Hence, we have

$$U(x,s)=-\frac{1}{s}e^{-\sqrt{s}x}+\frac{1}{s}$$

Taking inverse Laplace transform we find the desired solution as

$$u(x,t)=1-L^{-1}\left[\frac{1}{s}e^{-\sqrt{s}x}\right]=1-\left[1-\text{erf}\left(\frac{x}{2\sqrt{t}}\right)\right]=\text{erf}\left(\frac{x}{2\sqrt{t}}\right)$$

48.2.4 Problem 4

Solve the one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial t^2},\quad x>0,\; t>0$$

with the initial conditions

$$y(x,0)=1,\; y_t(x,0)=0$$

and boundary conditions

$$y(0,t)=\sin\omega t,\; \lim_{x\to\infty} y(x,t)=0$$

Solution: Taking Laplace transform we get

$$s^2Y(x,s)-sy(x,0)-y_t(x,0)-a^2\frac{d^2}{dx^2}Y(x,s)=0$$

With the given initial condition we have the resulting differential equation

$$\frac{d^2Y}{dx^2}-\frac{s^2}{a^2}=0$$

Its general solution is given as

$$Y(x,s)=c_1e^{\frac{s}{a}x}+c_2e^{-\frac{s}{a}x}$$

The given boundary conditions provides

$$\lim_{x\to\infty}Y(x,s)=0 \;\Rightarrow\; c_1=0,$$

and

$$Y(0,s) = \frac{\omega}{s^2+\omega^2} \quad \Rightarrow \quad c_2 = \frac{\omega}{s^2+\omega^2}$$

Thus we have

$$Y(x,s) = \frac{\omega}{s^2+\omega^2} e^{-\frac{s}{a}x}$$

Taking inverse Laplace transform we obtain

$$y(x,t) = \sin\left[\omega\left(t-\frac{x}{a}\right)\right] H\left(t-\frac{x}{a}\right).$$

Suggested Readings

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QUIZ

1. Laplace transform of the rectangular wave function $W(t-a,b)=\begin{cases}0, & 0\le t, t>b\\ 1, & a<t\le b\end{cases}$ is

(a) $\frac{1}{s}\left(e^{as}-e^{bs}\right)$ (b) $\frac{1}{s}\left(e^{-as}-e^{-bs}\right)$

(c) $\frac{1}{s}\left(e^{-bs}-e^{-as}\right)$ (d) $\frac{1}{s}\left(e^{bs}-e^{as}\right)$

2. Laplace transform of the function $\sin^2 pt$ is

(a) $\frac{4p^2}{s\left(s^2+4p^2\right)}$ (b) $\frac{2p^2}{s\left(s^2+2p^2\right)}$

(c) $\frac{2p^2}{s\left(s^2+4p^2\right)}$ (d) $\frac{2p}{s\left(s^2+4p^2\right)}$

3. Laplace transform of the function $\sin at\cos bt$ is

(a) $\frac{a^2\left(s^2+a^2-b^2\right)}{\left(s^2+(a+b)^2\right)\left(s^2+(a-b)^2\right)}$ (b) $\frac{a\left(s^2+a^2-b^2\right)}{\left(s^2+(a+b)^2\right)\left(s^2+(a-b)^2\right)}$

(c) $\frac{a\left(s^2+a^2-b^2\right)}{\left(s^2+(a+b)\right)\left(s^2+(a-b)\right)}$ (d) $\frac{a\left(s^2+a^2-b^2\right)^2}{\left(s^2+(a+b)^2\right)\left(s^2+(a-b)^2\right)}$

4. Which of the following function is not piecewise continuous?

(a) $f(t)=\frac{1}{t-2},\ t\neq 2$ (b) $f(t)=\begin{cases}2t, & t\le 1\\ 1+t^2, & t>1\end{cases}$

(c) $f(t)=\begin{cases}\frac{1-e^{-t}}{t}, & t\neq 0\\ 0, & t=0\end{cases}$ (d) $f(t)=\begin{cases}t\sin\left(\frac{1}{t}\right), & t\neq 0\\ 0, & t=0\end{cases}$

5. Laplace transform of $t^2 \cos at$ is

(a) $2s\left(s^2+a^2\right)^{-2}-8a^2s\left(s^2+a^2\right)^{-3}$

(b) $2s\left(s^2+a^2\right)^{-2}-8a^2s\left(s^2+a^2\right)^{-2}$

(c) $2s\left(s^2+a^2\right)^{-2}-8a\ s\left(s^2+a^2\right)^{-3}$

(d) $2s^2\left(s^2+a^2\right)^{-2}-8a^2s\left(s^2+a^2\right)^{-3}$

6. Laplace transform of $f(t)=|\sin(t)|,\quad t>0$ is

(a) $\dfrac{1-e^{-2\pi s}}{\left(1+e^{-2\pi s}\right)\left(s^2+1\right)}$

(b) $\dfrac{1-e^{-\pi s}}{\left(1+e^{-2\pi s}\right)\left(s^2+1\right)}$

(c) $\dfrac{1+e^{-\pi s}}{\left(1-e^{-\pi s}\right)\left(s^2+1\right)}$

(d) $\dfrac{1-e^{-\pi s}}{\left(1+e^{-\pi s}\right)\left(s^2+1\right)}$

7. Laplace transform of $f(t)=t\,H(t-a),\quad t>0$ is

(a) $\dfrac{e^{as}}{s^2}[1-as]$

(b) $\dfrac{e^{as}}{s^2}[1+as]$

(c) $\dfrac{e^{-as}}{s^2}[1-as]$

(d) $\dfrac{e^{-as}}{s^2}[1+as]$

8. Which of the following functions does not possess the Laplace transform?

(a) $e^t \operatorname{erfc}\left(\sqrt{t}\right)$

(b) $\sin\left(e^{t^2}\right)$

(c) e^{t^2}

(d) $te^{t^2}\sin\left(e^{t^2}\right)$

9. The value of $L^{-1}\left[\dfrac{2s+3}{s^2+4}\right]$ is

(a) $\cos 2t+\dfrac{3}{2}\sin 2t$

(b) $2\cos 2t+\dfrac{3}{2}\sin t$

(c) $2\cos 2t+\dfrac{3}{4}\sin 2t$

(d) $2\cos 2t+\dfrac{3}{2}\sin 2t$

10. The inverse Laplace transform of $\dfrac{2-5s}{(s-6)(s^2+11)}$ is

(a) $\dfrac{1}{47}\left[-28e^{6t}+28\cos\sqrt{11}\,t-\dfrac{67}{\sqrt{11}}\sin\sqrt{11}\,t\right]$

(b) $\dfrac{1}{47}\left[-28e^{6t}-28\cos\sqrt{11}\,t+\dfrac{67}{\sqrt{11}}\sin\sqrt{11}\,t\right]$

(c) $\dfrac{1}{47}\left[28e^{6t}+28\cos\sqrt{11}\,t-\dfrac{67}{\sqrt{11}}\sin\sqrt{11}\,t\right]$

(d) $\dfrac{1}{47}\left[28e^{6t}-28\cos\sqrt{11}\,t-\dfrac{67}{\sqrt{11}}\sin\sqrt{11}\,t\right]$

11. The value of $L^{-1}\left[\dfrac{3s-5}{4s^2-4s+1}\right]$ is

(a) $\dfrac{1}{8}e^{\frac{t}{2}}(6-7t)$, (b) $\dfrac{1}{8}e^{-\frac{t}{2}}(6-7t)$,

(c) $\dfrac{1}{8}e^{\frac{t}{2}}(6+7t)$, (d) $\dfrac{1}{8}e^{-\frac{t}{2}}(6+7t)$,

12. The value of $F(s)=\dfrac{e^{-\pi s}}{s^2-2}$ is

(a) $\dfrac{1}{\sqrt{2}}H(t-\pi)\cosh\left\{\sqrt{2}(t-\pi)\right\}$ (b) $\dfrac{1}{\sqrt{2}}H(t-\pi)\sin\left\{\sqrt{2}(t-\pi)\right\}$

(c) $\dfrac{1}{\sqrt{2}}H(t-\pi)\cos\left\{\sqrt{2}(t-\pi)\right\}$ (d) $\dfrac{1}{\sqrt{2}}H(t-\pi)\sinh\left\{\sqrt{2}(t-\pi)\right\}$

13. The solution of the initial value problem $\frac{d^2y}{dt^2}+\frac{dy}{dt}=\left(1-H(t-1)\right);\ \ y(0)=1, y'(0)=-1$ is

(a) $y=t-1+2e^{-t}-H(t-1)\left(t-2+e^{-t+1}\right)$ (b) $y=t-1+2e^{-t}-H(t-1)\left(t+2+e^{-t+1}\right)$

(c) $y=t-1+2e^{-t}-H(t-1)\left(t-2+e^{t-1}\right)$ (d) $y=t-1+2e^{-t}-H(t-1)\left(t+2+e^{t-1}\right)$

14. The solution of the initial value problem

$$\frac{d^2y}{dt^2}+y=f(t);\ \ y(0)=y'(0)=0;\ \ \text{where}\ \ f(t)=\begin{cases}\cos t, & 0\le t\le\pi\\ 0, & t>\pi\end{cases}$$

(a) $y=\frac{1}{2}\left[t\cos t+H(t-\pi)(t-\pi)\sin(t-\pi)\right]$

(b) $y=\frac{1}{2}\left[t\sin t+H(t-\pi)(t-\pi)\sin(t-\pi)\right]$

(c) $y=\frac{1}{2}\left[t\sin t-H(t-\pi)(t-\pi)\sin(t-\pi)\right]$

(d) $y=\frac{1}{2}\left[t\cos t-H(t-\pi)(t-\pi)\sin(t-\pi)\right]$

15. The solution of the integral equation $y(t)=\sin t+2\int_0^t y(u)\cos(t-u)du$ is

(a) $y(t)=te^{-t}$ (b) $y(t)=t^2e^{t}$

(c) $y(t)=te^{t}$ (d) $y(t)=t^2e^{-t}$

16. The solution of the integro-differential equation $\frac{dy(t)}{dt} + 3y(t) + 2\int_0^t y(u)du = t;\ \ y(0) = 1$ is

(a) $y(t) = \frac{1}{2} - 2e^{-t} + \frac{5}{2}e^{2t}$ (b) $y(t) = \frac{1}{2} - 2e^{t} + \frac{5}{2}e^{-2t}$

(c) $y(t) = \frac{1}{2} - 2e^{t} + \frac{5}{2}e^{2t}$ (d) $y(t) = \frac{1}{2} - 2e^{-t} + \frac{5}{2}e^{-2t}$

17. The solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2};\ \ 0 < x < 1,\ t > 0$$

with the following initial and boundary conditions

$y(x, 0+) = \sin \pi x, \ \ 0 < x < 1,$ $y_t(x, 0+) = 0, \ \ 0 < x < 1$

$y(0, t) = 0, \ \ t > 0,$ $y(1, t) = 0, \ \ t > 0$

is

(a) $y(x, t) = \sin \pi x \cos \pi t$ (b) $y(x, t) = \sin 2\pi x \cos \pi t$

(c) $y(x, t) = \sin \pi x \cos 2\pi t$ (d) $y(x, t) = \sin 2\pi x \cos 2\pi t$

18. The solution of the integral equation $F(t) = t + 2\int_0^t \cos(t-u)F(u)du$

(a) $2e^t(t-1) - 2 + t$ (b) $2e^t(t+1) + 2 + t$

(c) $2e^t(t-1) + 2 + t$ (d) $2e^{-t}(t-1) + 2 + t$

19. The value of the integral $\int_0^\infty e^{-t}\frac{\sin t}{t}dt$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

20. The value of the integral $\int_0^\infty t\,e^{-3t}\cos t\,dt$ is

(a) $\frac{1}{25}$ (b) $\frac{2}{25}$

(c) $\frac{3}{25}$ (d) $\frac{4}{25}$

Answers:

1. b	2. c
3. b	4. a
5. a	6. c
7. d	8. c
9. d	10. a
11. a	12. d
13. a	14. b
15. c	16. d
17. a	18. c
19. b	20. b