Engineering Mechanics

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Engineering Mechanics
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MODULE 1. BASIC CONCEPTS

LESSON 1. BASIC CONCEPTS

1.1 INTRODUCTION

Mechanics is the physical science concerned with the behaviour of bodies that are acted upon by forces.

Statics is the study which deals with the condition of bodies in equilibrium subjected to external forces.

In other words, when the force system acting on a body is balanced, the system has no external effect on the body, the body is in equilibrium.

Dynamics is also a branch of mechanics in which the forces and their effects on the bodies in motion are studied. Dynamics is sub-divided into two parts: (1) Kinematics and (2) Kinetics

Kinematics deals with the geometry of motion of bodies without and application of external forces.

Kinetics deals with the motion of bodies with the application of external forces.

Hydromechanics is the study which deals with the conditions of fluid under which it can remain at rest or in motion. Hydromechanics can be divided into hydrostatics and hydrodynamics.

Hydrostatics is the study of fluid at rest.

Hydrodynamics is the study of fluid in motion.

- A body is said to be rigid if it retain its shape and size even if the external forces are applied on it. It is called a rigid body.

1.2 SOME BASIC TERMS USED IN MECHANICS

The followings are the basic terms which are used in mechanics:

Mass: The quantity of the matter possessed by a body is called mass. The mass of a body can not change unless the body is damaged and part of it is physically separated.

Length: It is a concept to measure linear distances.

Time: Time is the measure of succession of events. The successive event selected is the rotation of earth about its own axis and this is called a day.

Space: Any geometric region in which the study of a body has been done is called space.

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Displacement: It is defined as the distance moved by a body/particle in the specified direction.

Velocity: The rate of change of displacement with respect to time is defined as velocity.

Acceleration: It is the rate of change of velocity with respect to time.

Momentum: The product of mass and velocity is called momentum. Thus

Momentum = Mass × Velocity

Particle: It can be defined as an object which has only mass and no size.

- Such a body cannot exist theoretically.

- When we deal with the problems involving distances considerably larger compared to the size of the body, the body may be treated as particle.

1.3 LAWS OF MECHANICS

The following are the fundamental laws of mechanics:

(i) Newton's first law

(ii) Newton's second law

(iii) Newton's third law

(iv) Newton's law of gravitation

(v) Law of transmissibility of forces

(vi) Parallelogram law of forces

Law of transmissibility of forces and law of parallelogram of forces will be discussed in coming lessons. Let us discuss the remaining laws:

(i) Newton's first law: It states that everybody continues in its state of rest or of uniform motion in a straight line unless it is compelled by an external agency acting on it.

(ii) Newton's second law: It states that the rate of change of momentum of a body is directly proportional to the impressed force and it takes place in the direction of the force acting on it.

According to this law,

Force = rate of change of momentum. But momentum = mass × velocity

As mass do not change,

Force = mass × rate of change of velocity

i.e., Force = mass × acceleration

F = m × a

(iii) Newton's third law: It states that for every action there is an equal and opposite reaction.

1.4 UNITS AND DIMENSIONS OF QUANTITIES

1.4.1 Units

Measurements are always made in comparison with certain standards. For example, when we say that cloth piece is 2.5 metres long, the measurement of length is with respect to a scale on which graduations are marked. In turn, the graduation of the scale must have been made according to a national or an international standard. The standard so chosen for the measurement of length is called the unit of length. In this example, 'metre' is the unit of length.

Similarly, for the measurement of time, weight, current, speed etc, different units are used.

Each physical quantity is measured for the purpose of analysis, study, comparison, experimentation/results, design etc. with the help of measuring units by comparison.

There are four systems of units used for the measurement of physical quantities. viz. FPS (Foot – Pound – Second) system, CGS (Centimetre – Gram – Second) system, MKS (Meter - Kilogram – Second) system and SI (System international d'units – the French name)

The SI system of units is said to be an absolute system.

S.I Units (International System of Units)

The fundamental units of the system are metre (m) for length, kilogram (kg) for mass and second (s) for time.

The unit for force is newton (N). One newton is the amount of force required to induce an acceleration of 1 m/sec^2 on one kg mass. Weight of a body (in N) = Mass of the body (in kg) × Acceleration due to gravity (in m/sec^2).

1.4.2 Dimensions

The branch of mathematics dealing with dimensions of quantities is called dimensional analysis. There are two systems of dimensional analysis viz. absolute system and *gravitational system.*

Absolute system (MLT system)

- A system of units defined on the basis of length, time and mass is referred to as an absolute system.

- According to SI system of units, three basic units metre, second and kilogram can be used. In MLT system, M refers to Mass, L refers to Length and T refers to Time.

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Gravitational system (FLT system)

- A system of units defined on the basis of length, time and force is referred to as a gravitational system.

- In this system, force is measured in a gravitational field. Thus, its magnitude depends upon the location where the measurement is made. FLT system refers to the Force-Length-Time system.

The dimensions of basic quantities in MLT and FLT systems are shown in Table 1.1.

Table 1.1 Dimensions of quantities in MLT and FLT systems

Quantity	MLT-System	FLT-system
Length	L	L
Mass	M	$FL^{-1}T^2$
Area	L^2	L^2
Volume	L^3	L^3
Velocity	LT^{-1}	LT^{-1}
Acceleration	LT^{-2}	LT^{-2}
Momentum	MLT^{-1}	FT
Stress	$ML^{-1}T^{-2}$	FL^{-2}
Weight	MLT^{-2}	F
Force	MLT^{-2}	F
Power	ML^2T^{-3}	FLT^{-1}
Density	ML^{-3}	$FL^{-4}T^3$

1.5 VECTORS:

Various quantities used in engineering mechanics may be grouped into scalars and vectors.

Scalar Quantity: A quantity is said to be scalar if it is completely defined by its magnitude alone. Examples of scalar quantities are:

Area, length, Mass, Moment of inertia, Energy, Power, Volume and Work etc.

Vector Quantity: A quantity is said to be vector if it is completely defined only when its magnitude as well as direction are specified. Examples of vector quantities include:

Force, Moment, Momentum, Displacement, Velocity and Acceleration.

MODULE 2. SYSTEM OF FORCES

LESSON 2. FORCE SYSTEM

2.1 INTRODUCTION

Definition of 'force' can be given in several ways. Most simply it can be defined as 'the cause of change in the state of motion of a particle or body'. It is of course, the product (multiplication) of mass of the particle and its acceleration.

Force is the manifestation of action of one particle on the other. It is a vector quantity.
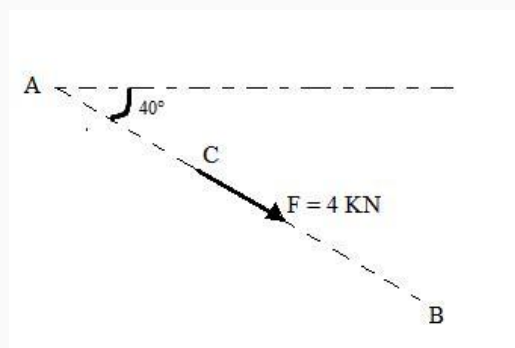
2.2 CHARACTERISTICS OF A FORCE

A Force has following basic characteristics

i) Magnitude

ii) Direction

iii) Point of application

iv) Line of action

Force is represented as a vector .i.e an arrow with its magnitude.

e.g. for the force shown in Fig. 2.1, magnitude of force is 4KN, direction is 40° with the horizontal in fourth quadrant, point of application is C and line of action is AB.
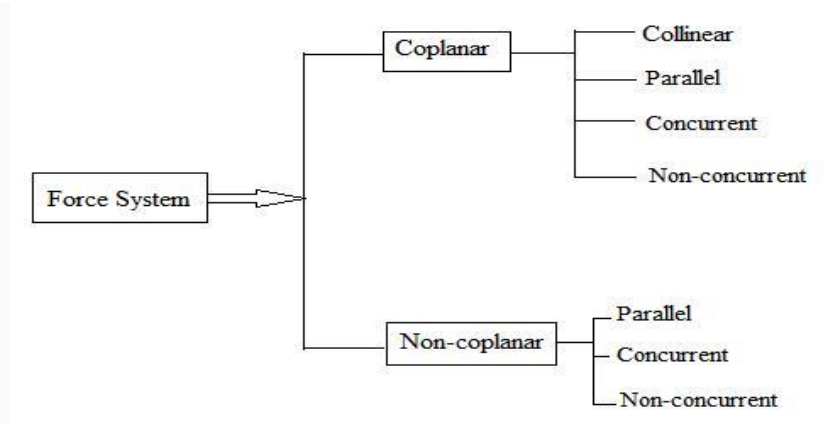

Fig.2.1 Characteristics of a force

Smaller magnitudes of forces are measured in newton (N) and larger in kilonewton (KN).
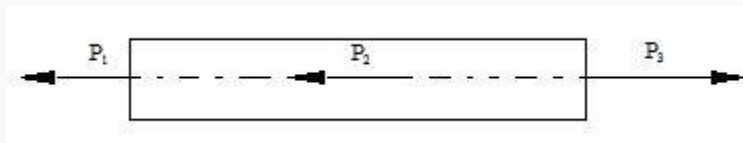
2.3 SYSTEMS OF FORCES

When a mechanics problem or system has more than one force acting, it is known as a 'force system' or 'system of force'.
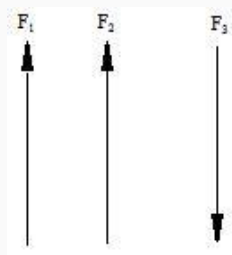
Fig.2.2 Force System

2.3.1 Collinear Force System

When the lines of action of all the forces of a system act along the same line, this force system is called collinear force system.

Fig.2.3 Force System

2.3.2 Parallel Forces

Fig.2.4 Force System

2.3.3 Coplanar Force System

When the lines of action of a set of forces lie in a single plane is called coplanar force system.

2.3.4 Non-Coplanar Force System

When the line of action of all the forces do not lie in one plane, is called Non-coplanar force system
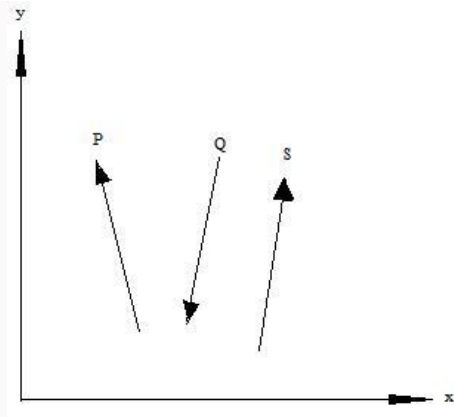
Fig.2.5 Force System

2.3.5 Concurrent Force System

The forces when extended pass through a single point and the point is called point of concurrency. The lines of actions of all forces meet at the point of concurrency. Concurrent forces may or may not be coplanar.

2.3.6 Non-concurrent Force System

When the forces of a system do not meet at a common point of concurrency, this type of force system is called non-concurrent force system. Parallel forces are the example of this type of force system. Non-concurrent forces may be coplanar or non-coplanar.

2.3.7 Coplanar and concurrent force system

A force system in which all the forces lie in a single plane and meet at one point, For example, forces acting at a joint of a roof truss (see fig.2.6)

P = External force

F_1 to F_5 = Member forces (internal) R_A and R_B = Reactions
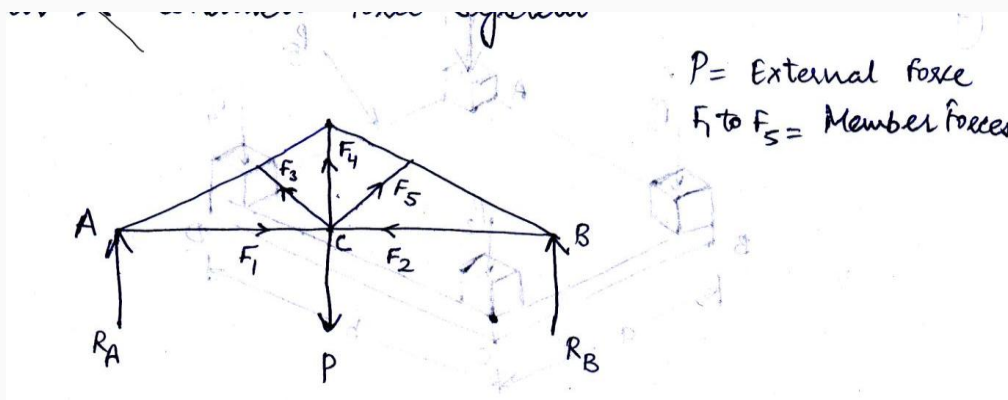
C = Point of concurrency

Fig.2.6 Coplanar concurrent force system

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2.3.8 Coplanar and non-concurrent force system

These forces do not meet at a common point; however, they lie in a single plane, for example, forces acting on a beam as shown in Fig.2.7:
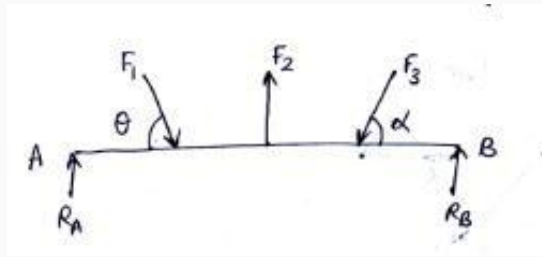

Fig.2.7 Coplanar non-concurrent force system

2.3.9 Non-coplanar and concurrent force system

In this system, the forces lie in a different planes but pass through a single point. Example is forces acting at the top end of an electrical pole (see Fig.2.8)
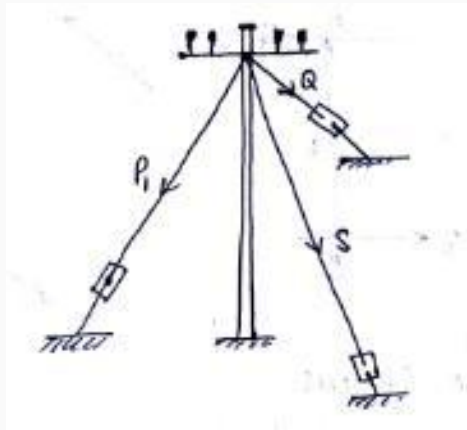

Fig.2.8 Force System

Example 2.1: The tension in the guy wires OA and OB of the electrical pole are 500 N and 300 N respectively as shown in Fig.2.9. Determine the horizontal and vertical components of these tensions exerted by the guy wires on the pole at O.

Fig 2.9

Solution: The tensions exerted by the guy wires on the pole at O are acting as shown in the above figure. The components of each of the forces are determined as given in the following table:

Cable	Force P	Inclination with x-axis θ	x-component Px = P cos θ	y-component Py = P sin θ
OB	500 N	$\tan^{-1}$ 6/2 = 71.57°	500 cos 71.57° = 158.07 N (→)	500 sin 71.57° = 474.36 N (↓)
OA	300 N	$\tan^{-1}$ 6/1.5 = 75.96°	300 cos 75.96° = 72.78 N (←)	300 sin 75.96° = 291.04 N (↓)

2.3.10 Non-coplanar and non-concurrent force system

The forces which do not lie in a single plane and do not pass through a single point are known as non-coplanar and non-concurrent forces. Example is the loads transferred through columns to the rectangular mat foundation as shown in Fig.2.10.

Non- Coplanar and non-Concurrent force system
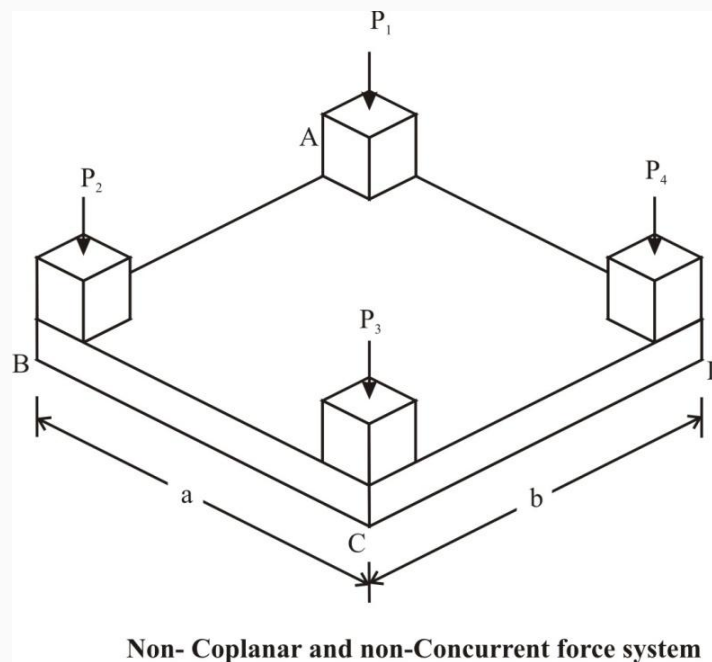
Fig. 2.10 Non-coplanar non-concurrent force system

<center>**LESSON 3. PRINCIPLE OF SUPERPOSITION OF FORCES**</center>

3.1 PRINCIPLE OF SUPERPOSITION OF FORCES

This principle states that the combined effect of force system acting on a particle or a rigid body is the sum of effects of individual forces.

Consider two forces P and Q acting at A on a boat as shown in Fig.3.1. Let R be the resultant of these two forces P and Q. According to Newton's second law of motion, the boat will move in the direction of resultant force R with acceleration proportional to R. The same motion can be obtained when P and Q are applied simultaneously.
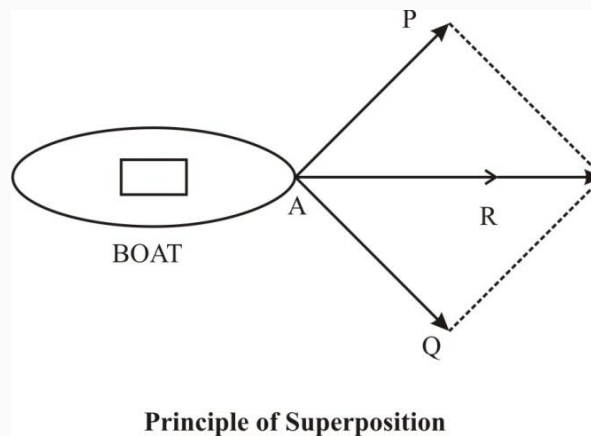

Principle of Superposition

Fig. 3.1 Principle of superposition

3.2 COUPLE

A system of two equal parallel forces acting in opposite directions is said to form a couple. Fig.3.2 shows a couple formed by horizontal, vertical and inclined forces.
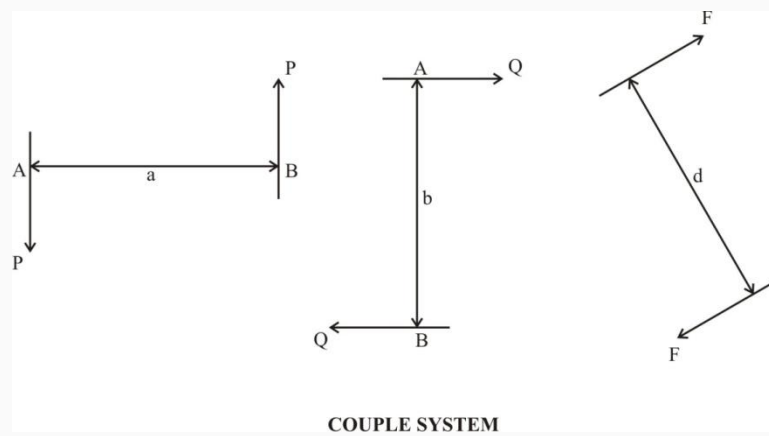

COUPLE SYSTEM

Fig. 3.2 Couple systems

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The plane in which the two forces forming a couple lie is called the plane of the couple and the distance between their line of action is called the arm of the couple. Any couple acting on a rigid body produces only rotation to the body. This rotation is measured by the moment of the couple, which is product of magnitude of the force and the distance between the two forces (arm of the couple). In contrast, the couple does not cause any translation to the rigid body.

The magnitude of the moment of the couple is determined by using the principle of superposition. That is, the moment of the couple is equal to the sum of the moment of the two forces of the couple about any point. As seen in Fig.3.3, the moment of couple about O_1 is given by

$$M_{O1} = +F(d_1) - F(a+d_1) = -F \times a \tag{2.1a}$$

Similarly, the moment of the couple about point O_2 is

$$M_{O2} = -F(a-d_2) - F(d_2) = -F \times a \tag{2.1b}$$
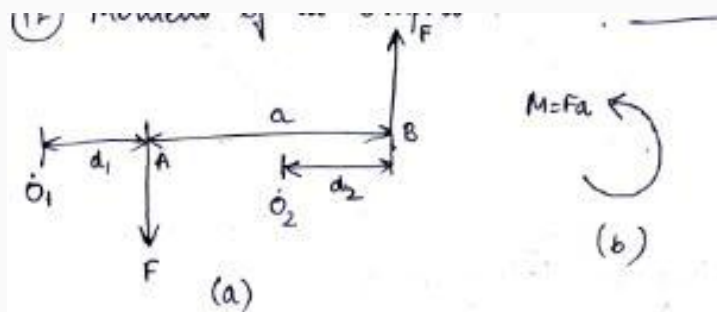
Fig.3.3 Moment of a couple

It is clear that the moment of a couple about any point is always constant. Interestingly, couple can also be diagrammatically shown by a rotation arrow as shown in Fig.3.3(b) indicating the magnitude of the moment of a couple, $M = Fa$.

3.2.1 Characteristics of a Couple

A couple is completely defined by following elements:

i) The magnitude of its moment

ii) The plane in which it acts defined by the direction of the normal to the plane.

iii) The direction of rotation in the plane that is the sense of the couple.

Moment of a couple is a vector quantity having the direction normal to the plane in which it acts.

3.3 EQUIVALENT FORCE-COUPLE SYSTEM

A force at any given point on a rigid body can always be replaced by another force of same direction but acting at different point along with an associated couple.

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Let P be a force acting on a rectangular plane at point A as shown in Fig.3.4(a). Introducing two collinear forces of magnitude P acting opposite to one another at point B and parallel to the one acting at A as indicated in fig. 12(b), the condition remain same as in Fig.3.4(a) itself. According to the principle of superposition, the systems shown in Fig.3.4(a) and (b) are statically equivalent. Subsequently, the force P acting at A and the one acting at B opposite to that at A can be combined together to form a couple, the moment of which is $M = Pa$ acting in counter-clockwise direction [see fig. 3.4(c)]. This couple can be applied at any point on the plate and is shown in fig. 3.4(a) is statically equivalent to the force-couple system shown in Fig. 3.4. This indicates that any given force can be reduced into an equivalent force-couple system and vice versa.
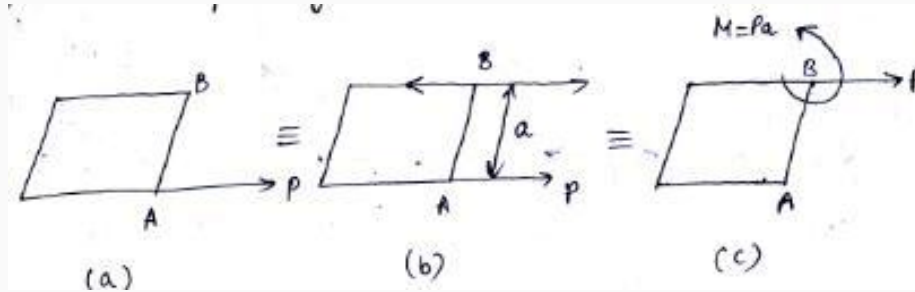

Fig.3.4 Force-couple system

<div align="center">**LESSON 4 RESOLUTION OF A FORCE INTO COMPONENTS**</div>

4.1 RESOLUTION OF A FORCE INTO COMPONENTS

A given force *F* can be resolved into (or replaced by) two forces, which together produces the same effects that of force *F*. These forces are called the components of the force *F*. This process of replacing a force into its components is known as resolution of a force into components. A force can be resolved into two components, which are either perpendicular to each other or inclined to each other. If the two components are perpendicular to one another, then they are known as rectangular components and when the components are inclined to each other, they are called as inclined components. The resolution of force into components is illustrated as follows.

4.1.1 Resolution of a Force into Rectangular Components

Consider a force *F* acting on a particle O inclined at an angle *Θ* as shown in Fig.4.1(a). Let *x* and *y* axes can be the two axes passing through O perpendicular to each other. These two axes are called rectangular axes or coordinate axes. They may be horizontal and vertical or inclined as shown in Fig. 4.1(b).
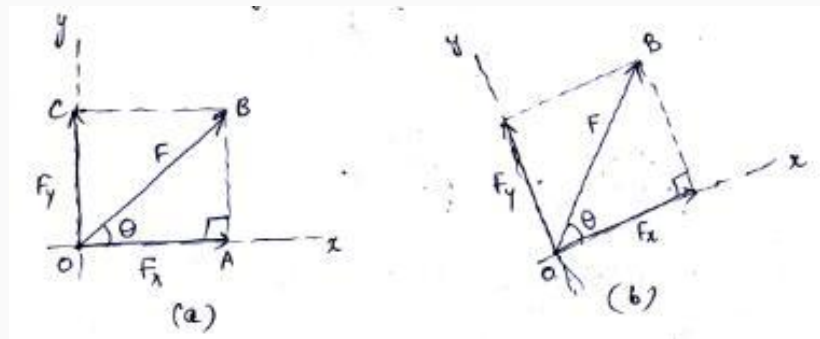

<div align="center">**Fig. 4.1 Resolution of force into rectangular components**</div>

The force *F* can now be resolved into two components F_x and F_y along the *x* and *y* axes and hence, the components are called rectangular components. Further, the polygon constructed with these two components as adjacent sides will form a rectangle OABC and, therefore, the components are known as rectangular components.

From the right angled triangle OAB, the trigonometrical functions can be used to resolve the force as follows:

$$\cos \Theta = {OA \over OB}$$

Therefore,

$$OA = OB \times \cos \Theta$$

Or

$$F_x = OA = F \cos \varTheta \hspace{4cm} (4.1a)$$

$$\sin \varTheta = \backslash [\{\{AB\} \backslash over \{OB\}\} \backslash]$$

Therefore,

$$AB = OB \times \sin \varTheta$$

$$F_y = OC = AB = F \sin \varTheta \hspace{3cm} (4.1b)$$

Therefore, the two rectangular components of the force F are:

$F_x = F \cos \varTheta$ and $F_y = F \sin \varTheta$

The conventional coordinate directions are used for the sign conventions of the components of the force. That is, the components along the coordinate directions are considered as positive components and the one in the opposite direction as negative components. The sign conventions shown in Fig.4.2 are used in general.
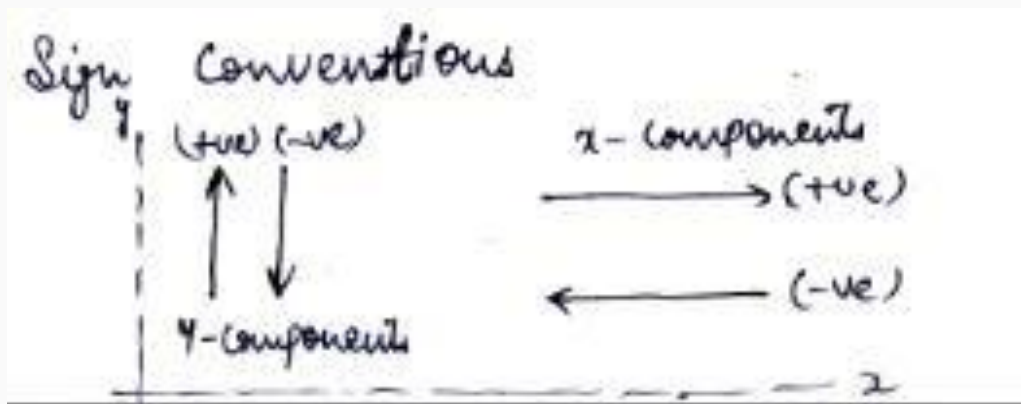

Fig.4.2 Sign conventions

Example 4.1: Determine the components of force P = 40 kN along x and y as shown in Fig.4.3.

Fig.4.3

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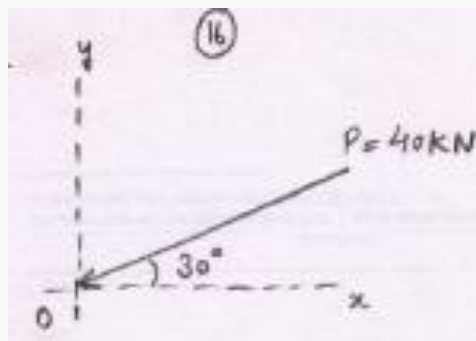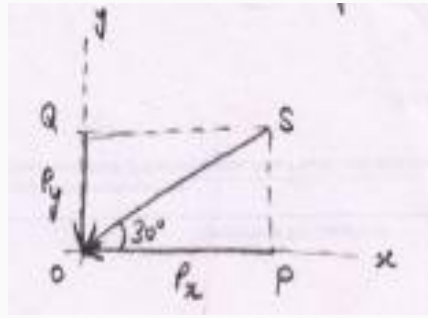
Solution: Plot a rectangle OPSQ taking the force P (that is OS) as the diagonal as illustrated in Fig.4.4, the two components P_x and P_y can be obtained.

Fig.4.4

Consider the right angle triangle OPQ in which

$$\cos 30° = {OP \over OS}$$

Or

$$OP = OS \cos 30°$$

Therefore,

$$P_x = P \cos 30° = 40 \cos 30° = 34.64 \text{ KN } (\leftarrow)$$

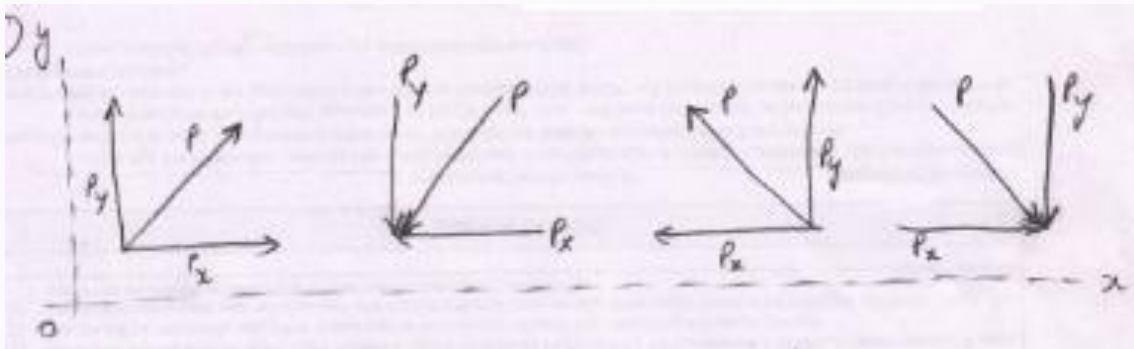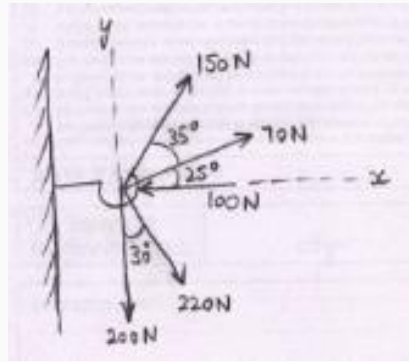$$\sin 30° = {PQ \over OQ} = {OS \over OQ}$$

Hence,

$$OS = OQ \sin 30°$$

$$P_y = P \sin 30° = 40 \sin 30° = 20 \text{ KN } (\downarrow)$$

Note: The directions of P_x and P_y are obtained based on the direction of P as shown follows:

Fig.4.5

**Example 4.2:** Determine the x and y components of each of the forces shown in the following Fig.4.6

Fig.4.6

Solution: The components of each of the forces are shown in the following table:

Force P	Inclination with x-axis Θ	x-component $P_x = P \cos Θ$	y-component $P_y = P \sin Θ$	Remarks
100 N	O°	100 cos 0° = 100 N (←)	100 sin 0° = 0	The force acts along x-axis, Θ = 0°
70 N	25°	70 cos 25° = 63.44 N(→)	70 sin 25° = 29.58 N (↑)	-
150 N	25° + 35° = 60°	150 cos 60° = 75 N (→)	150 sin 60° = 129.90 N (↑)	-
220 N	90° - 30° = 60°	220 cos 60° = 110 N (→)	220 sin 60° = 190.53 N (↓)	The angle is given with y-axis
200 N	90°	200 cos 90° = 0	200 sin 90° = 200 N (↓)	-

Example 4.3: A force of 150 N is acting on a block as shown in Fig.4.7. Find the components of forces along the horizontal and vertical axes.

Fig.4.7

Engineering Mechanics
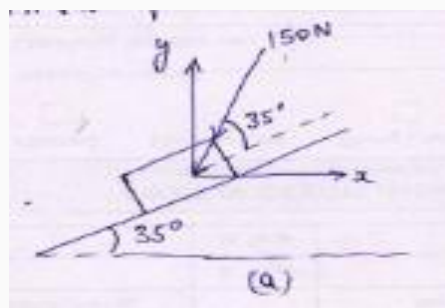
Solution: As given in the Fig.7, the force 150 N makes an angle 35° to the plane (shown by a dotted line) and the plane makes an angle 35° to the horizontal that is x-axis. Therefore, the total inclination of the force 150 N with x-axis is 70° [see Fig.4.8]
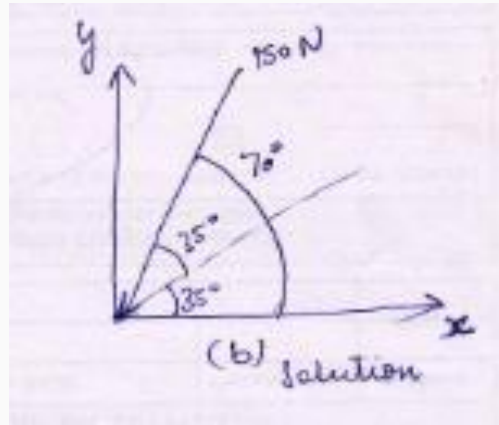
Fig.4.8

Hence the components are:

The x-component of 150 N is,

P_x = 150 cos 70° = 51.30 N ($\leftarrow$)

The y-component of 150 N is,

P_y = 150 sin 70° = 140.95 N ($\downarrow$)

4.1.2 Resolution of Force into Inclined Components

Sometimes, it is essential to know the components of a force, which are not perpendicular to one another. Such components are known as inclined components or non-rectangular components and they are determined either by triangular law of a force or by using law of parallelogram of forces.

4.1.2.1 Triangular Law of Forces

If two forces P and Q are acting on a particle A, then the two forces can be added or combined to form a single force F by arranging the forces in tip-to-tail fashion and then the single force is obtained by connecting the tail of the first force to the tip of the second force. Fig. 4.9(a) shows two forces acting on A. The two forces can be added either as shown in Fig. 4.9(b) or in Fig. 4.9(c). Considering the force Q as the first force acting at A, its tail end is at A and tip will be, say at $B1$. Now the tail end of force P is merged with the tip of force Q, that is at $B1$, and the force P is drawn. The tip of P will be at C (say). Therefore, the force F (addition of forces P and Q) is obtained by joining A (tail of Q) and C (tip of P). The same can also be achieved by considering the force P first and then Q as shown in Fig. 4.9(c). Since, the three forces P, Q and F from the three sides of a triangle, the law is therefore known as triangular law of forces. The single force F combining two forces P and Q is called resultant force.
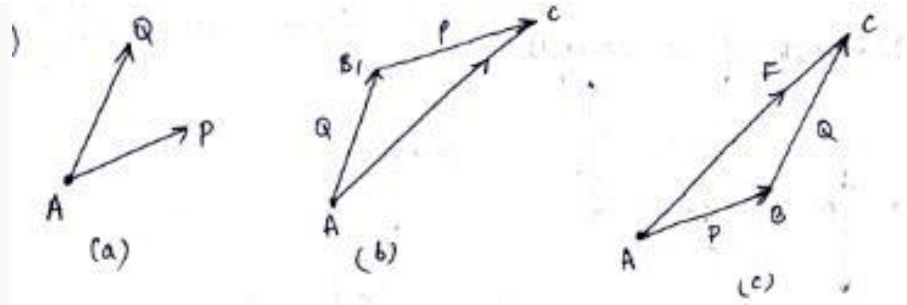
Fig. 4.9 Triangular law of forces

The triangular law of forces is used for the addition of two forces. However, it can be extended to add more than two forces, which extend the law into polygonal law of forces. Fig.4.10 illustrates the addition of four forces P, Q, S and T. As shown in Fig. 4.10(b), F_1, is the addition of forces P and Q. The resultant of forces P, Q and S is obtained by adding F_1 and S as presented in Fig.4.10(c) in which triangular law of forces is applied to combine F_1 and S. Similarly, F_3 is the resultant of P, Q, S and T, which is an addition of F_2 and T. It is pertinent to mention that the forces can be taken in any order.

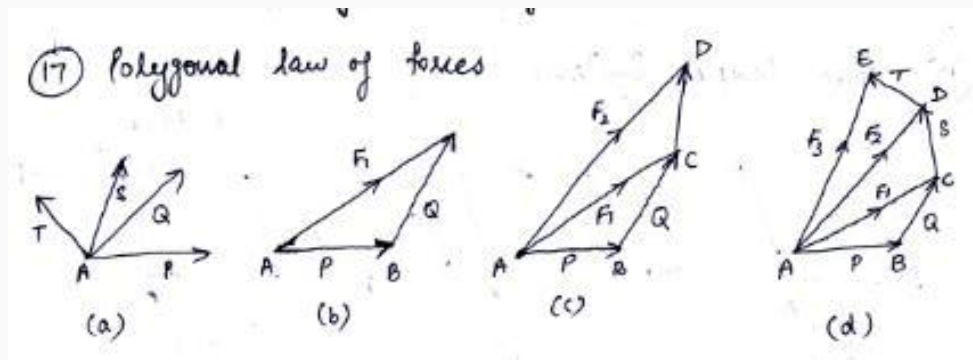**Fig. 4.10 Polygonal law of forces**

4.1.2.2 Law of Parallelogram of Forces

This law states that two forces acting on a particle may be replaced by a single force (called resultant of the two forces) obtained by drawing the diagonal of a parallelogram whose two adjacent sides are equal to the given two forces.

Let P and Q be two forces acting on a particle A as shown in Fig.4.11. Constructing a parallelogram ABCD taking P and Q as its adjacent sides, the diagonal AC gives the resultant force R of the two forces P and Q. The expressions for the magnitude and direction of R are obtained as follows:

Considering the right angled triangle ACE, from the Pythagoras theorem,
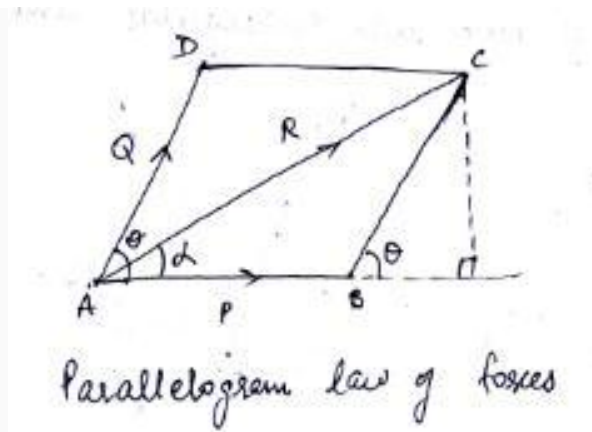
Fig. 4.11 Parallelogram law of forces

$$(AC)^2 = (AE)^2 + (EC)^2$$

Also,

$$AE = AB + BE$$

Therefore,

$$(AC)^2 = (AB+BE)^2 + (EC)^2 = (AB)^2 + 2 \times (AB) \times (BE) + (BE)^2 + (EC)^2$$

Further, from right angled triangle BCE,

$$(BE)^2 + (EC)^2 = (BC)^2$$

Hence,

$$(AC)^2 = (AB)^2 + 2 \times (AB) \times (BE) + (BC)^2$$

Substituting AB = P, BC = AD = Q, BE = BC (cos θ) = Q cos θ and AC = R, the magnitude of the resultant force R of the two forces P and Q is

$$R^2 = P^2 + 2PQ \cos \theta + Q^2$$

Or

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$
(4.2a)

The direction of resultant R, defined by the angle a, which the resultant makes with the force P is obtained from right angled triangle ACE, that is

$$\tan a = \frac{CE}{AE} = \frac{CE}{AB + BE}$$

Substituting CE = BC (sin θ) = Q sin θ along with the values of AB and BE, it yields

$$\tan a = \frac{Q\sin\theta}{P + Q\cos\theta}$$

or $\qquad\qquad a = \[{\tan ^{ - 1}}\left({{{Q\sin \theta } \over {P + Q\cos \theta }}} \right)\]$ (4.2b)

The above two laws are used to determine the components of a given force into two inclined components, which are not perpendicular to each other.

Example 4.4: A small block of weight 150 N is placed on an inclined plane which makes an angle, Θ = 30° with the horizontal. What is the component of this weight parallel to inclined plane and perpendicular to inclined plane?
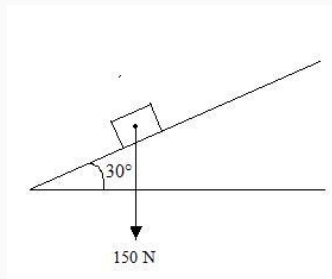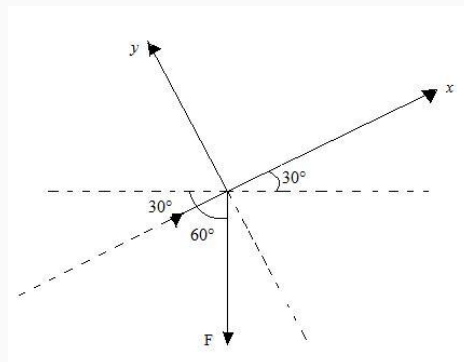

Fig.4.12

Solution: 1. Select the axis

        x-axis parallel to inclined plane

        y-axis perpendicular to inclined plane

2. Draw the force diagram,

Fig. 4.13

 3. Find the force components,

F= 150 N

α = 60°

F_x = F cos α = 150 cos 60° = 75 N

F_y = F sin α = 150 sin 60° = 129.90 N

4.2 LAMI'S THEOREM

It states that," If three forces acting at a point are in equilibrium each force will be proportional to the sine of the angle between the other two forces."

Suppose the three forces P, Q and R are acting at a point O and they are in equilibrium as shown in Fig.14.

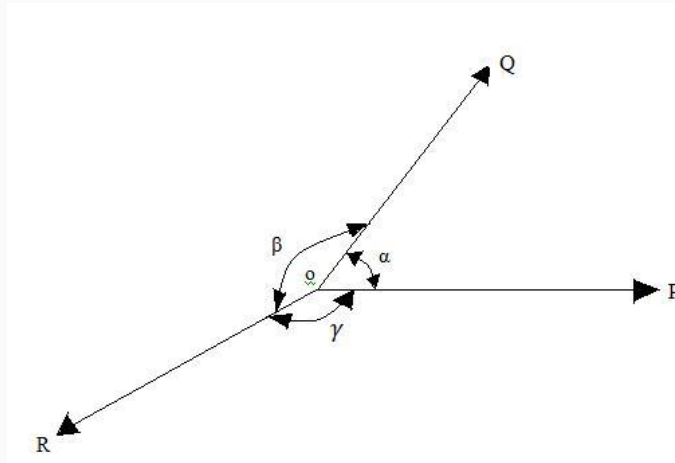**Fig. 4.14**

Let α = Angle between force P and Q.

β = Angle between force Q and R.

γ = Angle between force R and P.

Then according to Lami's Theorem,

P α sine of angle between Q and R α sinβ.

Therefore, $\frac{P}{\sin\beta}$ = constant

Similarly = $\frac{Q}{\sin\gamma}$ constant and $\frac{R}{\sin\alpha}$ = constant

Or $\frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin\alpha}$

LESSON 5. SYSTEM OF FORCES

5.1 FREE BODY DIAGRAM

A free body diagram is a sketch of a body, a portion of a body, or two or more bodies completely isolated or free from all other bodies, showing the forces exerted by all other bodies on the one being considered.Characteristics of free body diagram:-

1. It is a diagram or sketch of a body.
2. The body is shown completely separated from all other bodies.
3. The action on the body of each body removed in the isolating process is shown as a force or forces on the diagram.

5.2 EQUILIBRIUM OF FORCES

Equilibrium is defined as the condition of a body, which is subjected to a force system whose resultant force is equal to zero. It means the effect of the given force system is zero and the particle or rigid body is said to be in equilibrium.

For example, a particle subjected to two forces will be in equilibrium when the two forces are equal in magnitude, opposite in direction and act along the same line of action as shown in Figure.

Fig. 5.1 Equilibrium of forces
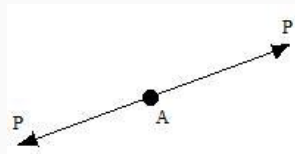
5.2.1 Equations of equilibrium for a concurrent, coplanar force system

The resultant of a concurrent, coplanar force system is a single force through the point of concurrence. When the resultant force is zero, the body on which the force system acts in equilibrium.

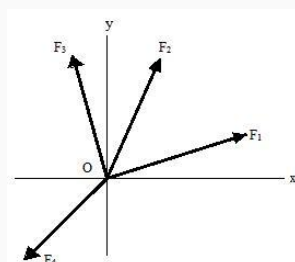Consider the force system as shown in figure:

Fig.5.2 Equilibrium of concurrent and coplanar Force system

Engineering Mechanics

If the sum of the x components of the forces of the system is equal to zero, the resultant can act only along the y axis.

If in addition, the sum of the y components of the forces of the system is equal to zero, the resultant must be zero. Consequently, one complete set of equations of equilibrium for a concurrent, coplanar force system is

$$\sum F_x = 0, \quad \sum F_y = 0 \qquad (5.1)$$

Again, if the sum of the x components of the forces of the system is equal to zero, the resultant can be only a force along the y axis and if the sum of the moments of the forces of the system with respect to an axis through A is equal to zero where A is any point not on the y axis is not zero. Thus, another set of equations which assure equilibrium for this system is

$$\sum F_x = 0, \quad \sum M_A = 0 \qquad (5.2)$$

Where A is not on the y axis.

In a similar manner, a third set of independent equations can be shown to be

$$\sum M_A = 0, \quad \sum M_B = 0 \qquad (5.3)$$

Where line AB does not pass through the point of concurrence of the forces of the system. There are only two independent equations of equilibrium for a concurrent, coplanar force system. When a force system of this type contains not more than two unknowns (two magnitudes, one magnitude and one slope, or two slopes), they can be determined directly from the equations of equilibrium.

When a concurrent, coplanar force system contains more than two unknowns, they cannot all be determined from the equations of equilibrium alone, and the force system is said to be statically indeterminate.

For a collinear force system, Eq.(5.1) reduces to one equation,

$$\sum F_x = 0$$

Where the x axis is parallel to the forces. Likewise, Eq.(5.2) can be reduced to the equation

$$\sum M_A = 0$$

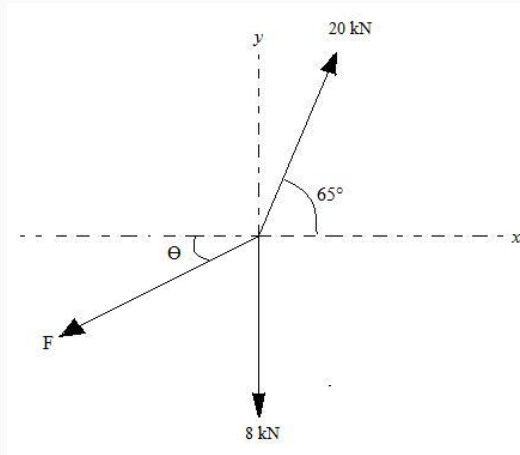**LESSON 6. General Procedure for the solution of problems in Equilibrium**

6.1 GENERAL PROCEDURE FOR THE SOLUTION OF PROBLEMS IN EQUILIBRIUM

The following sequence of steps is designed to aid in organizing the analysis and solution of any problem in equilibrium. The steps are listed in the order in which they should be performed in the solution.

6.1.1 Step by Step Procedure

1. Determine carefully what data are given and what results are required.

2. Draw a free body diagram of the member or group of members on which some or all of the unknown forces are acting.

3. Observe the type of force system which acts on the free body diagram drawn.

4. Note the number of independent equations of equilibrium available for the type of force system involved.

5. Compare the number of unknowns on the free body diagram with the number of independent equations of equilibrium available for the force system.

6. (a) If there are as many independent equations of equilibrium as unknowns, proceed with the solution by writing and solving the equations of equilibrium.

   (b) If there are more unknowns to be evaluated than independent equations of equilibrium available, draw a free body diagram of another body and repeat steps 3,4 and 5 for the second free body diagram drawn.

7. (a)If there are as many independent equations of equilibrium as unknowns for the second free body diagram, proceed with the solution by writing and solving the necessary equations of equilibrium.

   (b) If there are more unknowns to be evaluated than independent equations of equilibrium for the second free body diagram, compare the total number of unknowns on both free body diagrams with the total number of independent equations of equilibrium available for both diagrams.

8. If there are as many independent equations of equilibrium as unknowns for both diagrams, proceed to solve the problem by writing and solving the equations of equilibrium. If there are more unknowns than independent equations of equilibrium, repeat step 6(b) and 7. If there are still too many unknowns after as many free body diagrams have been drawn as there are individual bodies in the problem, then the problem is statically indeterminate i.e not all the unknowns can be evaluated by statics alone.

Engineering Mechanics

Example 6.1: Determine the value of F and Θ so that particle A is in equilibrium.

Fig. 6.1

Solution:

1. $\sum$ Fx = 20 cos 65° + F cos (180° + Θ) + 16 cos 270°

    = 4.23 – F cos Θ – 0

    = 0

Therefore, F cos Θ = 4.23

2. $\sum$ Fy = 20 sin 65° + F sin (180° + Θ) + 16 sin 270°

    = 18.13 – F sin Θ – 0

    = 0

Therefore, F sin Θ = 18.13

3. tan Θ = $\left[\frac{F\sin\theta}{F\cos\theta}\right]$ = $\left[\frac{18.13}{4.23}\right]$ = 4.29

Θ = 76.86°

F cos Θ = 4.23

F = $\left[\frac{4.23}{\cos 76.86^\circ}\right]$ = 18.63 N

MODULE 3.

LESSON 7. Centroid of a Quadrant of a Circle

7.1 INTRODUCTION

The force of attraction of the earth for a particle is called the weight of particle. A body consists of a number of particles each of which has a weight or force of attraction directed towards the centre of the earth. The resultant of this parallel system of gravitational forces in space is the weight of the body.

- The resultant weight does, however, pass through one point in the body, or the body extended, for all orientations of the body, and this point is defined as the centre of gravity or centre of mass for the body.

In the case of plane figures, the notion of centre of gravity can be modified as centroid of the plane figure. Hence, the centroid may be defined as that point through which the total area of the given figure may be imagined to be acting.

- The difference between centre of gravity and centroid is that the centre of gravity applies to the bodies with mass and weight, while the centroid refers to the plane areas, lines and volumes of the body.

The position of the centroid of a plane area is defined analytically with reference to the coordinate axes as shown in Fig.7.1:
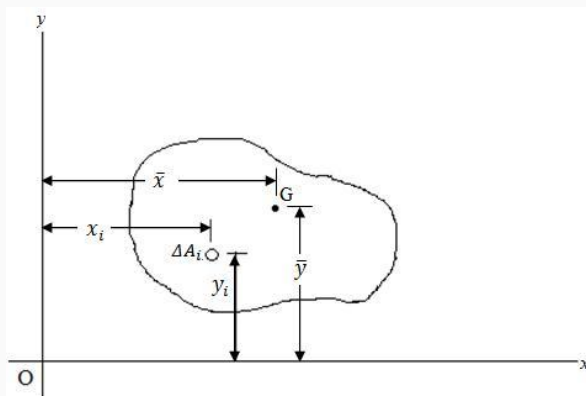

Fig.7.1 Centroid of plane figure

Let ΔA_i be the area of an elemental part of plane figure having total area A and x_i, y_i be the coordinates of the centre of the element with respect to the coordinate axes as shown in figure. The centroidal coordinates of the total area are given as:

$$\bar x = {{\sum\nolimits (\Delta A_i x_i)} \over {\sum\nolimits (\Delta A_i)}} \qquad (7.1a)$$

$$\bar y = {{\sum\nolimits (\Delta A_i y_i)} \over {\sum\nolimits (\Delta A_i)}} \qquad (7.1b)$$

The summations indicate the inclusion of all elements of the area within the boundary of the area. When the elements are considered to be of sizes, which are smaller and smaller, the equation (7.1) becomes

$$\bar x = {\smallint xdA \over \smallint dA} \qquad (7.2a)$$

$$\bar y = {\smallint ydA \over \smallint dA} \qquad (7.2b)$$

The numerators of equations (7.1) and (7.2) are called the first moments of areas. That is

$\sum (\Delta A_i x_i) = \int xdA$ is called the first moment of area about y-axis and is denoted by Q_y and $\sum(\Delta A_i y_i) = \int ydA$ is called the first moment of area about x-axis Q_x. The denominator is the total area of the plane figure A. Therefore, Eq. (7.2) can be written as

$$\bar x = {Q_y \over A} \qquad (7.3a)$$

$$\bar y = {Q_x \over A} \qquad (7.3b)$$

If the plane area is made up of composite parts consisting of geometrical figures such as rectangle, triangle, circle etc. then the centroid of such composite plane figures is obtained from:

$$\bar x = {\sum A_i x_i \over \sum A_i} \qquad (7.4a)$$

$$\bar y = {\sum A_i y_i \over \sum A_i} \qquad (7.4b)$$

Equations (7.1) – (7.4) can be used to determine the centroid of any plane area.
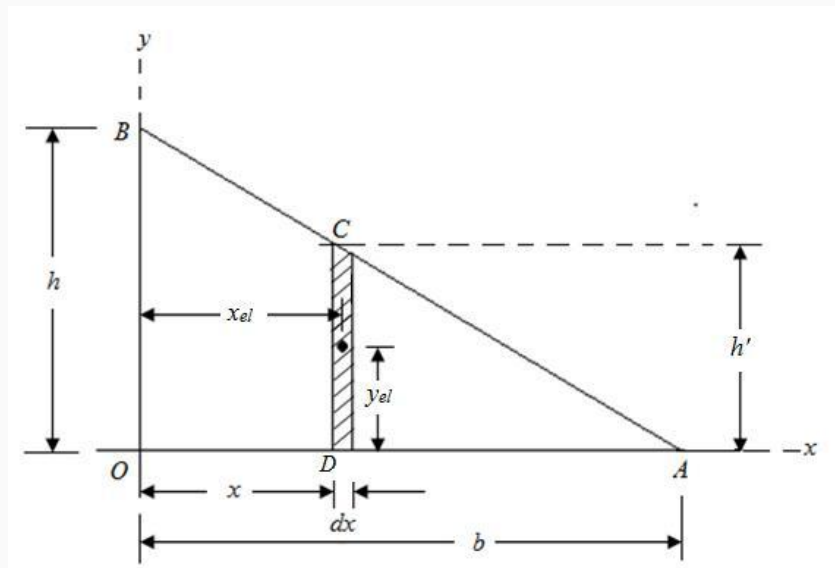
7.2 CENTROID OF A TRIANGLE

Fig. 7.2 Centroid of a Triangle

Let us consider a triangular plane area of width b and height h as shown in Figure 7.2. Taking a differential strip of thickness dx and depth h' at a distance x from O as shown, the area of the strip is given as

$$dA = h' \times dx \qquad\qquad (7.5a)$$

The height h' can be expressed in terms of h and b from similar triangles OAB and DAC as

$$h' = (b\text{-}x) \qquad\qquad (7.5b)$$

Substituting Eq. (7.5b) into Eq. (7.5a) yields

$$dA = (b\text{-}x)\, dx \qquad\qquad (7.5c)$$

The coordinates of the centre of the elemental strip are $x_{el} = x$ and $y_{el} = h'/2$. The first moments of the elemental strip about x and y axes are respectively given as

$$dQ_x = y dA = h'\, dx = h^2/2b^2\, (b\text{-}x^2)\, dx \qquad (7.5d)$$

$$dQ_y = x dA = x\, h'dx = (b\text{-}x)\, x dx \qquad (7.5e)$$

To determine the centroidal coordinates of the triangle OAB, Eqs. [7.5(c), (d) and (e)] are substituted in Eq. (7.2) and integrated between the limits of x from 0 to b.

$$\bar{x} = \frac{\int x dA}{\int dA} = \frac{\int_0^b \frac{bh}{b}(b-x)x dx}{\int_0^b \frac{bh}{b}(b-x)dx}$$

$$\bar{x} = \frac{\int_0^b (b-x)x dx}{\int_0^b (b-x)dx} = \frac{\frac{b^3}{2} - \frac{b^3}{3}}{b^2 - \frac{b^2}{2}} = \frac{\frac{b^3}{6}}{\frac{b^2}{2}}$$

$$\bar{x} = \frac{b}{3} \qquad\qquad (7.6a)$$

$$\bar{y} = \frac{\int y dA}{\int dA} = \frac{\int_0^b \frac{bh^2}{2b^2}(b-x)^2 dx}{\int_0^b \frac{bh}{b}(b-x)dx}$$

$$\bar{y} = \frac{\frac{h}{2b}\int_0^b (b-x)^2 dx}{\int_0^b (b-x)dx} = \frac{\frac{h}{2b}(b^3 - b^3 + \frac{b^3}{3})}{b^2 - \frac{b^2}{2}} = \frac{\frac{hd^2}{6}}{\frac{b^2}{2}}$$

$$\bar{y} = \frac{h}{3} \qquad\qquad (7.6b)$$

7.3 CENTROID OF A QUADRANT OF A CIRCLE

Figure shows a quadrant of a circle of radius R. The differential element shown by shaded portion subtends an angle $d\theta$ at the centre and is located at θ with x-axis. The area of the element is approximated as the area of the triangle given by
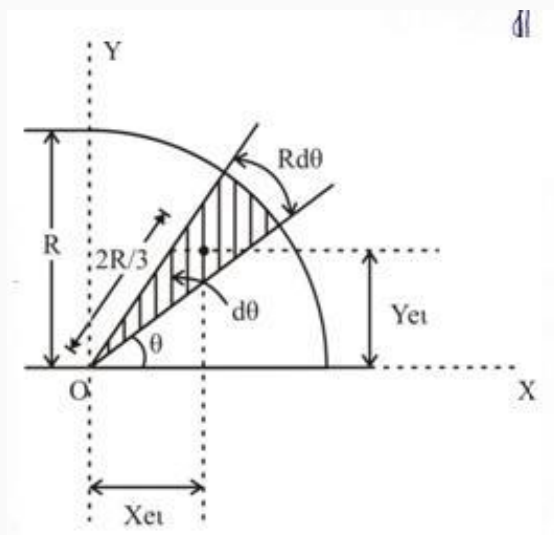

Fig. 7.3 Centroid of a quadrant of a circle

$$dA = \frac{1}{2} R^2\, d\theta \tag{7.7a}$$

The centroidal coordinates of the differential element with reference to x and y axes are:

$$x_{el} = \frac{2}{3} R \cos\theta \text{ and } y_{el} = \frac{2}{3} R \sin\theta$$

The first moments of the differential strip about x and y axes are computed as

$$dQ_x = y\,dA = \left(\frac{2}{3}R \sin\theta\right)\left(\frac{1}{2}R^2\,d\theta\right) = \left(\frac{R^3}{3}\sin\theta\right)d\theta \tag{7.7b}$$

$$dQ_y = x\,dA = \left(\frac{2}{3}R \cos\theta\right)\left(\frac{1}{2}R^2\,d\theta\right) = \left(\frac{R^3}{3}\cos\theta\right)d\theta \tag{7.7c}$$

Therefore, the coordinates of the centroid of the quadrant of a circle are obtained by substituting Eqs. (7.7a) through (7.7c) in Eq. (7.2) and integrating between the limits of θ from o to $\Pi/2$ as follows:

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^{\Pi/2} \frac{R^3}{3}\cos\theta\,d\theta}{\int_0^{\Pi/2} \frac{R^2}{2}\,d\theta}$$

$$\bar{x} = \frac{\frac{R^3}{3}[\sin\theta]}{\frac{R^2}{2}[\theta]} = \frac{2R}{3}\frac{1}{\Pi/2}$$

$$\bar{x} = \frac{4R}{3\Pi} \tag{7.8a}$$

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$$\bar{y} = \frac{\int y \, dA}{\int dA} = \frac{\int_0^{\Pi/2} \frac{R^3}{3} \sin\theta \, d\theta}{\int_0^{\Pi/2} \frac{R^2}{2} \, d\theta}$$

$$\bar{y} = \frac{\frac{R^3}{3}[-\cos\theta]}{\frac{R^2}{2}[\theta]} = \frac{2R}{3} \frac{1}{\Pi/2}$$

$$\bar{y} = \frac{4R}{3\Pi} \qquad\qquad (7.8b)$$

7.4 CENTROID OF A SEMICIRCLE

The centroidal coordinates of a semicircle are obtained similar to the quadrant of a circle, using Eqs.(7.7a) through (7.7c) in Eq. (7.2). But the limit of integration of θ is from 0 to Π.
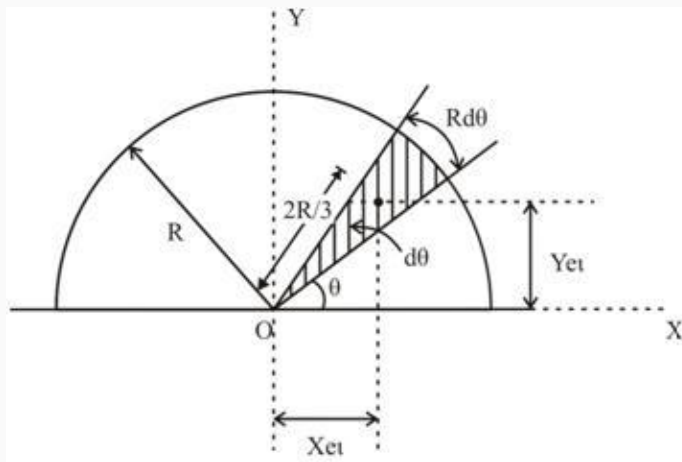

Fig.7.4 Centroid of a semicircle

$$\bar{x} = \frac{\int x \, dA}{\int dA} = \frac{\int_0^{\Pi} \frac{R^3}{3} \cos\theta \, d\theta}{\int_0^{\Pi} \frac{R^2}{2} \, d\theta} = \frac{\frac{R^3}{3}[\sin\theta]}{\frac{R^2}{2}[\theta]}$$

$$\bar{x} = 0 \qquad\qquad (7.9a)$$

This indicates that the centroid lies on y-axis.

$$\bar{y} = \frac{\int y \, dA}{\int dA} = \frac{\int_0^{\Pi} \frac{R^3}{3} \sin\theta \, d\theta}{\int_0^{\Pi} \frac{R^2}{2} \, d\theta}$$

$$\bar{y} = \frac{\frac{R^3}{3}[-\cos\theta]}{\frac{R^2}{2}[\theta]} = \frac{2R}{3} \frac{-(\cos\Pi - \cos 0^0)}{\Pi}$$
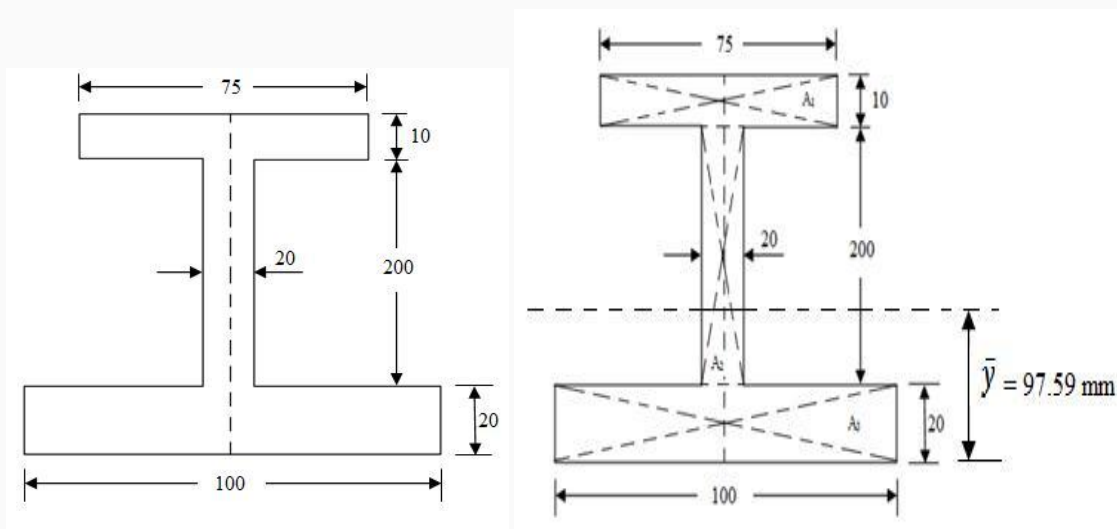
$$\bar{y} = \frac{4R}{3\Pi} \qquad\qquad (7.9b)$$

Table.1

Centroids of common shapes of areas

Type	Shape	$\bar{x}$	$\bar{y}$	Area A
Rectangle		$\dfrac{b}{2}$	$\dfrac{d}{2}$	bd
Right angled triangle		$\dfrac{b}{3}$	$\dfrac{h}{3}$	$\dfrac{bh}{2}$
Symmetrical triangle		0	$\dfrac{h}{3}$	$\dfrac{bh}{2}$
Circle		0	0	Πr^2
Semicircle		0	$\dfrac{4r}{3\Pi}$	$\dfrac{\Pi r^2}{2}$
Quadrant			$\dfrac{4r}{3\Pi}$	$\dfrac{\Pi r^2}{4}$
		$\dfrac{4r}{3\Pi}$		
General Spandrel		$\left(\dfrac{n+1}{n+2}\right)a$	$\left(\dfrac{n+1}{4n+2}\right)b$	$\dfrac{ab}{n+1}$
Circular sector		$\dfrac{2r\,sin\alpha}{3\alpha}$	0	ar^2

Example: Find the position of the centroid of I-section as shown in Figure.

(a)   (b)

(all dimensions are in mm)

Fig.7.5(a) and (b)

Sol: First, divide the figure into standard areas means rectangles.

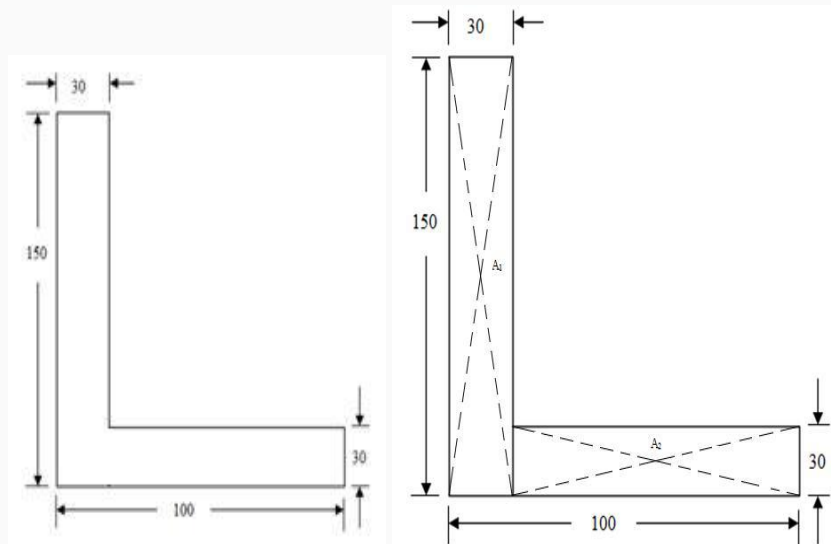I-section is symmetrical about y-axis.

A_1 = area of the top flange

A_2 = area of the web

A_3 = area of the bottom flange

Rectangles	Area (mm²)	Centroidal distance y_i from x axis	$A_i y_i$ (mm³)
A_1	75×10 = 750	20+200+5 = 225	168750
A_2	200×20 = 4000	20+100 = 120	480000
A_3	100×20 = 2000	= 5	10000
	$\Sigma A_i = 6750$		$\Sigma A_i y_i = 678750$

$$\bar{y} = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{658750}{6750} = 97.59 \text{ mm}$$

**Example** : Find the position of the centroid of the given Figure.

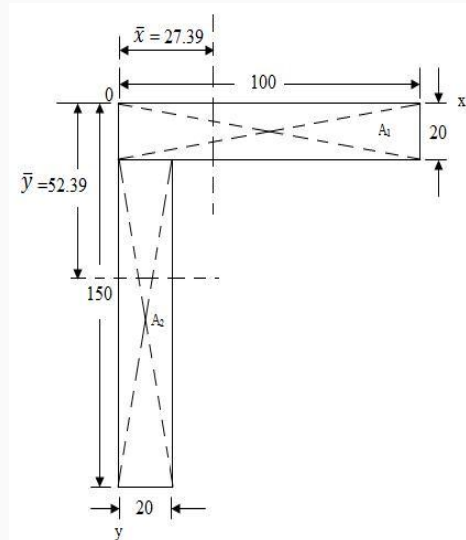(a)   (b)

Fig.7.6

Sol: First, divide the whole Figure into standard rectangle areas.

Rectangles	Area (mm²)	Centroidal distance x_i from y axis	Centroidal distance y_i from x axis	$A_i x_i$ (mm³)	$A_i y_i$ (mm³)
A_1	150×30 = 4500	15	75	67500	337500
A_2	70×30 = 2100	30+35 = 65	15	136500	31500
	$\Sigma A_i = 6600$			$\Sigma A_i x_i = 204000$	$\Sigma A_i y_i = 369000$

Example : Find the position of centroids in the following Figure.7.7(a):
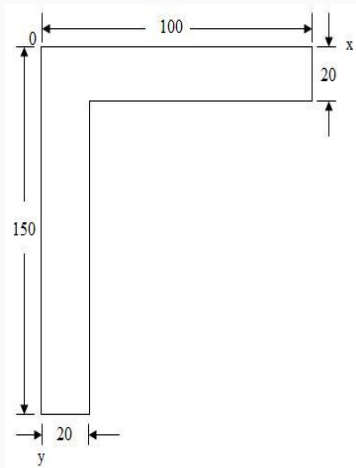
(a)   (b)

Fig.7.7

Sol: Divide the whole Figure into rectangular areas as shown in Figure.7.7 (b).

Rectangular sections	Area (mm^2)	Centroidal distance x_i from y axis	Centroidal distance y_i from x axis	$A_i x_i$ (mm^3)	$A_i y_i$ (mm^3)
A$_1$	100×20 = 2000	50	-10	100000	-20000
A$_2$	130×20 = 2600	10	-85	26000	-221000
	$\sum A_i = 4600$			$\sum A_i x_i =$ 126000	$\sum A_i y_i = -$ 241000

$$\bar x = {{\sum{A_i}{x_i}} \over {\sum{A_i}}} = {{126000} \over {4600}} = 27.39mm$$

$$\bar y = {{\sum{A_i}{y_i}} \over {\sum{A_i}}} = {{241000} \over {4600}} = 52.39mm$$

LESSON 8. MOMENT OF INERTIA

8.1 INTRODUCTION

As per Newton's first law of motion, it is the property of a matter by virtue of which it offers resistance to any change in its state is called Inertia. Usually, the moment of inertia refers to the mass moment of inertia. On the other hand, the moment of inertia of an area refers to the resistance of the cross-sectional area to bending or flexure. It represents the flexural strength of a cross-sectional area. The flexural strength is expressed in terms of bending formula given by

$$\frac{M}{I} = \frac{\sigma}{y} \qquad (8.1)$$

Where M is bending moment, I is the moment of inertia of the section about the axis of bending called *neutral axis* (axis at which the bending stress is zero), σ is the bending stress at any element at distance y from neutral axis. Equation (11.1) can be written as

$$\sigma = \frac{M}{I} \times y \qquad (8.2)$$

Equation (8.2) indicates the larger the moment of inertia of an area, the smaller the bending stress, which signifies more rigidity.

Fig.8.1 Moment of Inertia
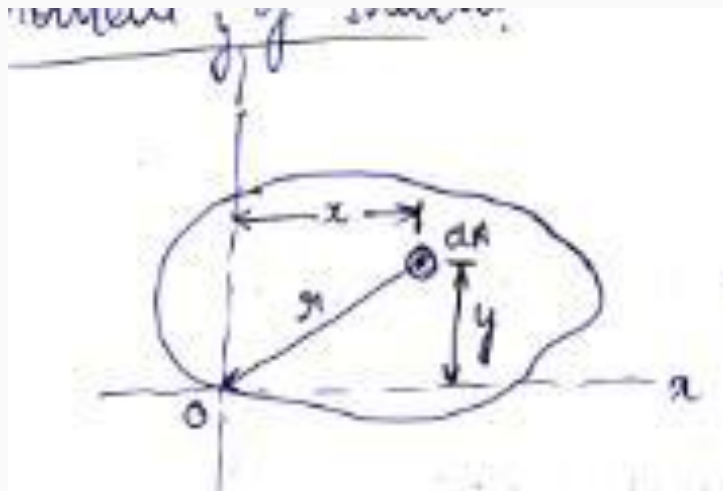
The moment of inertia of an area about x and y axes as shown in Figure 8.1 are defined by

$$I_x = \int y^2 dA \qquad (8.3a)$$

$$I_y = \int x^2 dA \qquad (8.3b)$$

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Where dA is the area of an element x, y stands for distance of the element from y and x axes respectively. Moment of inertia of an area is expressed as fourth power of the distance, that is cm^4, mm^4 or m^4.

In case of shafts subjected to torsion or twisting moment, the moment of inertia of the cross-sectional area about its centre O is considered. This moment of inertia about 0 is *called polar moment of inertia or moment of inertia about pole.* It is denoted by J. The problems in which J is used are cylindrical shafts subjected to torsion, slabs subjected to rotation etc. The polar moment of inertia is given by

$$J = \int r^2 \, dA \qquad\qquad (8.4)$$

Where r is the distance of an element area from O.

8.2 MOMENT OF INERTIA OF GEOMETRICAL FIGURES USING METHOD OF INTEGRATION
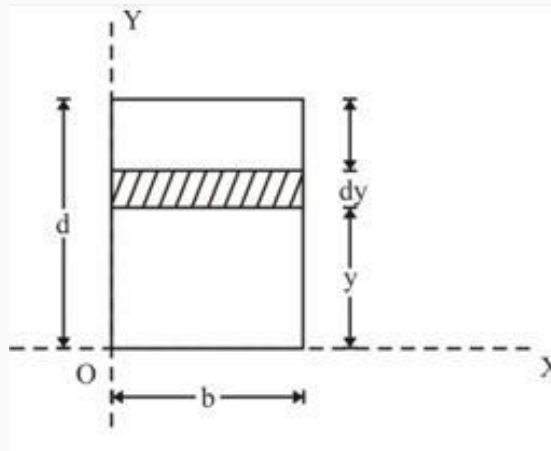
8.2.1 Rectangular Area

Fig. 8.2 Moment of inertia of a rectangular area

Consider an elemental strip of width b and thickness dy in a rectangular area as shown in Fig.8.2 . Area of the element and its centroidal coordinates from x and y axes are given as

$$dA = b \times dy$$

$$x = \[{b \over 2}\]$$
$$y = y + \[{dy \over 2}\]$$

Here $dy/2$ is very small and hence neglected. Substituting these values in Eq. (8.3a), the moment of inertia of rectangular about x-axis is determined as

$$I_x = \int y^2 \, dA = \int y^2 \, bdA$$

Integrating between the limits of y from 0 to d, yields

$$I_x = \smallint\limits_0^d b y^2 dA = \frac{b d^3}{3}$$
$$I_x = \frac{1}{3} bd^3 \tag{8.5}$$

Similarly, moment of inertia of rectangular about y-axis is

$$I_y = \frac{1}{3} db^3$$
$$\tag{8.6}$$

8.2.2 Triangular area

Consider a triangular plane area of base width b and height d as shown in Fig.8.3. From the similar triangles ACD and AOB, the width CD of the element can be written as
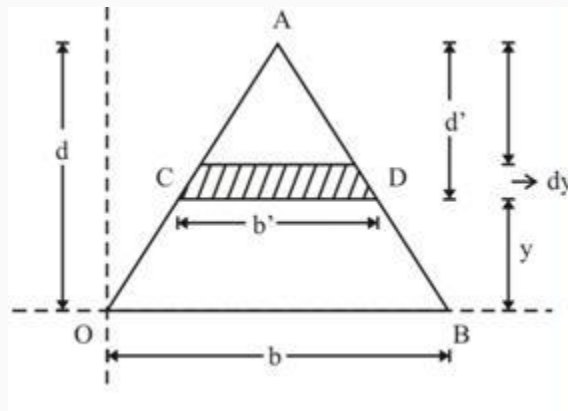

Fig.8.3 Moment of inertia of a triangle about its base

The area of the elemental strip (shown hatched) is

$$dA = b' \times dy = \frac{b}{d}\left(d - y \right)dy$$

The moment of inertia of the triangular area about x-axis (base of the triangle) is determined by

$$I_x = \smallint y^2\, dA = \smallint y^2 \frac{b}{d}\left(d - y \right)dy$$

Integrating between the limits of y from 0 to d, gives

$$I_x = \frac{b}{d}\smallint \limits_0^d \left(d - y \right) y^2 dy$$

$$I_x = \frac{b}{d}\left[\frac{d y^3}{3} - \frac{y^4}{4} \right] = \frac{b}{d}\left[\frac{d^3}{3} - \frac{d^4}{4} \right]$$

$$I_x = \frac{b d^3}{12} \tag{8.7}$$

8.2.3 Circular Area

Consider a circular area of radius R as shown in Fig.8.4(a) and 4(b), with an elemental area as indicated by shaded part. Fig.8.4(b) gives the enlarged view of the element ABCD.
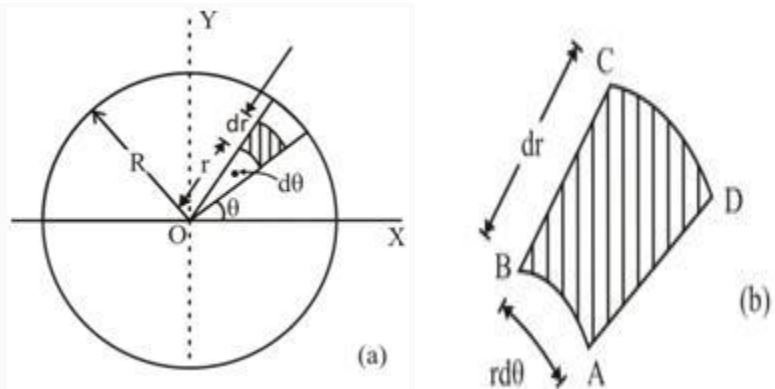
Fig.8.4 Moment of inertia of a circle

The area of the element,

$$dA = \text{Area of ABCD}$$

$$= \text{arc length AB} \times \text{BC}$$

$$dA = rd\theta \times dr$$

The centroidal coordinates of the element from x and y axes are given as

$$x = \left(r + {dr \over 2} \right)\cos\left(\theta + {d\theta \over 2}\right) \approx r\cos\theta$$

and

$$y = \left(r + {dr \over 2} \right)\sin\left(\theta + {d\theta \over 2}\right) \approx r\sin\theta$$

Using Eq. (8.3), the moment of inertia of circular area about x and y axes are determined as:

$$I_x = \int y^2 \, dA = \int r^2 \sin^2\theta \times rd\theta \times dr$$

$$I_x = \int r^3 \, dr \times \int \sin^2\theta d\theta$$

With the integration limits of r from 0 to R and θ from 0 to 2Π results

$$I_x = \int_0^R r^3 \, dr \times \int_0^{2\Pi} \sin^2\theta d\theta = {R^4 \over 4} \int_0^{2\Pi} \left({1-cos2\theta \over 2} \right) d\theta$$

$$I_x = {R^4 \over 8}\left[\theta - {sin2\theta \over 2} \right] = {R^4 \over 8} \times 2\Pi$$

$$I_x = {\Pi R^4 \over 4} \tag{8.8a}$$

and $\qquad\qquad\qquad\qquad I_y = \int x^2 \, dA = \int r^2 \cos^2\theta \times rd\theta \times dr$

Engineering Mechanics

$$I_y = \int_0^R r^3 \, dr \times \int_0^{2\Pi} \cos^2\theta \, d\theta = \frac{R^4}{4} \int_0^{2\Pi} \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$I_y = \frac{R^4}{8} \left[\theta + \frac{\sin 2\theta}{2} \right] = \frac{R^4}{8} \times 2\Pi$$

$$I_y = \frac{\Pi R^4}{4} \tag{8.8b}$$

As both x and y axes pass through the centroid of the circular area, Equations (8.8a) and (8.8b) give the moment of inertia of circle about its centroidal axes.

The above concept can be extended to obtain the moment of inertia of semicircular and quarter circular area as given below.
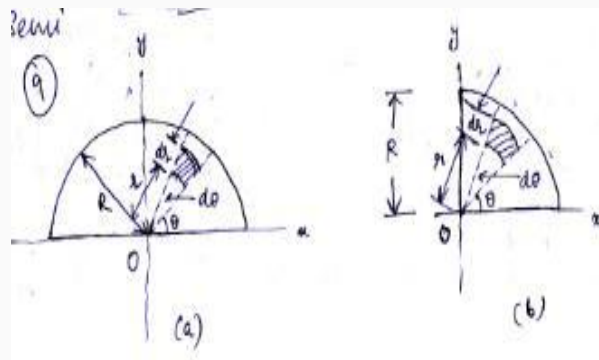
Fig.8.5 Moment of inertia of : (a) semicircle, and (b) quarter circle
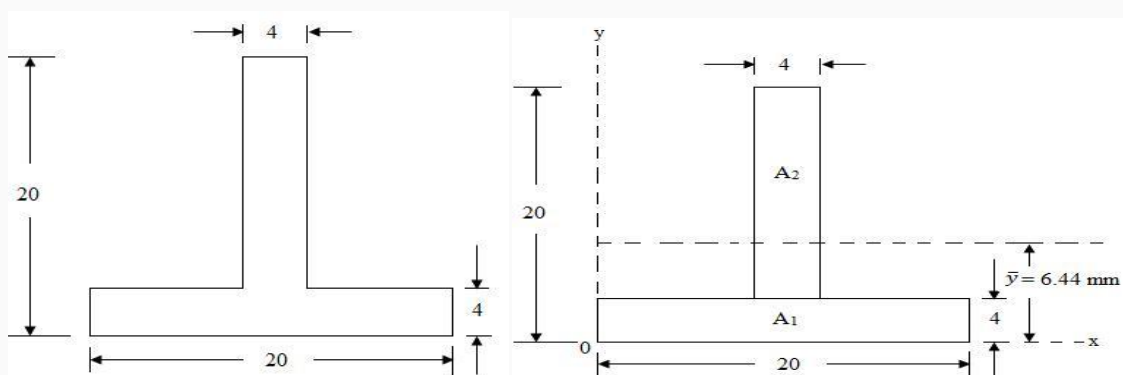
Moment of inertia of semicircular area about x and y axes is

$$I_x = I_y = \frac{R^4}{8} \tag{8.9}$$

Moment of inertia of quarter circular area about x and y axes is

$$I_x = I_y = \frac{R^4}{16} \tag{8.10}$$

Example: A T-section is 20×20×4 as shown in Figure. Calculate the moment of inertia of the section about the *x-x* axis parallel to the base of T-section passing through its centroid.

(a)   (b)

(all dimensions are in mm)

Fig. 8.6

Sol: The given T-section is symmetrical about the y-y axis. Therefore, $\bar x = {20 \over 2}$ = 10mm

For finding ,

Rectangular sections	Area (mm²)	y_i (mm)	$A_i y_i$ (mm³)
A₁	20×4 = 80	2	160
A₂	16×4 = 64	4+8 = 12	768
	$\Sigma A_i = 144$		$\Sigma A_i y_i = 928$

$\bar y = {\sum A_i y_i \over \sum A_i}$ = ${928 \over 144}$ = 6.44mm

The moment of inertia of the entire area about the centroidal x-x axis is given by

I_{xx} = $\left[{20 \times 4^3 \over 12} + \left(20 \times 4 \right){\left(4.44 \right)}^2 \right] + \left[{4 \times 16^3 \over 12} + \left(4 \times 16 \right){\left(5.56 \right)}^2 \right]$

= [10667 + 1577.09] + [1365.33 + 1978.47]

= 12244.09 + 3343.8

= 15587.89 mm⁴

Example: Determine the moment of inertia about the horizontal axis passing through the centroid of the section as shown in Fig.7.

(a)   (b)
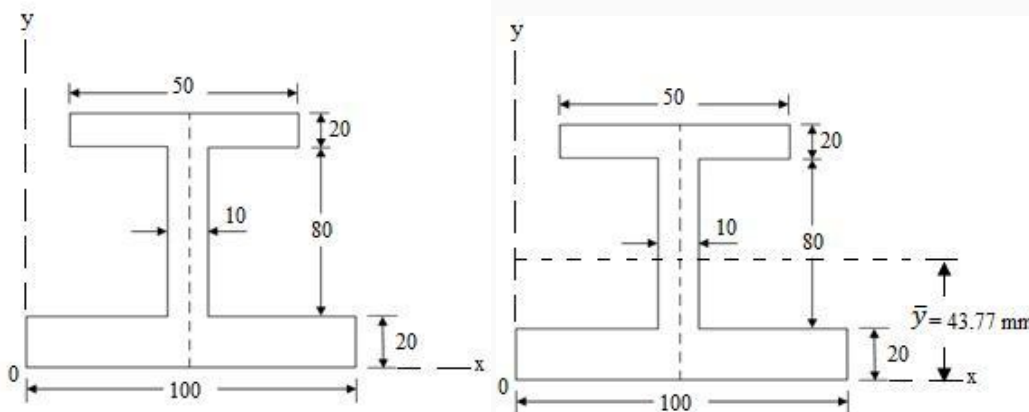
(all dimensions are in mm)

Fig.8.7

Sol: The given I-section is symmetrical about the y-y axis, therefore,

$$\bar x = {100 \over 2} = 50 \text{ mm}$$

To find $\bar x$, we have to divide the whole Figure into standard areas.

Rectangular sections	Area (mm²)	y_i (mm)	$A_i y_i$ (mm³)
A_1	100×20 = 2000	10	20000
A_2	80×10 = 800	20+40 = 60	4800
A_3	50×20 = 1000	20+80+10 = 110	110000
	$\Sigma A_i = 3080$		$\Sigma A_i y_i = 134800$

$$\bar y = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{134800}{3080} = 43.77 \text{ mm}$$

The moment of inertia of the entire area about the centroidal x-x axis is given by

$$I_{xx} = \left[\frac{100 \times 20^3}{12} + (100 \times 20)(33.77)^2\right] + \left[\frac{10 \times 80^3}{12} + (10 \times 80)(16.23)^2\right] + \left[\frac{50 \times 20^3}{12} + (50 \times 20)(66.23)^2\right]$$

$$= [66666.67 + 2280825.8 + 426666.67 + 210730.32 + 33333.33 + 4386412.9]$$

$$= 7404635.69 \text{ mm}^4$$

Example: Find the moment of inertia of a plate with a circular hole about its centroidal x axis as shown in Fig.8.

(a)   (b)
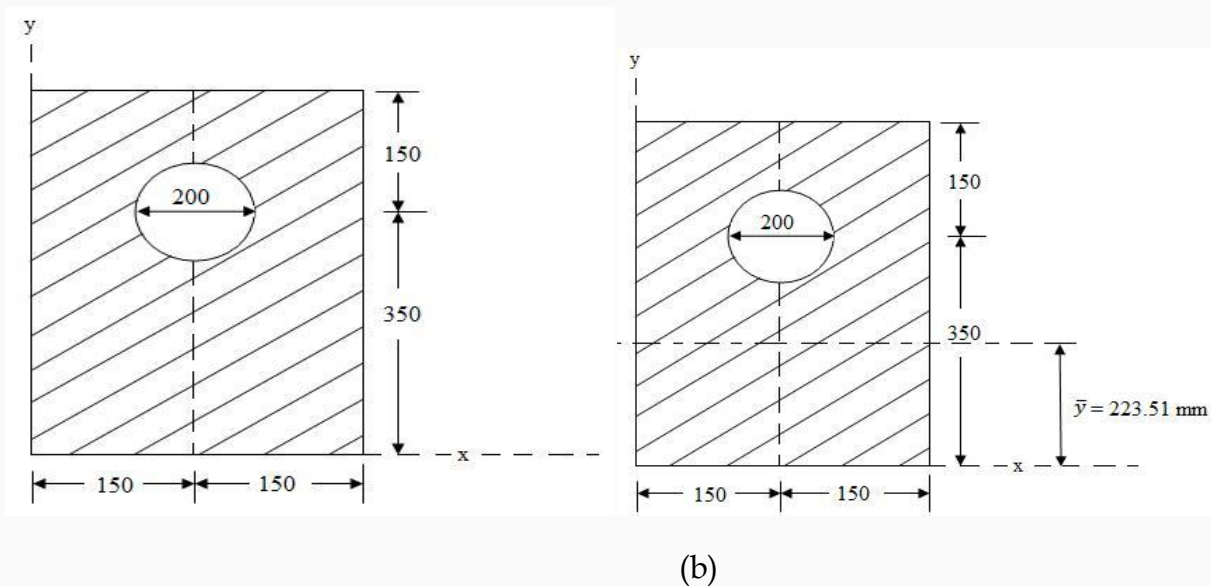
(all dimensions are in mm)

Fig.8.8

Engineering Mechanics

Sol: The given plate is symmetrical about the y-y axis, therefore $\bar x$ = 150 mm.

To find , $\bar y$

sections	Area (mm²)	y_i (mm)	$A_i y_i$ (mm³)
Rectangular Section	300×500 = 150000	250	37500000
Circular Section	$-\Pi r^2$ = -31415.93	350	-10995575.5
	ΣA_i = 118584.07		$\Sigma A_i y_i$ = 26504424.5

$$\bar y = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{26504424.5}{118584.07} = 223.51 \text{ mm}$$

The moment of inertia of the entire area about the centroidal x-x axis is given by

$$I_{xx} = \left[\frac{300 \times 500^8}{12} + (300 \times 500)(26.49)^2 \right] - \left[\frac{\Pi \times 200^4}{64} + (31415.93)(126.49)^2 \right]$$

$= [(3125000000 + 105258015) - (78539816.34 - 502646086.7)]$

$= [(3230258015) - (581185903)]$

$= 2649072112 \text{ mm}^4$

$= 2.65 \times 10^9 \text{ mm}^4$

LESSON 9. Radius of Gyration of an area about an Axis

9.1 PARALLEL AXIS THEOREM

Statement – Moment of inertia of an area about any reference axis is equal to the sum of moment of inertia of the same area about its centroidal axis parallel to reference axis and the product of area and the square of the distance between the reference and centroidal axes.

$$I_{ref} = I_{cent} + Ah^2 \qquad\qquad (9.1)$$
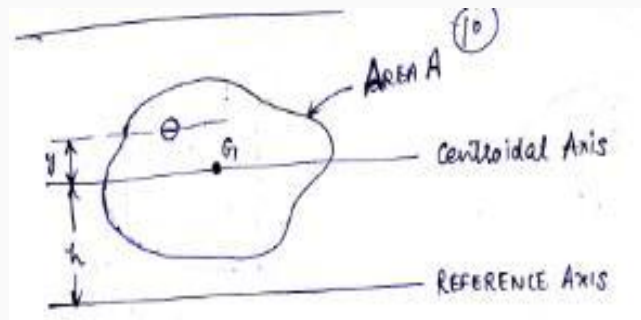

Fig.1 Parallel Axis Theorem

An axis passing through the centroid C of an area is called centroidal axis. The theorem can be proved as:

The moment of inertia of an area about a given reference axis is written using Eq.(8.3a) as

$$I_{ref} = \int (h+y)^2 \, dA$$

Here

$$(h+y)^2 = h^2 + 2hy + y^2$$

Therefore,

$$I_{ref} = \int h^2 \, dA + \int 2hy \, dA + \int y^2 \, dA$$

$$I_{ref} = h^2 \int dA + 2h \int y \, dA + \int y^2 \, dA$$

In the above expression $\int y^2 dA = I_{cent}$ as per Eq.(8.3a), $\int dA = A$ and $\int y dA$ = first moment of an area about centroidal axis which will be equal to zero.

Hence, $\qquad\qquad I_{ref} = h^2 A + I_{cent}$

or $\qquad\qquad I_{ref} = I_{cent} + Ah^2$

Example: The area in Fig… is symmetrical with respect to the x and y axis. The area is 150 m^2 and the moment of inertia with respect to the 'b' axis is 4200 m^4. Determine the moment of inertia of the area with respect to the 'a' axis.

Engineering Mechanics

Solution: According to the parallel axis theorem,

$I_b = I_{cent} + Ah^2$

$4200 = I_{cent} + 150(4)^2$

$I_{cent} = 2400 - 4200$

$I_{cent} = 1800 \text{ m}^4$

$I_a = I_{cent} + Ah^2$

$= 1800 + 150(6)^2$

$I_a = 7200 \text{ m}^4$

9.2 PERPENDICULAR AXIS THEOREM

Statement: Moment of inertia of a plane area about an axis perpendicular to the plane of the figure (z-axis) is equal to the sum of moment of inertia of the same area about two rectangular axes in the plane of the area (x and y axes).
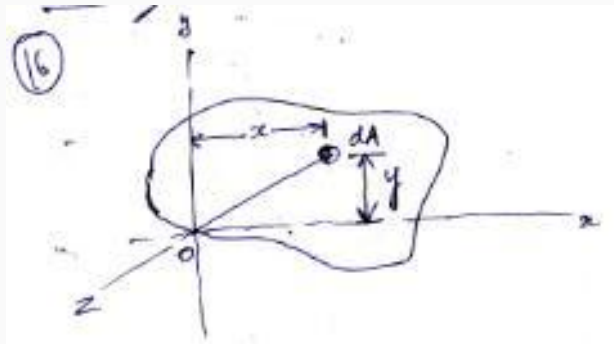
Fig.2 Perpendicular axis theorem

$$I_z = I_x + I_y \qquad\qquad (9.2)$$

According to the definition of moment of inertia [Eq.8.3a] about z-axis is given by

$$I_z = \int r^2 dA$$

From Fig. 2, $r^2 = x^2 + y^2$ can be used.

Therefore,

$$I_z = \int (x^2 + y^2) dA$$

$$Iz = \int x^2 dA + \int y^2 dA$$

$$I_z = I_y + I_x$$

Or $\qquad\qquad Iz = I_x + I_y$

9.3 MOMENT OF INERTIA OF GEOMETRICAL FIGURES ABOUT CENTROIDAL AXES

Using parallel Axis Theorem and moment of inertia of geometrical figures about one of their edges, the moment of inertia about their centroidal axis can be determined. Centroidal moment of inertia for a rectangle, triangle, semicircle and quarter circle are obtained as follows:

9.3.1 Rectangular Area

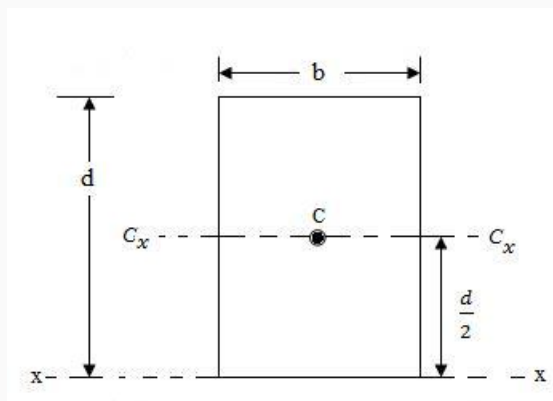Moment of inertia of a rectangle about x-axis as shown in Fig.3 is given as [Eq.(8.5)]

$I_x = \left[{{b{d^3}} \over {12}} \right]$

Fig.3 Moment of inertia of a rectangle about its centroidal axis

From parallel axis theorem,

$$I_{cent} = I_{ref} - Ah^2$$

$Ic_x = I_x - Ah^2 = \left[{{b{d^3}} \over 3} - bd \right] \left[{\left({{d \over 2}} \right)^2} \right]$

$$I_{cx} = \left[{{b{d^3}} \over {12}} \right]$$
(11.12a)

Similarly

$$I_{cx} = \left[{{b{d^3}} \over {12}} \right]$$
(11.12b)

LESSON 10. Triangular Area

Triangular Area

From Eq. (11.7), moment of inertia of a triangle about its base *x-x* as shown in Fig.12 is
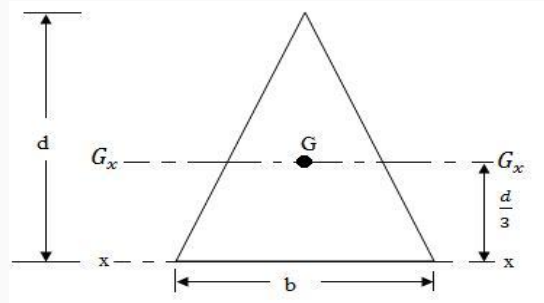
$$I_x = {{b{d^3}} \over {12}}$$

Fig.12 Moment of inertia of a triangle about its centroidal axis

The moment of inertia about centroidal axis is

$$I_{Gx} = I_x - Ah^2 = Ah^2 {{b{d^3}} \over {12}} - {{bd} \over 2}{\left({{d \over 3}} \right)^3}$$

$$I_{Gx} = {{b{d^3}} \over {36}} \qquad (11.13)$$

Semicircular Area

Equation (11.9) gives the moment of inertia of semicircular area about *x* and *y* axes as given in Fig.13. That is

$$I_x = I_y = {{{R^4}} \over 8}$$

Fig.13 Moment of inertia of a semicircular about its centroidal axis
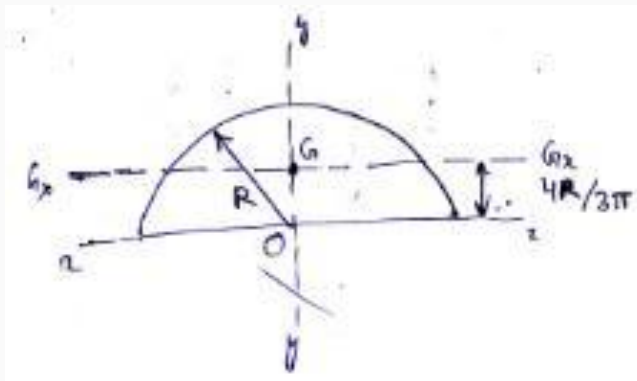
However, y-axis passes through the centroid G, hence, moment of inertia about centroidal y-axis is same as

$$I_{Gy} = I_y = \frac{R^4}{8}$$

Moment of inertia about centroidal x-axis is

$$I_{Gx} = I_x - Ah^2 = \frac{R^4}{8} - \frac{R^2}{2}\left(\frac{4R}{3}\right)^2$$

$$I_{Gx} = 0.11R^4 \tag{11.14}$$

Quarter Circular Area

Moment of inertia of quarter circular area about x and y axes [Eq.(11.10)] as shown in Fig.14 is
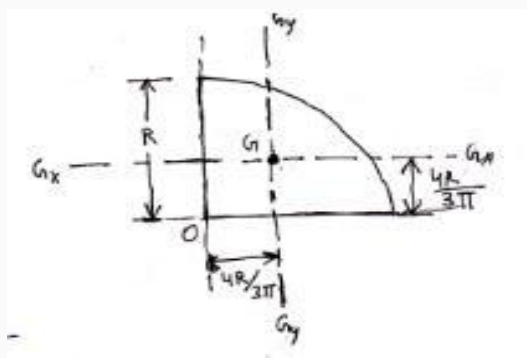
$$I_x = I_y = \frac{R^4}{16}$$

Fig.14 Moment of inertia of a quarter circle about its centroidal axis

Therefore, moment of inertia about centroidal axes (G_x and G_y) is determined as

$$I_{Gx} = I_x - Ah^2 = \frac{R^4}{16} - \frac{R^2}{4}\left(\frac{4R}{3}\right)^2$$

$$I_{Gx} = 0.055R^4 \tag{11.15a}$$

Similarly, I_{Gy} will be obtained and it is equal to I_{Gx} itself.

$$I_{Gy} = 0.055R^4 \tag{11.15b}$$

Engineering Mechanics

Moment of Inertia of hollow rectangular and circular sections about their centroidal axes are given in Fig.15.
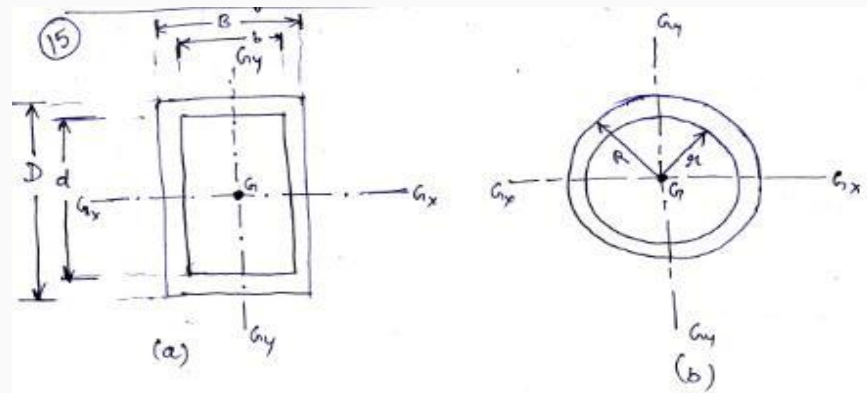

Fig.15 (a) Hollow rectangular section and (b) Hollow circular section

Hollow rectangular section:

$$I_{Gx} = \frac{1}{12}\left(BD^3 - bd^3 \right) \quad (11.16a)$$

$$I_{Gy} = \frac{1}{12}\left(DB^3 - db^3 \right) \quad (11.16b)$$

Hollow circular section:

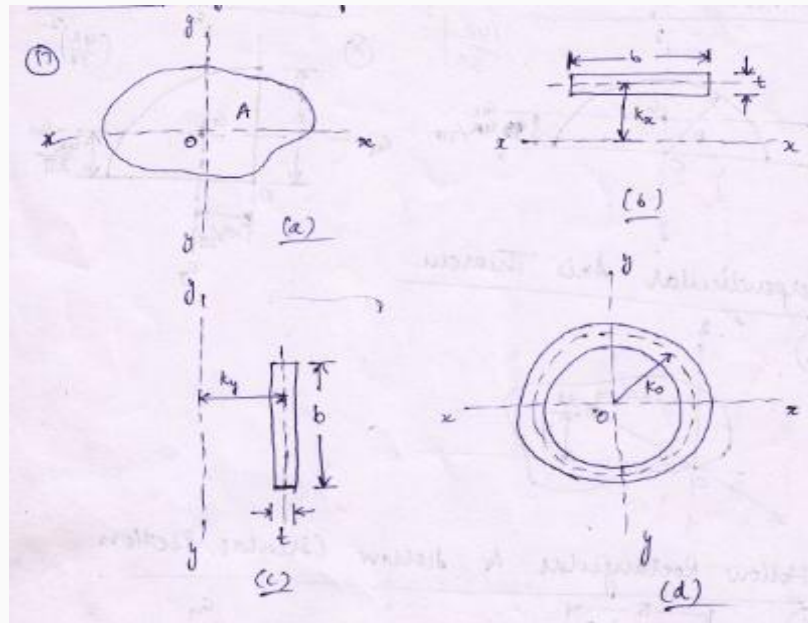$$I_{Gx} = I_{Gy} = \frac{\pi}{4}\left(R^4 - r^4 \right) \quad (11.17)$$

LESSON 11. RADIUS OF GYRATION OF AN AREA ABOUT AN AXIS

11.1 RADIUS OF GYRATION OF AN AREA ABOUT AN AXIS

Consider an area A whose moment of inertia about x and y axes is I_x and I_y. Suppose the area concentrated into a thin strip parallel of to x-axis as shown in Fig.11.1. such that its moment of inertia about x-axis is same as I_x. Then the distance at which this strip is to be placed from x-axis is called *radius of gyration* of the area about x-axis, denoted by R_x.

Fig.11.1 Radius of gyration

According to the parallel axis theorem, the moment of inertia of the strip about x-axis is

$$I_{xx} = I_{cent} + AR_x^2 \qquad (11.19)$$

But $I_{cent} = {{b t^3} \over {12}}$ is negligible as t is small, t^3 is negligibly smaller.

Therefore,

$$I_{xx} = AR_x^2$$

Or $\qquad R_x = \sqrt{{{I_{xx}} \over A}} \qquad (11.20)$

It is expressed in mm or cm or m.

Hence, the radius of gyration of an area about an axis is defined as the distance from the axis where thin strip (having an area equal to the given area) is to be placed so that the moment of inertia of the strip about the axis is equal to the moment of inertia of the given area about that axis.

Engineering Mechanics

Similar to Eq. (11.20), the radius of gyration of the area about y-axis [Figure 11.13(c)] and pole or polar radius of gyration [Figure 11.13(d)] are obtained as given in Eqs. (11.21) and (11.22).

$$R_y = \sqrt{\frac{I_{yy}}{A}} \tag{11.21}$$

$$R_0 = \sqrt{\frac{J}{A}} \tag{11.22}$$

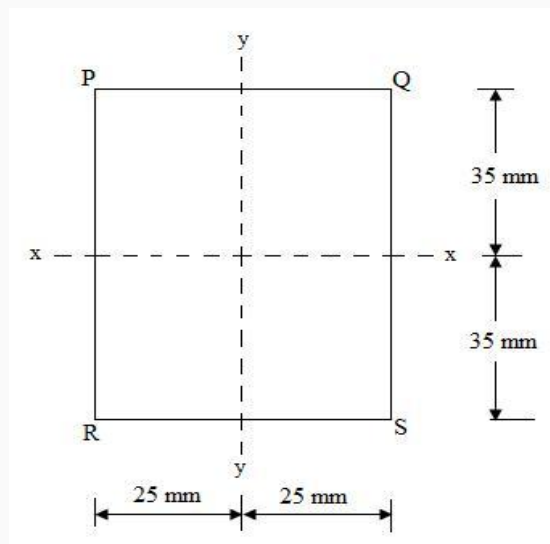Where J = polar moment of inertia using perpendicular axis theorem

$$J = I_{xx} + I_{yy}$$

$$AR_0{}^2 = AR_x{}^2 + AR_y{}^2$$

Hence

$$R_0{}^2 = R_x{}^2 + R_y{}^2 \tag{11.23}$$

Example 1: Find the moment of inertia of the rectangular section as shown in Figure------- about the faces PQ and RS.

Fig.11.2

Sol: From the Figure, width b = 50 mm and depth d = 70 mm

The moment of inertia of the entire rectangular area about PQ is given by

$I_{PQ} = I_{xx} + Ah^2$

$$= \frac{b d^3}{12} + \left(b \times d \right) h^2$$

$$= \frac{50 \times 70^3}{12} + \left(50 \times 70 \right) \times \left(35 \right)^2$$

$$=1429166.67 + 4287500$$

$$= 5716666.67 \text{ mm}^4$$

$I_{BC} = I_{yy} + Ah^2$

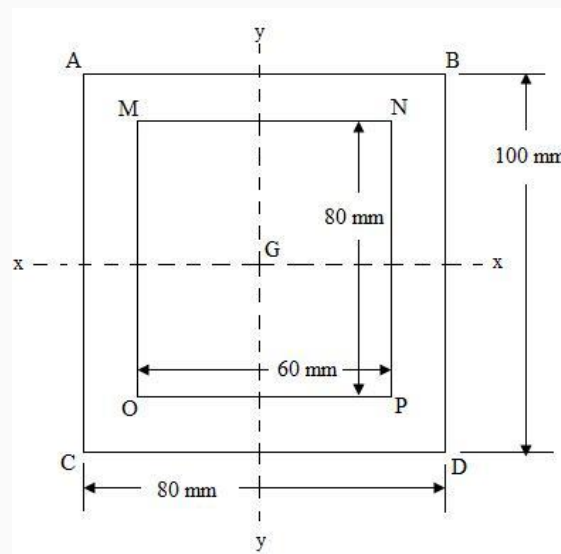$$= \frac{b d^3}{12} + \left(b \times d \right) h^2$$

$$= \frac{70 \times 50^3}{12} + \left(70 \times 50 \right) \times \left(25 \right)^2$$

$$= 729166.67 + 2187500$$

$$= 2916666.67 \text{ mm}^4$$

Example 2: Find the moment of inertia of a hollow rectangular section about two mutually perpendicular axes in its plane and passing through its centroid . The dimensions of the outer rectangle are 80 mm×100 mm and that of the inner rectangle is 60 mm×80 mm.

Fig.11.3

Sol: Outer rectangle: B = 80 mm and D = 100 mm

   Inner rectangle: b = 60 mm and d = 80 mm

The formula is $I_{Gx} = \frac{1}{12}\left(B D^3 - b d^3 \right)$

$$= \frac{1}{12}\left(80 \times 100^3 - 60 \times 80^3 \right)$$

$$= 4106666.67 \text{ mm}^4$$

$$I_{Gy} = \frac{1}{12}\left(D B^3 - d b^3 \right)$$

$$= \frac{1}{12}\left(100 \times 80^3 - 80 \times 60^3 \right)$$

= 2826666.67 mm⁴

Example 3: Determine the moment of inertia and radius of gyration about the horizontal axis passing through the centroid of the section as shown in Figure.
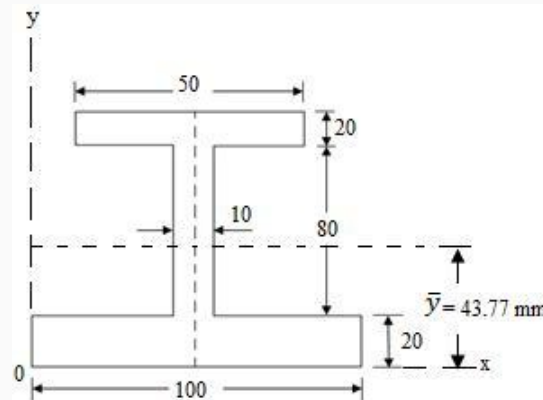

Fig.11.4

Sol: The given I-section is symmetrical about the y-y axis, therefore,

$$\bar x = {100 \over 2} = 50 \text{ mm}$$

To find , we have to divide the whole Figure into standard areas.

Rectangular sections	Area (mm²)	y_i (mm)	$A_i y_i$ (mm³)
A_1	100×20 = 2000	10	20000
A_2	80×10 = 800	20+40 = 60	4800
A_3	50×20 = 1000	20+80+10 = 110	110000
	$\Sigma A_i = 3080$		$\Sigma A_i y_i = 134800$

$$\bar y = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{134800}{3080} = 43.77 \text{ mm}$$

The moment of inertia of the entire area about the centroidal x-x axis is given by

$$I_{xx} = \left[\frac{100 \times 20^3}{12} + (100 \times 20)(33.77)^2\right] + \left[\frac{10 \times 80^3}{12} + (10 \times 80)(16.23)^2\right] + \left[\frac{50 \times 20^3}{12} + (50 \times 20)(66.23)^2\right]$$

$$= [66666.67 + 2280825.8 + 426666.67 + 210730.32 + 33333.33 + 4386412.9]$$

$$= 7404635.69 \text{ mm}^4$$

Radius of gyration, $Rx = \sqrt{\frac{I_{xx}}{A}}$

$$\sqrt{\frac{7404635.69}{3080}} = 49.03 \text{ mm}$$

MODULE 4. FRICTION AND FRICTIONAL FORCES

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LESSON 12. FRICTION AND FRICTIONAL FORCES

</div>

12.1 INTRODUCTION

When two bodies in contact have a tendency to move over each other a resistance to the movement is set up. This resistance to the movement is called the Force of friction or simply friction. Friction depends upon the nature of the surface of contact. Friction acts parallel to the surface of contact. The direction of this frictional force on any one of the surfaces of contact will be opposite to the direction in which the contact surface tends to move. In other words, friction opposes motion.

- Friction is an important force in many aspects of everyday life.
- If there is too much friction, loss of energy, wear and tear of materials in contact occurs.
- If there is less friction or no friction, this would result in 'slipping' all around.
- For example oil in the engine of car is meant to minimize friction between moving parts in contact to reduce excessive friction for reducing loss of energy and material.
- We need the friction between the tires on the road surface, to let the wheels roll. Friction is caused due to the unevenness of the surface of contact of bodies tending to move past each other.

When one body moves relative to the other, the tangential forces will always be developed along the surfaces of contact. These tangential forces are called frictional forces.
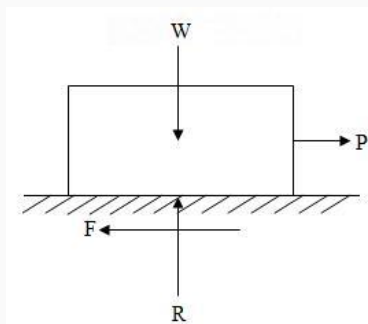

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Fig.12.1

</div>

Fig.12.1 shows a wooden block resting on a rough horizontal table. Let W be the weight of the block. Let the block be subjected to a horizontal force P. When this applied force is sufficiently small, the block will remain in equilibrium. The rough table surface will exert a normal reaction R and a tangential reaction (friction) F on the block so as to keep the block in equilibrium. Resolving, the forces on the block vertically and horizontally, we get

$$R = W$$

and $$F = P$$

Engineering Mechanics

Suppose the force P is gradually increased. The friction F will also increase, so that at every stage of equilibrium $F = P$. But there is a limit to which friction can increase. We can not expect the frictional resistance to go on increasing infinitely as the force P is increased. Let F_1 represent the greatest possible friction. Let P_1 be the applied force corresponding to this condition. At this stage,

$$F_1 = P_1$$

If the applied force exceeds P_1, the block will slip on the table since the friction resistance cannot increased beyond the value F_1. The greatest possible friction depends upon the normal reaction. In the example given above the normal reaction is equal to W. When the block is at the point of sliding the ratio of this friction to the normal reaction is found to be a constant which depends upon the surfaces of contact. This contact is called the coefficient of friction.

i.e. If F is the frictional resistance when the block is in the limiting equilibrium

$F = \mu R$ where μ is the coefficient of friction

12.2 ANGLE OF FRICTION

Consider the block resting on the horizontal rough surface. Let R' be the resultant reaction (resultant of the normal reaction R and friction F). The angle Θ between the resultant reaction and the normal to the surface is called the angle of friction.
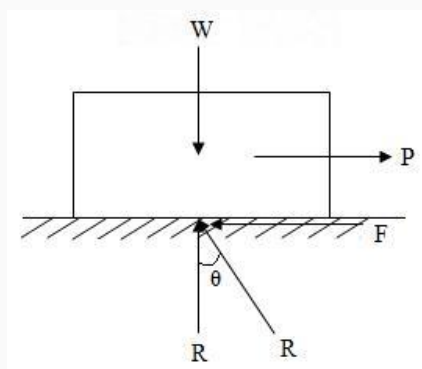

Fig.12.2

Obviously, $\tan\Theta = \left[{F \over R}\right]$

Corresponding to the limiting condition of equilibrium, the friction F will reach the maximum value. Corresponding this condition the angle of friction reaches a maximum value λ, so that
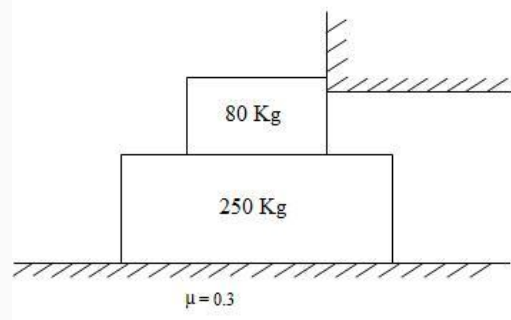
$$\tan \lambda = \left[{{F_{max}} \over R}\right] = \mu$$

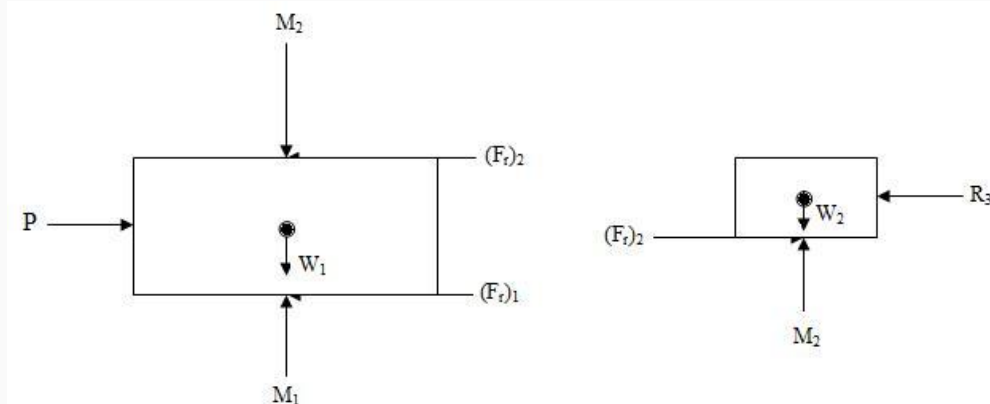Therefore $\qquad\qquad\qquad\qquad \lambda = \left[\tan^{-1}\mu\right]$

The inclination of the resultant reaction with the normal when the condition of limiting equilibrium is reached is called the *angle of limiting friction*.

Engineering Mechanics

Example: If coefficient of friction between all surfaces shown in Fig.12.3 is 0.30. What is the horizontal force required to get 250 kg block moving to the right?

Fig.12.3

Solution: In this problem 80 kg block is completely restrained against motion and as we apply force P on 250 Kg block as shown in Fig.3, there is no force acting vertically at the contact surfaces between the obstacle and 80 kg block. Hence frictional force acts only at bottom and top surfaces of 250 kg block while only at lower surface of 80 kg block. Refer Fig.12.4.

(a)   Lower block   (b) Upper block

Fig.12.4

Note that $\sum F_y = 0$ for upper block gives $M_2 = W_2$

Therefore, $M_2 = 80 \times 9.81 = 784.8$ N

For lower block, $\sum F_y = 0$ gives $M_1 = W_1 + M_2$

Therefore, $R_1 = (250 \times 9.81) + 784.8 = 3237.3$ N

Also $(F_r)_1 = \mu\, R_1 = (0.3)\,(3237.3) = 971.19$ R

and $(F_r)_2 = \mu\, R_2 = (0.3)\,(784.8) = 235.44$ R

$\sum F_x = 0$ for lower block gives $P = (F_r)_1 + (F_r)_2$

or $\qquad\qquad\qquad\qquad\qquad$ P = 1206.63 N

Note - $\sum F_x = 0$ is not necessary for upper block in this problem.

Example: A pull of 20 kN at 30° to the horizontal in necessary to move a block of wood on a horizontal table (Fig.12.5).If the coefficient of friction between the bodies in contact is 0.25, what is the weight of the block?
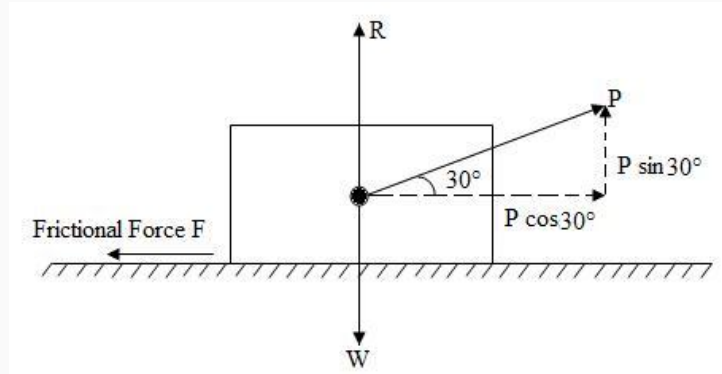
Fig.12.5

Solution: Pull (P) = 20 kN, inclination of the force θ = 30° to the horizontal

Coefficient of friction, μ = 0.25

Let W = unknown weight of the block.

Resolving the forces horizontally, we get

F = P Cos30°

μR = P Cos30° --(i)

Resolving the forces perpendicular to the plane vertically, we get

R + P Sin30° = W

R = W – P Sin30° ---(ii)

Substituting the value of R in Eqn.(i), we get

μ(W – P Sin30°) = P Cos30°

0.25(W – 20 Sin30°) = 20 Cos30°

0.25(W – 10) = 17.32

W = 79.28 kN

LESSON 13. ANGLE OF REPOSE

13.1 ANGLE OF REPOSE
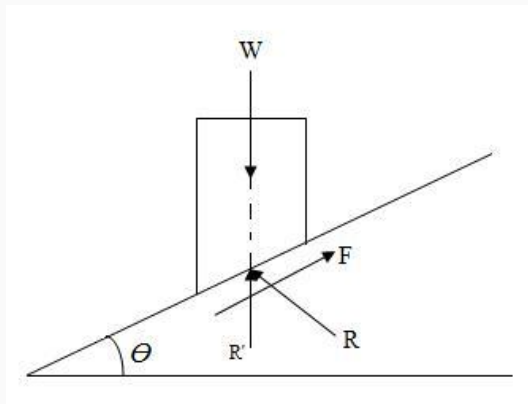
 Fig.13.1

Fig.13.1 shows a block of weight W resting on a rough inclined plane inclined at Θ with the horizontal plane. Let R be the normal reaction and F be the friction. Resolving the forces along the plane, $W \sin\Theta = F$ ....(i)

Resolving the forces normal to the plane, $W \cos\Theta = R$ ....(ii)

From equation (i) and (ii), $\tan\Theta = {F \over R}$

But we know that the tangent of the angle of friction is also equal to ${F \over R}$.

Therefore, Angle of the plane = Angle of friction

Suppose the angle of the plane, Θ is increased to a value ϕ so that the block is at the point of sliding. Corresponding to this condition,

$\tan\phi = {{F_{max}} \over R} = \mu = {\tan ^{-1}}\lambda$

Therefore, $\phi = \lambda$

The maximum inclination of the plane at which a body can remain in equilibrium over the plane entirely by the assistance of friction is called the angle of repose.

Obviously angle of repose ϕ = Angle of limiting friction λ.

Example: If the weight of the body is 130 N is at rest on a horizontal plane. A horizontal force of 100 N will just cause it to slide, Determine the limiting friction and coefficient of friction.

Solution: F_{max} = limiting friction

= Greatest force applied horizontally on the body resting on the horizontal

Plane = 100 N

Coefficient of friction = μ = $\left[\frac{F_{max}}{R}\right]$ = $\left[\frac{100}{130}\right]$ = 0.77

Example: The coefficient of friction between a body of weight 120 N and a horizontal plane on which it rests is 0.6. (a) Calculate the horizontal force which acting on the body can just cause it to slide, (b) What least horizontal force would cause the body to slide if an additional weight of 30 N be added to the body?

Solution: *(a)* Normal Reaction R = 120 N

Coefficient of friction = μ = 0.6

Therefore, Maximum friction possible = μR = 0.6 × 120 = 72 N

Therefore, Horizontal force required to just make the body slide = 72 N

(a) Normal Reaction = 120 + 30 = 150 N

(b) Maximum friction = μR = 0.6 × 150 = 90 N

Therefore, least force necessary to cause sliding = 90 N

13.2 LAWS OF FRICTION

The following are the laws of friction:

(i) Friction in non-limiting equilibrium

(ii) Friction in limiting equilibrium

(iii) Friction during motion

First Law (Applicable to non-limiting, limiting and dynamic condition). Friction always opposes motion. Frictional forces come into play only when a body is urged to move. Frictional force will always act in a direction opposite to that in which the body is urged to move.

Second Law (Applicable to non-limiting condition of equilibrium). The magnitude of the frictional force is just sufficient to prevent the body from moving. That is, only as much resistance as required to prevent motion will be offered as friction.

Third Law (Applicable to limiting condition of equilibrium). The limiting frictional resistance bears a constant ratio with the normal reaction. This ratio depends on the nature of the surfaces of contact. The limiting frictional resistance is independent of the area of contact.

Fourth Law (Friction during motion *i.e* Kinetic Friction). When motion takes place as one body slides over the other the magnitude of the frictional resistance will be less than that offered at the condition of limiting equilibrium. The magnitude of the friction will depend

61

only on the nature of the sliding and independent of the shape or the extent of the contact area.
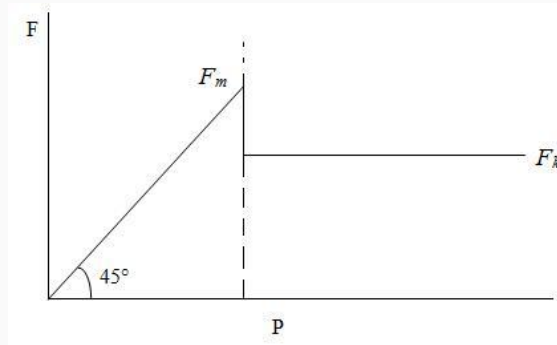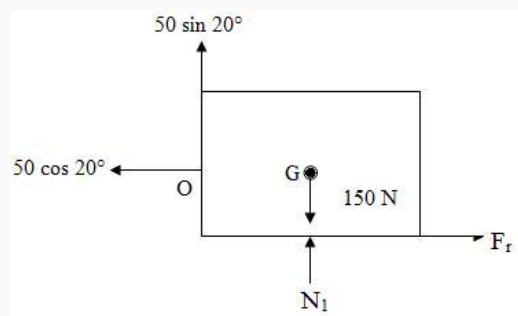

Fig.13.2

When a body resting on a horizontal surface is subjected to gradually increasing horizontal force P the condition $F = P$ is satisfied as long as the body is in equilibrium. In the condition of equilibrium, the maximum value of the friction is F_m which occurs when the body is in limiting equilibrium. If the horizontal force P on the body is further increased, then the equilibrium of the body is broken and the body moves over the surface. In this condition the value of the friction is F_k which is less than F_m.

Example: A 150 N block is placed on a rough horizontal surface as shown in Fig.13.3 knowing that block just slides for P = 50 N and Θ = 20°, determine μ. For same Θ =20° and same μ, determine magnitude of P to just slide the same block, 'P' being applied in .opposite direction, at same point O.
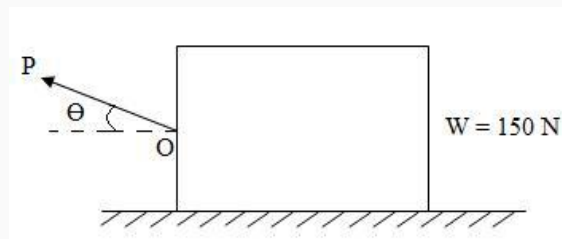
Fig.13.3

Solution: Refer Fig.13.3 showing Free Body Diagram of the block for case I

Fig.13.3(Case I)

$\sum F_x = 0$ gives $F_r = 50 \cos 20°$

Where $F_r = \mu\, R_1$

$R_1 = 150 - 50 \sin 20° = 132.9$ N

$\mu\, (132.9) = 50 \cos 20°$

$\mu = 0.353$

Fig.3(Case II) shows Free Body Diagram of same block for 'P' in opposite direction.

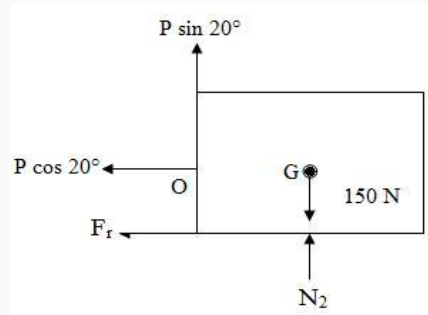**Fig.13.3(Case II)**

Here $R_2 = 150 + P \sin 20°$ And $F_r = P \cos 20° = R_2$

$P\, (0.939) = 0.353\, (150 + 0.342P)$

$P = 64.73$ N

Example: Two bodies weighing 120 kN and 100 kN rest on an inclined plane and are connected by a chord which is parallel to the plane. The body weighing 100 kN is below the one weighing 120 kN and coefficient of friction for 100 kN body is 0.2 and that for 120 kN is 0.3. Find the inclination of the plane to the horizontal and the tension in the chord when motion is about to take place, down the incline.
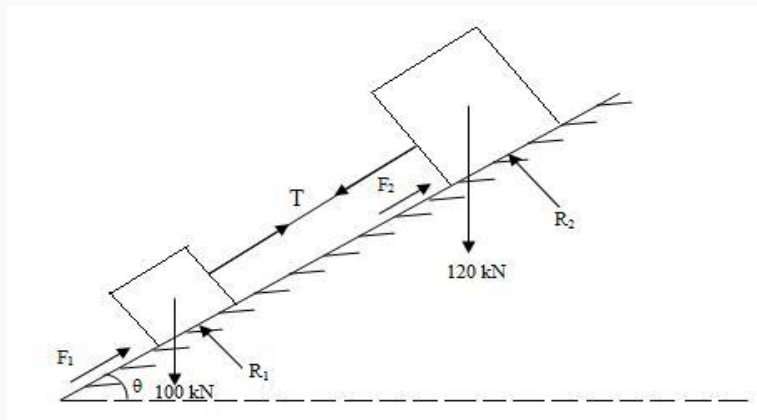
Fig.13.4

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Solution: Consider the weight of body A, i.e. 100 kN.

Resolving the forces parallel and perpendicular to the inclined plane, we get

$F_1 + T - 100 \sin\theta = 0$

$\mu R_1 + T - 100 \sin\theta = 0$

$0.20 R_1 + T - 100 \sin\theta = 0$

$T = 100 \sin\theta - 0.25 R_1$

$R_1 = 100 \cos\theta$

$100 \sin\theta - T = 0.20 \times 100 \cos\theta$ ---(i)

Consider the body B, i.e. 120 kN.

Resolving the forces parallel and perpendicular to the inclined plane, we get

$F_2 - T - 120 \sin\theta = 0$

$0.3 R_2 - T = 120 \sin\theta$

$R_2 = 120 \cos\theta$

$0.3 \times 120 \cos\theta - T = 120 \sin\theta$---(ii)

Solving Eqn. (i) and (ii), we get

$100 \sin\theta - T = 25 \cos\theta$

$36 \cos\theta - T = 120 \sin\theta$

$100 \sin\theta - 25 \cos\theta = 36 \cos\theta - 120 \sin\theta$

$61 \cos\theta = 220 \sin\theta$

$\tan\theta = \[{{61} \over {220}} = 0.277\]$

$\theta = 15.49°$

Therefore, T = 2.62 kN

LESSON 14. CONE OF FRICTION

14.1 CONE OF FRICTION
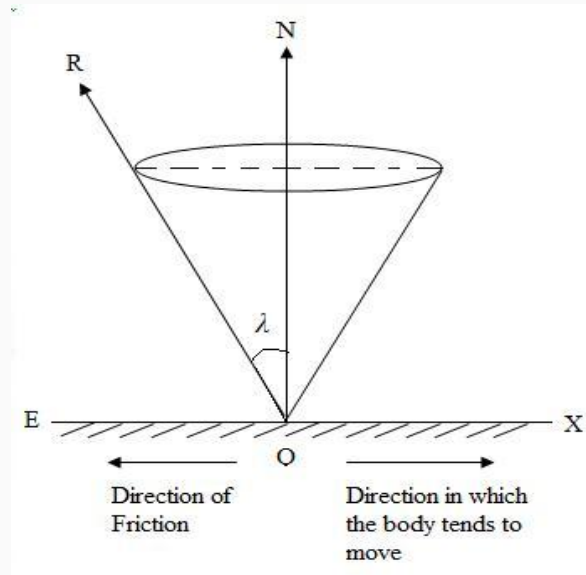
Fig.14.1

Let ON represent the normal reaction offered by a surface on a body (Fig.14.1). If OX is the direction in which the body tends to move then the force of friction acts in the opposite direction i.e, along OE. If the body be in limiting equilibrium the resultant R makes an angle λ with the normal ON.

Suppose the body is at the point of sliding in other direction, it is easily seen that the resultant reaction will make the same angle λ with the normal. Hence, when limiting friction is offered the line of action of the resultant reaction should always lie on the surface of an inverted right circular cone whose semi-vertex is λ. This cone is called the *cone of friction*.

Example: A block is weighing 50 kg is placed on a rough surface whose coefficient of friction is 0.30 and inclined force P is applied at its top corner as shown in Fig.14.2. Determine whether the block will tip or slide and the force P required to move the block.
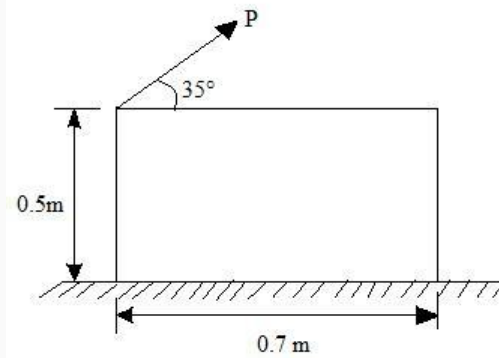
Fig.14.2

Solution: W = 50 × 9.81 = 490.5 N

By referring Fig.14.2

$\sum F_y = 0$

$R_1 - W + P \sin 35°$

$R_1 = 490.5 - 0.574\ P$

Or

Hence limiting friction force = $\mu\ R_1$

= 0.3 (490.5 – 0.574 P)

= 147.15 – 0.1722 P

Now F = P cos 35° using $\sum F_x = 0$

So, F = 0.819 P

Let us assume that block slides before tipping,

Then 0.819 P = 147.15 – 0.1722 P

P = 148.46 N

Now check the tipping of block,

$\sum M_0 = 0 = (148.46 \sin 35°)\ (0.35) + (148.46 \cos 35°)\ (0.5) - N_1\ (x)$

29.83 + 60.79 – $R_1\ x$ = 0

90.62 = $R_1\ x$

By putting the value of R_1, we get

90.62 = (490.5 – 0.574 P) x

x = 0.224 m

As x < 0.35 m, tipping will not occur.

Hence block slides with P = 148.46 N

Example: A ladder 5 m long weighing 200 N is resting against a wall at an angle 0f 60° to the horizontal ground. A man weighing 500 N climbs the ladder. At what position along the ladder from bottom does he induce slipping. The coefficient of friction for both the wall and the ground with the ladder is 0.2.
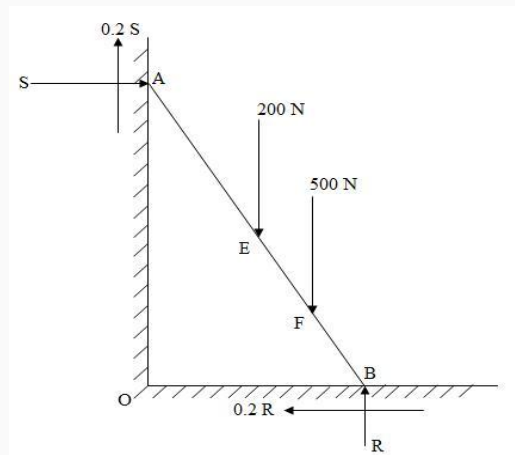
Fig.14.3

Solution: Let the ladder be at the point of sliding when the man is at a distance x metres from the foot of the ladder. See Fig.14.3.

Let F be the position of the man.

BF = x, BE = AE = 2.5m

Let the normal reactions at the floor and the wall be R and S. Friction at the floor and the wall will be 0.2R and 0.2S respectively.

Resolving the forces on the ladder horizontally and vertically,

$S = 0.2R$--------------(1)

$R + 0.2S = 700$ N-------------(2)

From equations (1) and (2), we get R = 673.08 N

$$S = 134.62 \text{ N}$$

Taking moments about the lower end of the ladder,

$$200 \times 2.5 \cos 60° + 500 \times x \cos 60° = S \times 5 \sin 60° + 0.2S \times 5 \cos 60°$$

$$250 + 250x = 2.5\ S + 0.5\ S$$

$$250 + 250x = S\ (2.5\ + 0.5)$$

$$250 + 250x = 134.62\ (2.5\ + 0.5)$$

$$250 + 250x = 650.23$$

$$x = 1.60\ \text{m}$$

Example: A block weighing 20 N is a rectangular prism resting on a rough inclined plane as shown in Fig.14.4.The block is tied up with a horizontal string which has a tension of 5 N. Find

(a) The frictional force on block

(b) Normal reaction of the inclined plane
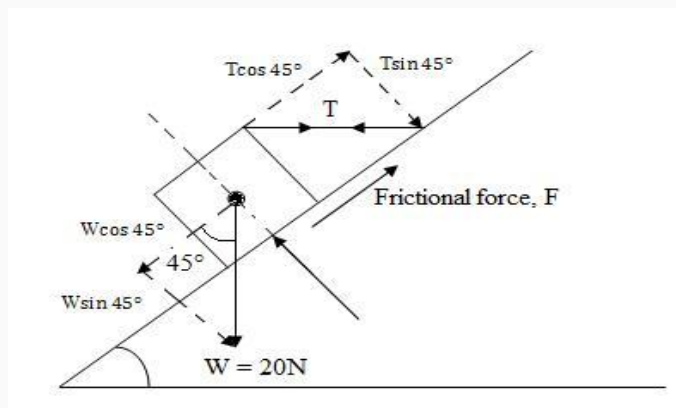
(c) The coefficient of friction between the surfaces of contact
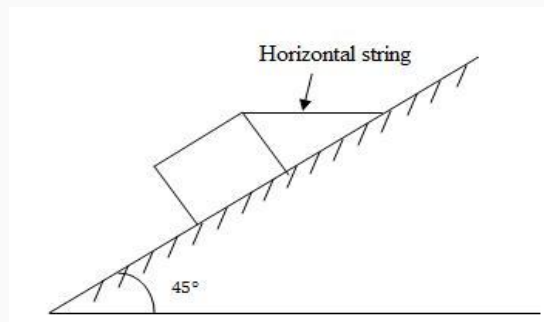
Fig.14.4

Solution: Weight of the block W = 20 N

Tension in the horizontal string T = 5 N

Angle of the inclined plane = 45°

Fig.14.5

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(dotted arrows indicate the components of the force along the tangential and normal to the inclined plane)

Resolving the force parallel to inclined plane, we get

F + T = W

By substituting the values, we get

F + 5 = 20

F = 10.6 N

(b) Resolving the force in the normal direction, we have

R = W + T

= 20 + 5

R = 17.68 N
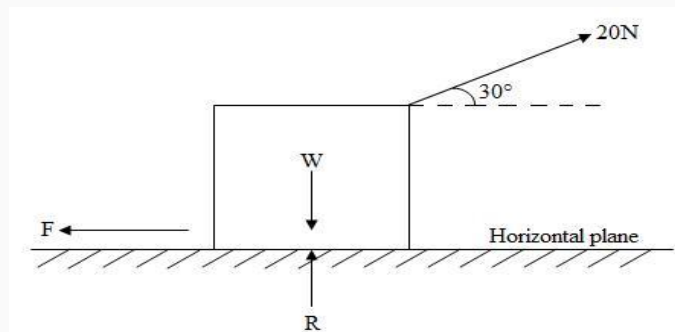
By using the relation, F = μR

10.6 = μ × 17.68

μ = 0.59

Example: A body resting on a rough horizontal plane required to pull 20N inclined at 30° to the plane just to remove it. It was found that a push of 25N inclined at 30° to the plane just removed the body. Determine the weight of the body and the coefficient of friction.

Fig.14.6

Solution: given, Pull = 20N, Push = 25N and θ = 30°

Let W = weight of the body in N, R = Normal reaction and μ = coefficient of friction

1. First of all,the pull acting on the body

Resolving the forces horizontally, we get

F = 20 = 20 × 0.866 = 17.32 N

Resolving the forces vertically, we get

R = W - 20 = W – 20 × 0.5 = (W – 10) N

According to the relation,

F = μ R

17.32 = μ (W – 10)…………………………………………………………………………..(i)

2. The push acting on the body

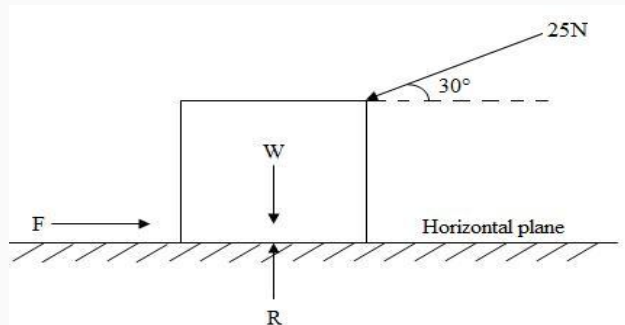**Fig.14.7**

Resolving the forces horizontally, we get

F = 25 = 25 × 0.866 = 21.65 N

Now, resolving the forces vertically, we get

R = W + 25 = W + 25 × 0.5 = (W + 12.5) N

By using the relation,

F = μ R

21.65 = μ (W + 12.5)…………………………………………………………………………...(ii)

Dividing the Eq. (i) to Eq. (ii), we get

$$\frac{17.32}{21.65} = \frac{\mu\left(W - 10\right)}{\mu\left(W + 12.5\right)} = \frac{W - 10}{W + 12.5}$$

By simplification, we get

W = 100 N

Now, substituting the value of W in both Eq. (i) and Eq. (ii), we get

μ = 0.192

MODULE 5.

**LESSON 15.**

15.1 INTRODUCTION

A plane truss or frame consists of several bars laying in one plane and connected by hinges or pins at their ends so as to provide a stable configuration. Frames are used in the roofs of sheds at Railway platform, workshops and in industrial buildings, bridges etc. Plane trusses are made of short thin members interconnected at hinges into triangulated patterns.

- The truss can have only hinged and roller supports.

- In field, usually joints are constructed as rigid by welding.

For analysis purpose we assume that the following conditions are satisfied.

(i) All the members are connected together at their ends by pin joints which are absolutely frictionless.

(ii) All loads and reactions act on the truss only at the joints.

(iii) The longitudinal centroidal axes of the members are absolutely straight, concident with the appropriate lines joining the joint centres and lie in the same plane of the lines of action of the loads and reactions.

The simplest frame is a triangle (Fig.15.1), consists of three members pin-jointed to each other. This can be easily analyzed by the condition of equilibrium. This frame is called the basic perfect frame. It has three members *AB*, *BC* and *CA* and three joints *A*, *B* and *C*.
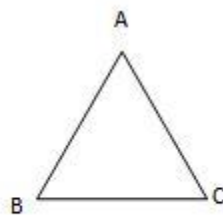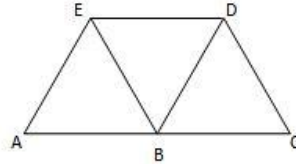
Fig.15.1 Basic perfect frame

Suppose we add two members *AD* and *CD* and a joint *D* to this basic perfect frame, we get a frame (Fig.15.2) which can also be analyzed by the condition of equilibrium. This frame is called a perfect frame.
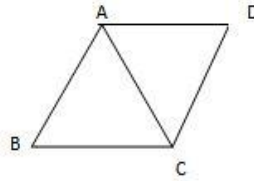
Fig.15.2 Perfect Frame

Suppose we add two members and a joint to this frame as shown in Fig.15.3, we again got a perfect frame.

Fig.15.3 Perfect Frame

In this way we can go on adding any number of sets and can obtain a perfect frame.
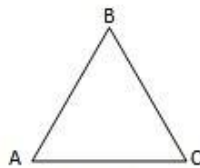
15.2 TYPES OF FRAMES

Following are the types of frames:

(a) Perfect Frame

(b) Deficient or Imperfect Frame

(c) Redundant Frame

(a) Perfect Frames :- Simplest perfect frame is a triangular assemblage of three member AB, BC, CA meeting at joints B, C and A as shown in Fig.15.4

Fig.15.4 Perfect Frame

Mathematically,

m = number of members

j = number of joints

Then m = 2 j – 3 is the condition for the frame to be a perfect frame.

- Hence for a stable frame the minimum number of members required = Twice the number of joints minus three.
- If the number provided is less than the above requirement equation then frame will not be stable.
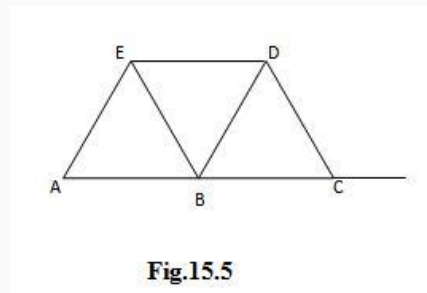
For the Fig.15.4, m = 3, j = 3

$$m = 2j - 3$$

$$3 = 2 \times 3 - 3$$

$$3 = 3 \quad \text{condition is satisfied.}$$

Now consider the frame in Fig.15.5

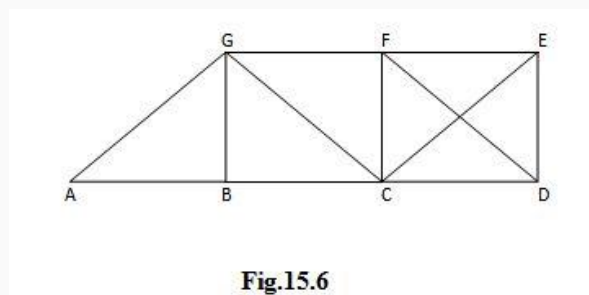Fig.15.5

Here   m = 7, j = 5

$$m = 2j - 3$$

$$7 = 2 \times 5 - 3$$

$$7 = 7$$

Hence the frame is perfect frame.

(a) Imperfect Frames :- (a) When the numbers are less than that required by equation m = 2j – 3 then frame is called imperfect or deficient frame. Such frame, cannot resist geometrical distortion under the action of loads.

(b) Redundant Frames :- If the number of members are more than that required by equation m = 2j -3, then such frames will be called as redundant frames.

Fig.15.6

In Fig.15.6,

m = 12, j = 7

m = 2j – 3

12 = 2 × 7 – 3

12 = 14 – 3 = 11

Hence, the frame is redundant to a single degree, because one member is more.

In general let a frame have j joints and n members.

- If $n = 2j - 3$, then the frame is perfect frame.

- If $n < 2j - 3$, then the frame is deficient frame.

- If $n > 2j - 3$, then the frame is redundant frame.

A perfect frame can always be analyzed by the condition of equilibrium. While a redundant frame cannot be fully analyzed by the condition of equilibrium. We will discuss the analysis of perfect frames only.

15.3 REACTIONS AT SUPPORT

Frames are usually provided with either

(i) Roller Supports

(ii) Hinged Support

(iii) Fixed Support

(i) Roller Support: Fig. 15.7 consists of support which is known as roller support. It is always a normal reaction R perpendicular to the surface of rolling. This support always gives one reaction component in perpendicular direction.
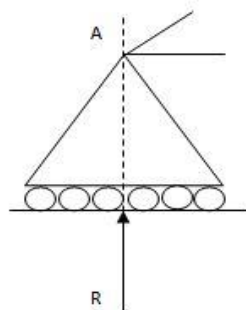
Fig.15.7 Roller Support

(ii) Hinged Support: This is the support at which inclined reaction R is developed. It has two components one is in vertical direction i.e. V and other is in horizontal direction which is H. Hence, hinged support always offers offers two reaction components V and H.
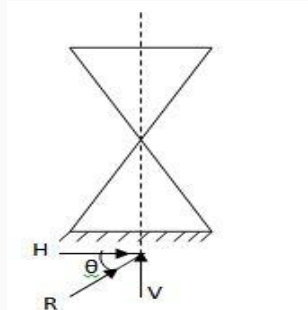

Fig.15.8 Hinged Support

(iii) Fixed Support: Fig. 15.9 shows a fixed support at which three reaction components are developed. One is in vertical direction i.e. V, one is in horizontal direction i.e. H and one is a moment M.
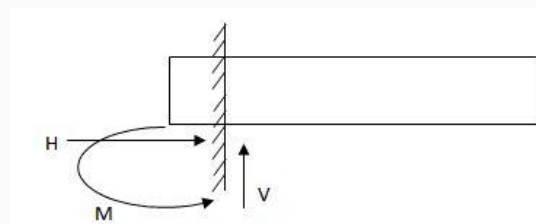

Fig.15.9 Fixed Support

In the cantilever frame shown in Fig. 15.10 the roller base at B is vertical and hence the reaction at this support is horizontal.

At a hinged support, the direction and the line of action of reaction will depend upon the load system on the structure.
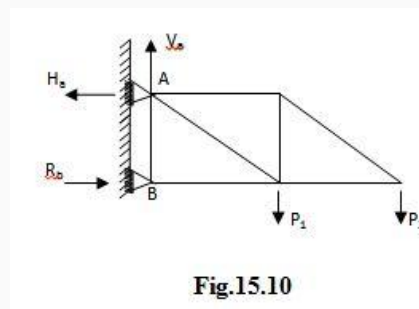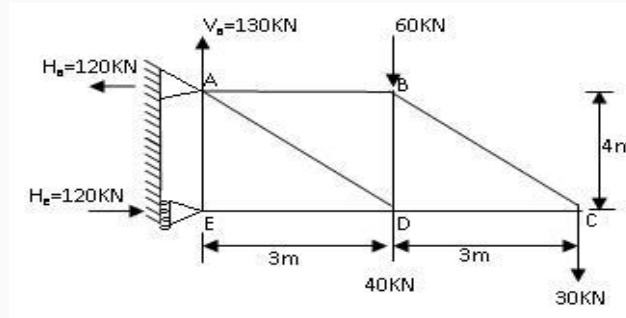

Fig.15.10

To determine the reactions

Reactions at the supports of a structure can be determined by the conditions of equilibrium. The external load system applied on the structure and the reactions at the supports must form a system in equilibrium.

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Consider the cantilever truss shown in Fig.15.11. The truss is provided with a hinged support at A and a roller support at E. The roller base at E being vertical the reaction at E is horizontal. Hence there will be no vertical reaction at E.

Fig.15.11

Taking moments about A.

$H_e \times 4 = (60 + 40) \, 3 + (30 \times 6)$

$H_e = 120 \text{ KN } \rightarrow$

Total applied vertical force = 60 + 40 + 30 = 130 KN $\downarrow$

Therefore, vertical reaction at A = V_a = 130 KN $\uparrow$

Resolving the forces horizontally, we get

$H_a = 120 \text{ KN} \leftarrow$

Thus the reaction at A consists of a vertical component V_a = 130 KN $\uparrow$ and a horizontal component
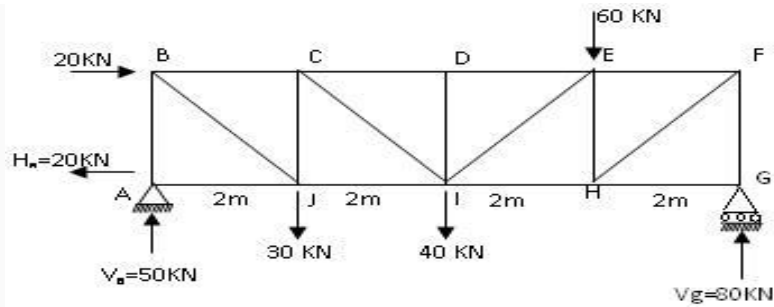
$H_a = 120 \text{ KN} \leftarrow$

Now consider the truss shown in Fig.15.12 provided with a hinged support at A and a roller support at G. The roller base at G is horizontal and hence the reaction at G is entirely vertical. There will be no horizontal reaction at G.

Taking moments about A

$V_g \times 8 = (20 \times 3) + (30 \times 2) + (40 \times 4) + (60 \times 6)$

Therefore, $V_g = 80 \text{ KN } \uparrow$

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Fig.15.12

Total applied vertical force = 30 + 40 + 60 = 130 KN ↓

Therefore, vertical reaction at $A = V_a$ = 130 – 80 = 50 KN ↑

Total applied horizontal force = 20 KN →

Therefore, Horizontal reaction at $A = H_a$ = 20 KN ←

To determine which member of a truss do not carry forces

In a truss carrying a load system some members may not carry forces. Such members can be identified by using the following principles.

(a) A single force cannot form a system in equilibrium. Means if there is only one force acting at a joint, then for the equilibrium of the joint, this force will be equal to zero as shown in Fig. 15.13.
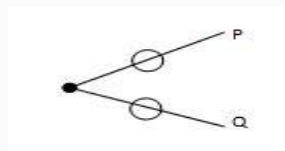
$P = 0$
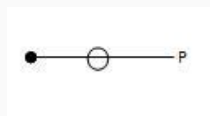
Fig.15.13

 (b) If two forces act at a joint, then for the equilibrium of the joint these two forces should act along the same straight line. The two forces will be equal and opposite. If these two forces are not along the same line. Then for equilibrium of the joint each force equals to zero as shown in Fig. 15.14.
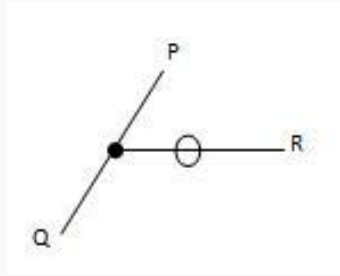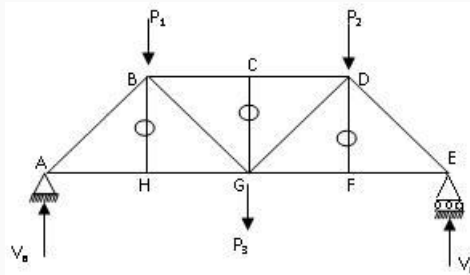
$P = 0$ and $Q = 0$

Fig.15.14

(c) If three forces act at a joint and two of them are along the same straight line, then for the equilibrium of the joint, the third force should be equal to zero as shown in Fig. 15.15.

$R = 0$

Fig. 15.15

For example, in the truss shown in Fig. 15.16

Fig. 15.16

Consider the joint H. Forces at this joint are

P_{ha} in the member HA.

P_{hg} in the member HG.

P_{hb} in the member HB.

Since P_{ha} and P_{hg} are in the same straight line, $P_{hb} = 0$. Similarly, $P_{cg} = 0$ and $P_{fd} = 0$.

 Assumptions in truss analysis

Trusses are analyzed based on the following assumptions:

(i) Each member of the truss is connected at its end by frictionless pins.

(ii) The truss is loaded as well as supported only at its joints.

(iii) The forces in the members of the truss are axial.

(iv) The self-weight of the members is neglected.

Engineering Mechanics

These assumptions only lead to idealization of a truss.

- In reality the steel trusses are not exactly pin-jointed. They are fabricated by riveting, bolting or welding the ends of the members to gusset plates.

- The truss joints are semi-rigid in reality and can transmit moments unlike frictionless pinned joints.

- In many situations the loads are not applied exactly at the joints.

- It should be remember that in an actual truss the centroidal axes of the members are not really concurrent at a joint thus inducing bending moments in the member.

LESSON 16.

16.1 ANALYSIS OF A TRUSS

The analysis of a truss consists of determination of reactions at supports and forces in the members of the truss.

- The reactions are determined by the condition that the applied load system and the induced reactions at the supports form a system in equilibrium.

- The forces in the members of the truss are determined by the condition that the every joint should be in equilibrium and so, the forces acting at every joint should form a system in equilibrium.

A truss can be analyzed by the following methods:

- Method of Joints

- Method of Sections

- Graphical Method

16.2 METHOD OF JOINTS

After determining the reaction at the supports, the equilibrium of each joint is considered one by one.

- Each joint will be in equilibrium if $\sum V = 0$ and $\sum H = 0$, these two conditions are satisfied.

- Forces in the members will either be tensile in nature or compressive in nature.

- The joint is selected in such a way that at any time there are not more than two unknowns.

The direction of an unknown force is assumed.

- If the magnitude of force comes out to be positive than assumed direction will be correct.

- If the magnitude of force comes out to be negative than assumed direction will be incorrect.

The process is continued until all the joints are considered thereby calculating the forces in all the members of the frame.
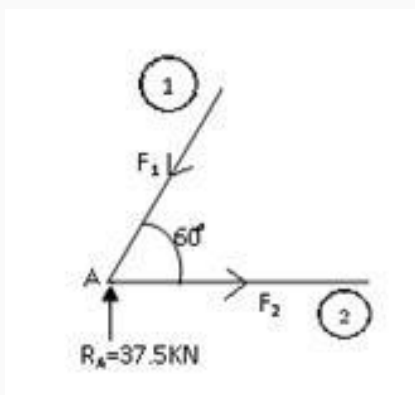
Engineering Mechanics

Example: Find the forces in the members *AB, BC, AC* of the truss shown below in Fig.16.1. End *A* is hinged and *B* is supported on rollers.
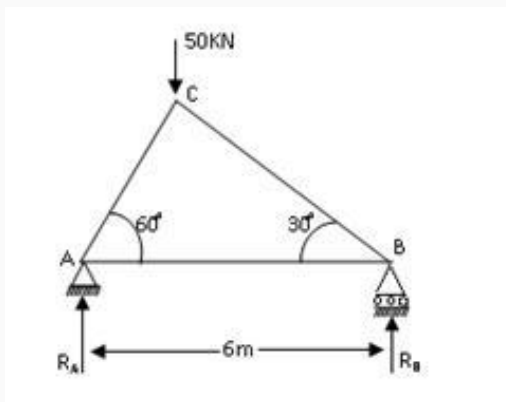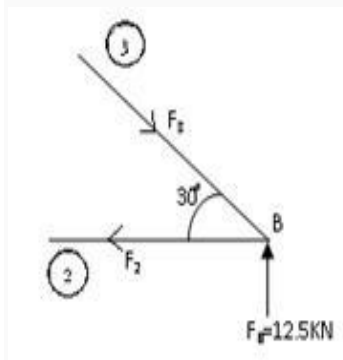

Fig.16.1

Solution: A roller offers a reaction perpendicular to plane of rolling. Let R_B is reaction at *B*. A hinge offers two reaction components one in vertical direction and another in horizontal direction. Since the load of 5O kN acts vertically downward, therefore only vertical direction R_A is developed and no horizontal reaction.

From the geometry of the figure, the distance of 50 kN load from *A* in horizontal direction along *AB* is *AC* cos60°.

Fig.16.2   **Fig.16.3**

In *ΔACB*,

Angle at *ACB* = 90°

$AC = AB \cos 60° = 6 \times \left[{1 \over 2}\right] = 3$ m

$BC = AB \sin 60° = 6 \times \left[{{\sqrt 3} \over 2}\right] = 3 \left[\sqrt 3\right]$ m

Distance of 50 kN load from $A = AC \cos 60° = 3 \times = 1.5$ m

Taking moments about *A*, we have

$R_B \times 6 = 50 \times 1.5$

Engineering Mechanics

$R_B = \dfrac{50 \times 1.5}{6} = 12.5$ kN

For equilibrium $\sum V = 0$, i.e.

$R_A + R_B = 50$

$R_A + 12.5 = 50$

$R_A = 37.5$ kN

Considering equilibrium of joint A, first because R_A is known and only two unknown forces F_1 and F_2 are there. At each joint two equation of statical equilibrium are available i.e.

$\sum V = 0$ and $\sum H = 0$.

Let F_1 is force produce in the member AC and F_2 is the force produced in the member AB as shown in Fig.2. Joint A has to be in equilibrium. Component of force F_1 in vertical direction will balance vertical reaction R_A. Therefore, the arrow is marked in member (1) in down direction. Applying condition $\sum V = 0$ at joint A.

$F_1 \sin 60° = 37.5$

$F_1 = 37.5 \times \dfrac{2}{\sqrt 3} = \dfrac{75}{\sqrt 3} = \dfrac{75}{\sqrt 3} \times \dfrac{\sqrt 3}{\sqrt 3}$

$= + 25\sqrt 3$ kN (+ve sign indicates that as assumed direction is correct)

$= 43.30$ kN

As the force F_1 is pushing joint A, therefore F_1 is compressive force. Mark arrow at joint C as pushing it to show that member AC is compression member (Fig.16.2)

Now applying condition $\sum H = 0$ at joint A,

$F_1 \cos 60° = F_2$

Therefore, $F_2 = 43.30 \times \dfrac{1}{2} = + 21.65$ kN

(again +ve sign indicated that the arrow marked in member AB is correct)

As the force F_2 is pulling the joint A, therefore F_2 is a tensile force. At B, place arrow marking away from B to show that member AB is a tension member.

Next consider joint B, as shown in Fig.16.3

Let F_3 is the force produced in the member BC. The joint B has to be in equilibrium. It must satisfy the two conditions of statical equilibrium viz. $\sum V = 0$ and $\sum H = 0$. Let us assume the direction of arrow towards B.

Applying $\sum V = 0$ at joint B.

Engineering Mechanics

$F_3 \sin 30° = 12.5$

Therefore $F_3 = + 25$ kN (+ve sign indicates that direction of F_3 is correct)

As F_3 pushes the joint B, Therefore it is a compressive force.

Now applying second condition $\sum H = 0$ at the joint B,

$F_3 \cos 30° = F_2$

$25 \times \left[\frac{\sqrt 3}{2}\right] = 21.65$

$21.65 = 21.65$ (Check)

Now the forces in the various members are tabulated in the following table.

Member	Force
AB	Tensile = 21.65 Kn
BC	Compressive = 25 kN
AC	Compressive = 43.30 kN

Forces are marked in the truss, as shown in Fig.16.4. If we take tensile forces as +ve, then compressive force will be –ve.
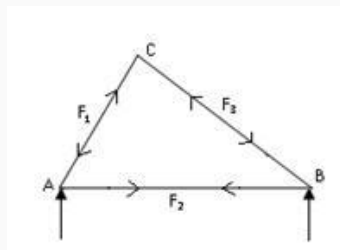

Fig.16.4

Example: Determine the forces in the members of the truss loaded as shown in the Fig.16.5. Also indicate the nature of the force (tensile or compressive).
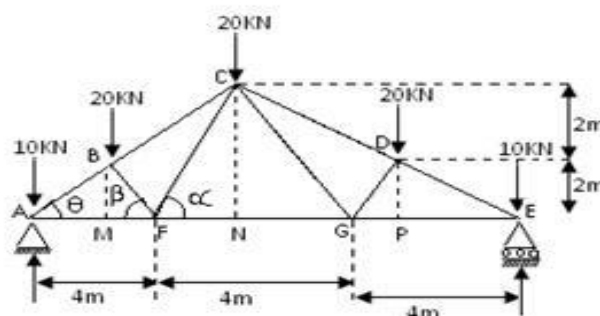

Fig.16.5

Solution: The truss is symmetrical and also loaded symmetrically therefore, reactions R_A and R_B will be equal. From the condition of equilibrium

$\sum V = 0$

i.e. $R_A + R_B = 10 + 20 + 20 + 20 + 10$

$2R_A = 80$

$R_A = 40$ kN, $R_B = 40$ kN

Draw BM and CN perpendicular to AE.

In $\Delta\ CAN$, tan $CAN = \ = 33.69°$

Q = angle at $CAN = 33.69°$

ΔCFN, tan $a = \ = 2$

$a = 63.43°$

$AC = \[\sqrt {A{N^2} + C{N^2}}\] = \[\sqrt {{6^2} + {4^2}}\] = 7.2$ m

$AB = 3.6$ m, $BM = 3.6$ sin $33.69° = 1.99$ m

$AM = AB$ cos $\Theta = 3.6$ cos $33.69° = 2.99$ m

$MF = AF - AM = 4 - 2.99 = 1.01$ m

$\tan \beta = \[{{BM} \over {MF}}\] = \[{2 \over {1.01}}\] = 1.98$

$\beta = 63.20° = a$

Considering joint A, (Fig.16.6). At this joint four forces are acting, out of which two forces AB and AF are unknown. Applying condition of equilibrium at joint A, $\sum V = 0$. Resolving all forces in a vertical direction, we have
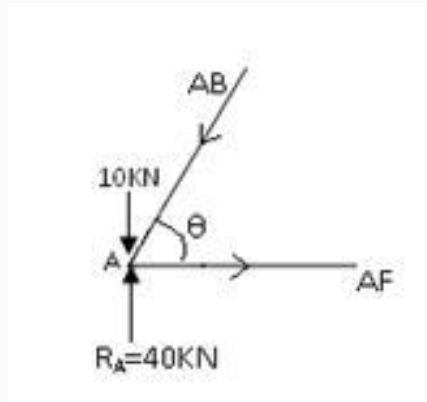

Fig.16.6

Engineering Mechanics

$10 + AB \sin \theta = 40$ ....(i)

$\sum H = 0$

Resolving all forces horizontally, we have

$AB \cos \theta = AF$.............................(ii)

(Here AF is assumed to be in tension and AB is assumed to be in compression)

From (i), $AB \times \sin 33.69° = 30$

Therefore, $AB = \dfrac{30}{\sin 33.69°} = \dfrac{30}{0.5547}$

$= + 54.08$ kN (compression)

From (ii), $AF = 54.08 \cos 33.69°$

$= 44.99$ kN (tensile)

Considering joint B, (Fig.16.7)
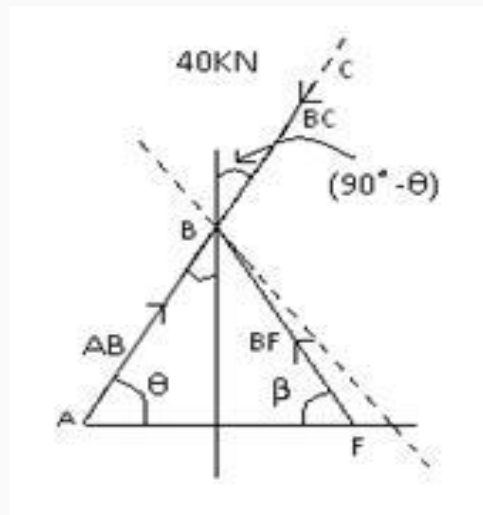
Fig.16.7

Angle $ABF = 180° - \theta - \beta$

$\qquad = 180° - 33.69° - 63.20°$

$\qquad = 83.11°$

Resolving all forces perpendicular to ABC,

$BF \sin 83.11° = 20 \sin (90 - \theta)$

$BF \times 0.9928 = 20 \cos \theta = 20 \times 0.8320$

BF = 16.76 kN (compressive)

Resolving all forces along the line ABC,

$AB + BF \cos 83.11° = BC + 20 \cos (90 – \theta)$

$54.08 + (16.76 × 0.1199) = BC + (20 × 0.5547)$

$56.089 = BC + 11.094$

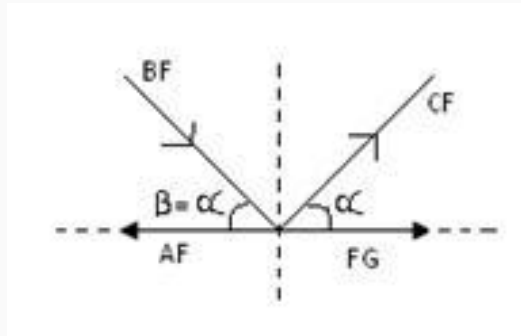BC = 44.49 KN (compressive)

Considering joint F, (Fig.16.8)

Fig.16.8

Resolving all forces vertically

$BF \sin a = CF \sin a$

Therefore, CF = 16.76 kN (tensile)

Resolving all forces in horizontal direction.

$FG + CF \cos a + BF \cos a – AF = 0$

Therefore, $FG + 16.76 \cos 63.20° + 16.76 \cos 63.20° - 44.99 = 0$

Therefore, $FG + 7.56 + 7.56 – 44.99 = 0$

$FG + 15.12 – 44.99 = 0$

FG = 29.87 KN (tensile)

We have analysed half the truss, other half is symmetrical, therefore

$AF = EG, AB = ED, BC = DC, BF = DG, CF = CG$

Engineering Mechanics

Member	Force	Nature
AB	54.08 KN	Compressive
BC	44.99 KN	Compressive
CD	44.99 KN	Compressive
DE	54.08 KN	Compressive
AF	44.99 KN	tensile
FG	29.87 KN	tensile
GE	44.99 KN	tensile
FB	16.76 KN	Compressive
FC	16.76 KN	tensile
GD	16.76 KN	Compressive
GC	16.76 KN	tensile

LESSON 17.

17.1 METHOD OF SECTIONS

For a member near to supports can be analysed with the help of method of joints and for members remote from supports can be quickly analysed with the help of method of sections.

- In this method a section line is passed through the members, in which forces are to be determined in such a way that not more than three members are cut.
- Then any of the cut part is then considered for equilibrium under the action of internal forces developed in the cut members and external forces on the cut part of the truss.
- The condition of equilibrium, i.e $\sum V = 0, \sum H = 0, \sum M = 0$ are applied to the cut part of the truss under consideration. As three equations are available, therefore, three unknown forces in the three members can be determined.
- If the magnitude of a force comes out to be positive then the assumed direction is correct. If the magnitude of a force is negative then reverse the direction of that force.

Example: Find the forces in the members PR and PQ of the truss loaded as shown in Fig.17.1, using method of sections.
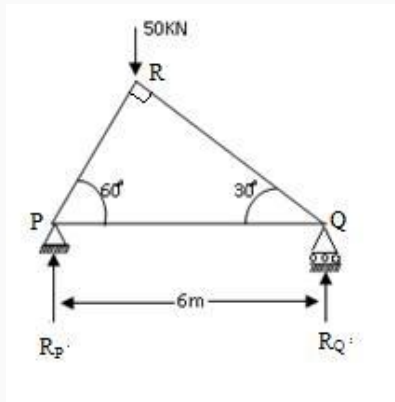

Fig.17.1

Solution: PR = 6 cos60° = 3 m

QR = 6 sin60° = 6 . $\left[{{\sqrt 3 } \over 2} \right]$ = 3 $\sqrt 3$ = 5.19 m

Determination of reactions.

Let R_P and R_Q be the reactions at P and Q.

Taking moments about P,

$R_Q \times 6 = 50 \times PR. \cos60°$

$6 R_Q = 50 \times 3 \times \left[{1 \over 2} \right]$
$R_Q = 12.5$ kN

$R_P + R_Q = 50$

$R_P = 50 – 12.5 = 37.5$ kN

Passing a section 1-1, thereby cutting the truss in two parts.
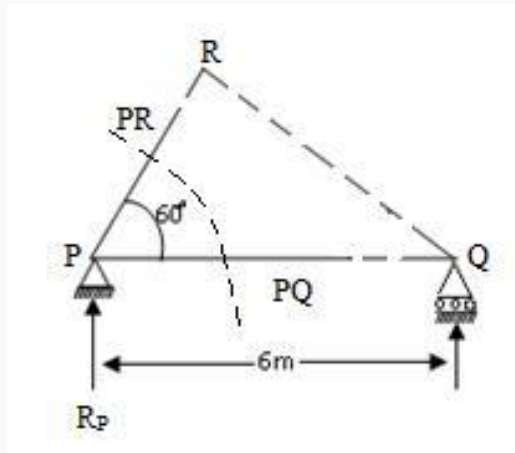

Fig.17.2

Considering equilibrium of the left part. The part part of the truss is shown in Fig.17.2. This part is in equilibrium under the action of one external force $R_P = 37.5$ kN and other two internal unknown forces PR and PQ in the members PR and PQ respectively. The directions of PR and PQ both are considered as tensile as marked in Fig.17.2

Determination of force PR

Taking moment of all forces about Q.

The moment of force PQ about point Q is zero.

Therefore, $R_P × 6 + (PR × QR) = 0$ (because QR is perpendicular distance between force PR and point Q i.e 5.19 m)

$(37.5 × 6) + PR × 6.\sin 60° = 0$

$PR = - \left[\frac{225}{6 \sin 60^\circ} \right] = - \left[\frac{37.5}{0.866} \right] = - 43.30$ kN

-ve sign indicates that the assumed direction is wrong. This force is actually compression force.

Hence, PR = 43.30 kN

Determination of force PQ

If we take the moments of all forces about point R, then PR will be eliminated and there will be only one unknown force PQ.

Hence, taking moment about point R,

-(PQ × PR.sin60°) + (R$_P$ × PR.sin60°) = 0 $\sum M = 0$

-(PQ × 3 × 0.866) + (37.5 × 3 × 0.5) = 0

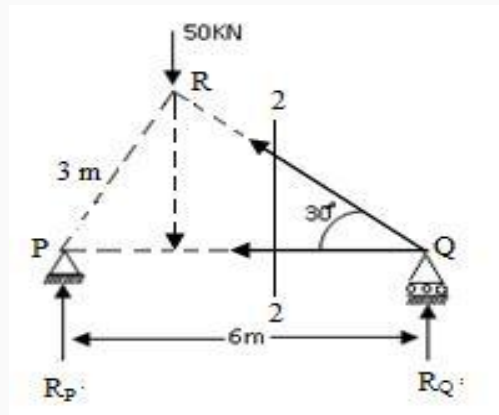2.598 PQ = 56.25

PQ = + 21.65 kN

Fig.17.3

The positive sign indicates that the assumed direction is correct. This force is tensile force.

Now if the in member QR is also to be determined then we will have to take another section 2-2 so as to cut the member QR and PQ, as shown in Fig.17.3 Now considering equilibrium of right part of truss, under the section of two internal forces QR , PQ and one external force R$_Q$= 12.5 kN, we can apply condition $\sum M = 0$, if we take moment about P, then forces PQ will be eliminated and only one unknown force QR will remain. Hence by taking moment about P, we get (QR × 3) + (12.5 × 6) = 0

QR = - $\frac{75}{3}$ = -25 kN

Negative sign indicates the assumed direction is wrong. This force is actually compressive. Similarly if we take moment about R, force QR is eliminated and PQ = 21.65 kN (tensile).

Example: Determine the forces in the members DE, BE and AB of the truss, shown in Fig.17.4.
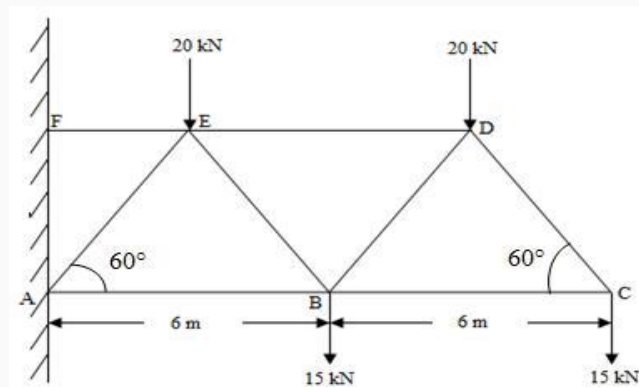

Fig.17.4

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Solution: Pass a section X-X in such a way so that three desired members DE, BE and AB are cut. Now consider the right part of the cut truss as shown in Fig.17.5.Let F_1, F_2 and F_3 be the forces in the members DE, BE and AB respectively.
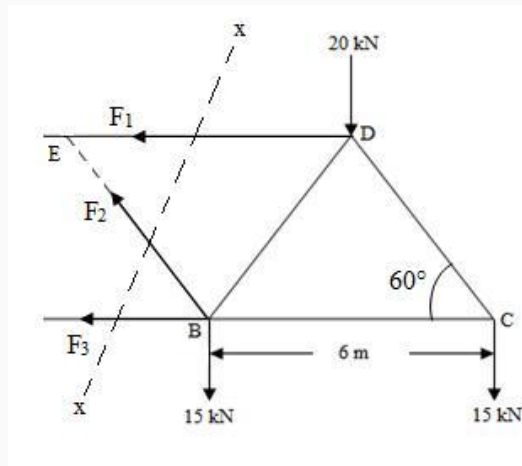

Fig.17.5

Determination of F_1,

Taking moments about B, so that moments of the forces F_2 and F_3 are eliminated

We have, $- F_1 \times 6 \sin60° + (20\times3) + (15\times6) = 0$

$-5.196 F_1 + 60 + 90 = 0$

$F_1 = 28.868$ kN (Tensile)

Determination of F_3,

Taking moments about E

$F_3 \times 6 \sin60° + (15\times3) + (20\times6) + (15\times9) = 0$

$5.196 F_3 + 45 + 120 + 135 = 0$

$F_3 = 57.737$ kN (Compressive)

Determination of F_2,

Applying equilibrium condition

$F_2 \sin60° = 20 + 15 + 15 = 50$

$F_2 = 57.735$ kN (Tensile)

LESSON 18.

18.1 INTRODUCTION

The forces in a perfect frame can also be determined by a graphical method. The analytical methods which are discussed in the last lessons give absolutely correct results, but sometimes there is not enough time to analyze the frame using those methods. Then graphical method is used conveniently to get the results.

18.2 GRAPHICAL METHOD

The naming of the various members of a frame are done according to Bow's notations. According to this notation a force is designated by two capital letters which are written on either side of the line of action of the force. A force with letters L and M on either side of the line of section is shown in Fig.18.1
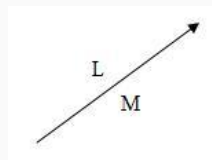

Fig.18.1

The following steps are necessary for the solution of a frame by graphical method:

(i) Making a space Diagram

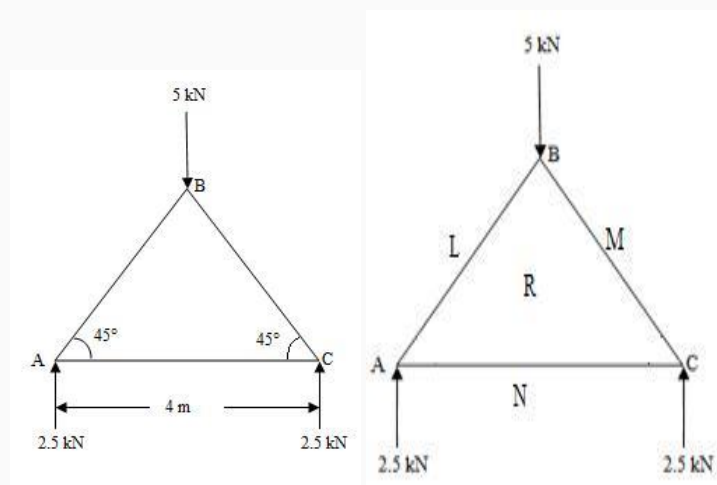(ii) Constructing a vector diagram

(iii) Preparing a force table

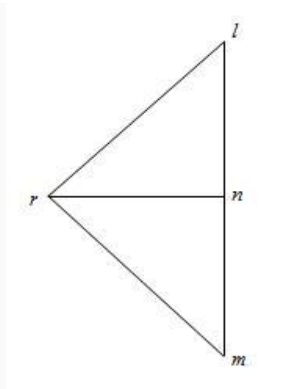**Fig.18.1(a) Given diagram**   **Fig.18.1(b) Space diagram**

Fig.18.1(c) Vector diagram

(i) Making a space diagram:

- The given truss or frame is drawn accurately to some linear scale.

- The loads and support reactions in magnitude and directions are also shown on the frame.

- Then the various members of the frame are named according to Bow's notation as shown in the Fig.18.1(b), members AB,BC and AC are to be determined. Fig.18.1(b) shows the space diagram to same linear scale.

- The member AB is named as LR and so on.

(ii) Constructing a vector diagram: Fig.18.1(c) shows a vector diagram.

- Take any point *l* and draw *lm* parallel to LM vertically downwards.   Cut *lm* = 5kN to same scale.

- Now from *m* draw *mn* parallel to MN vertically upwards and cut   *mn* = 2.5kN to the same scale.

- From *n* draw *nl* parallel to NL vertically upwards and cut *nl* = 2.5kN to the same scale.

- Now from *l*, draw a line *lr* parallel to LR and from *n* draw a line *nr* parallel to NR meeting the first line at *r*. This is vector diagram for joint A. Similarly the vector diagrams for joint B and Joint C can be drawn.

(iii) Preparing a force table: The magnitude of a force in a member is known by the length of the vector diagram for the corresponding member i.e. the length *lr* of the vector diagram will give the magnitude of force in the member LR of the frame.

Nature of the force is determined according to the following procedure:

(i) In the space diagram, consider any joint. Move round that joint in a clockwise direction. Note the order of two capital letters by which members are named. For example, the members at the joint A in space diagram are named as shown in Fig.18.1(b).(LR,RN and MR).

Engineering Mechanics

(ii) Now consider the vector diagram. Move on the vector diagram in the order of the letters (i.e. *lr*, *rn* and *nl*).

(iii) Now mark the arrows on the members of the space diagram of that joint.

(iv) Similarly all joints can be considered and arrows can be marked.

(v) If the arrow is pointing towards the joint, then the force in the member will be compressive whereas if the arrow is away from the joint, then the force in the member will be tensile.

Example: Find the forces in all the members of the given truss shown in Fig.........
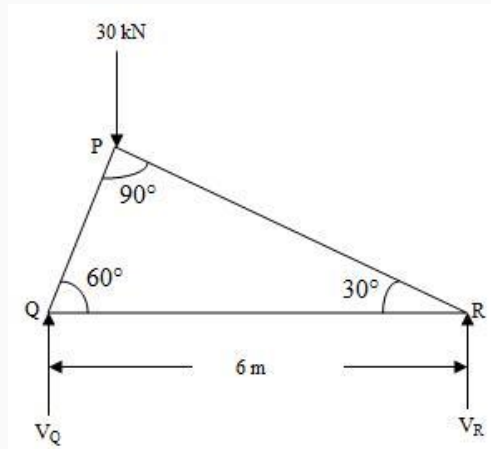
MODULE 6.

LESSON 19.

19.1 INTRODUCTION

When an external force acts on a body, the body tends to undergo some deformation. Due to cohesion between the molecules, the body resists deformation. This resistance by which material of the body opposes the deformation is known as strength of material.

Within the certain limit (i.e. in the elastic stage) the resistance offered by the material is proportional to the deformation brought out on the material by the external force. Also within the elastic limit the resistance is equal to the external force. But beyond the elastic stage, the resistance offered by the material is less than the external force. In such case, the deformation continues, until failure takes place.

Types of beams: A beam is a structural member subjected to a system of external forces at right angles to its axis.

(i) Cantilever beam: If such a member is fixed or built in at one end while its other end is free, the member is called a cantilever. Fig.19.1
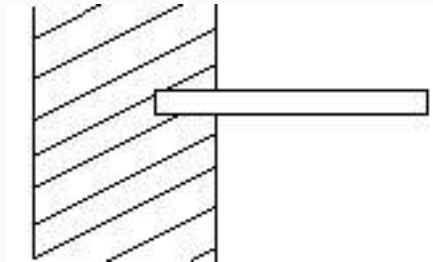

Fig.19.1

(ii) Simply or Freely Supported Beam: If the ends of a beam are made to freely rest on supports then the beam is called freely or simply supported beam. Fig.19.2
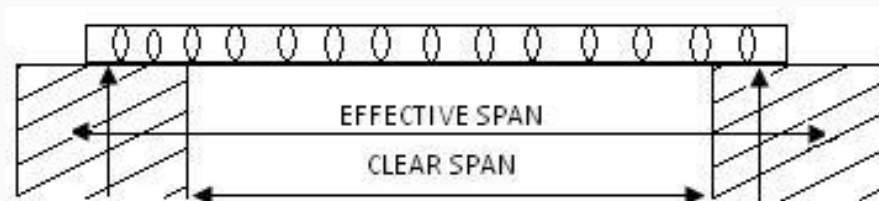

EFFECTIVE SPAN

CLEAR SPAN

Fig.19.2

(iii) Fixed Beam: If the beam is fixed at both its ends, it is called a built-in or fixed beam. Fig.19.3.
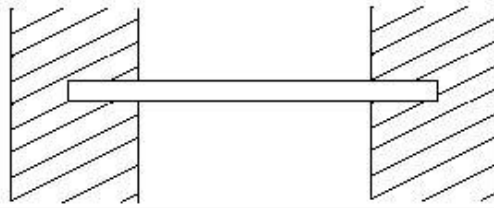

Fig.19.3

(iv) Continuous Beam: A beam which is provided with more than two supports is called a continuous beam. Fig.19.4.

Fig.19.4

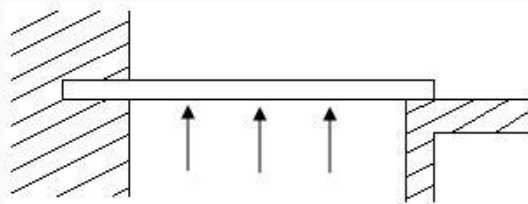**Types of Loads:** Beams may be subjected to various types of loads.

- **Concentrated or Point Load:** It is a load applied over a small area.
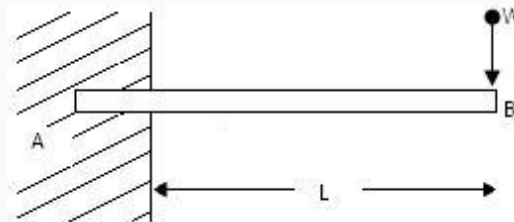-

Fig.19.5

- **Uniformly Distributed Load:** It is a load which is spread on some length of a beam. It is expressed by its intensity (Newton/meter). if the intensity of the distributed load is constant the load is called uniformly distributed load.
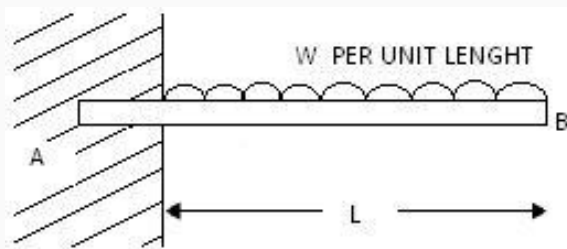
Fig.19.6

- **Uniformly Varying Load:** It has an intensity that varies according to some law along the length of the beam.
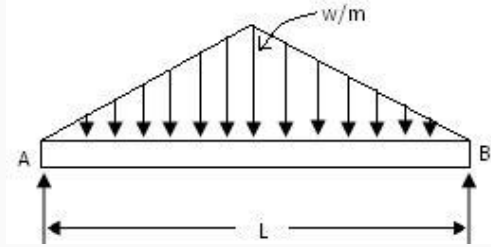
Fig.19.7

- **Gradually Varied Load:**
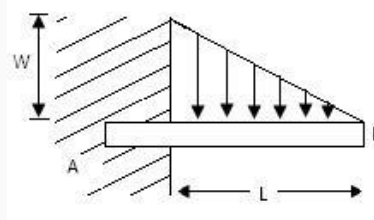
Fig.19.8

19.2 STRESS

The force of resistance offered by the body against the deformation is called the *stress*. The external force acting on the body is called the *load*. In other words,

- Under the influence of two equal and opposite forces P, the body in Fig.19.9 is in equilibrium. It deforms (stretches), and the stress (tensile) is developed.
-
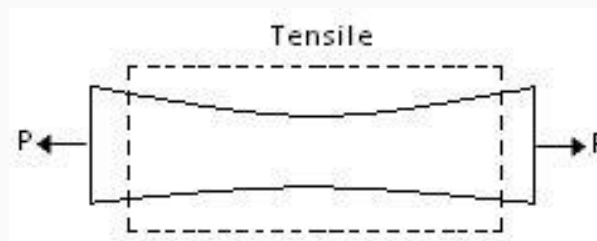
Fig.19.9

- Under the influence of two equal and opposite moments M, the body in Fig.19.10 deforms (bends in this case), and the stress (bending) is developed.
-

Fig.19.10

Engineering Mechanics

- A single force or a moment does not put the body under any stress. For a body to be under any stress, two equal and opposite forces or moments are necessary.

Mathematically Stress is written as

$$\sigma = \frac{P}{A}$$

where σ = Stress

P = External force or Load

A = Cross-sectional area

The SI units of the stress is N/m².

19.3 STRAIN

When a body is subjected to some external force, there is some change of dimensions of the body. The ratio of change of dimension of the body to the original dimension is known as *Strain*.

$$\text{Strain (e)} = \frac{Change\,in\,length}{Original\,length}$$

Strain is unit less.

19.4 TYPES OS STRESSES

The important types of simple stresses are:

(i) Tensile Stress

(ii) Compressive Stress

(iii) Shear Stress

(i) **Tensile Stress:** The stress induced in a body, when subjected to two equal and opposite pulls as shown in Fig.19.11 as a result of which , there is an increase in length, is known as tensile stress.

The ration of increase in length to the original length is known as *tensile strain*.

Fig.19.11

Let P = force acting on the body

   A = Cross-sectional area of the body

   L = Original length of the body

dL = increase in length due to pull P acting on the body

σ = stress induced in the body

e = strain

Tensile stress (σ) = $\left[{P \over A}\right]$

and Tensile strain (e) = $\left[{{increase\,in\,length} \over {Original\,length}}\right]$ = $\left[{dL \over L}\right]$

(ii) **Compressive Stress:** The stress induced in a body, when subjected to two equal and opposite pushes as shown in Fig.19.12 as a result of which there is decrease in length of the body, is known as compressive stress.

The ratio of decrease in length to the original length is known as *compressive strain.*

Fig.19.12

Compressive Stress (σ) = $\left[{P \over A}\right]$

Compressive Strain (e) = $\left[{{decrease\,in\,length} \over {Original\,length}}\right]$ = $\left[{dL \over L}\right]$

(iii) Shear Stress: The stress induced in a body, when subjected to two equal and opposite forces which are acting tangentially across the resisting section as a result of which the body tends to shear off across the section, is known as shear stress.

The corresponding strain is known as *shear strain.*

Fig.19.13 shows a rectangular block of height l and length L and width unity.
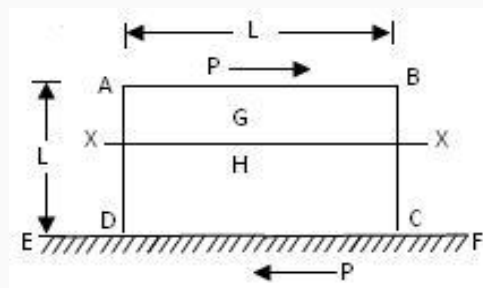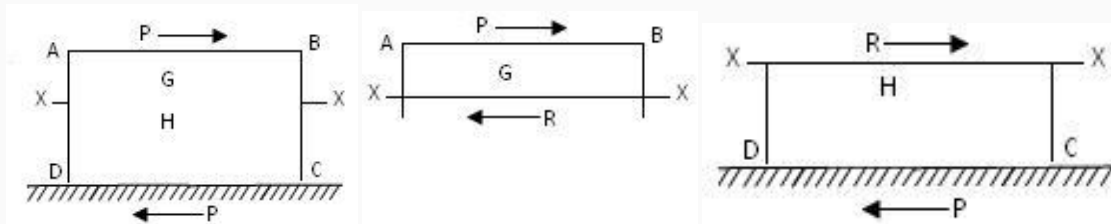

Fig.19.13

Let the bottom face of the block be fixed to a surface *EF*. Let a force *P* be applied tangentially along the top face of the block. Such a force acting tangentially along a surface is called a shear force.

Engineering Mechanics

For the equilibrium of the block, the surface *EF* will offer a tangential reaction *P* equal and opposite to the applied tangential force *P*. Let the block be taken to consist of two parts *G* and *H* to which it is divided by a section *XX*. Consider the equilibrium of the part *G* in Fig.19.14
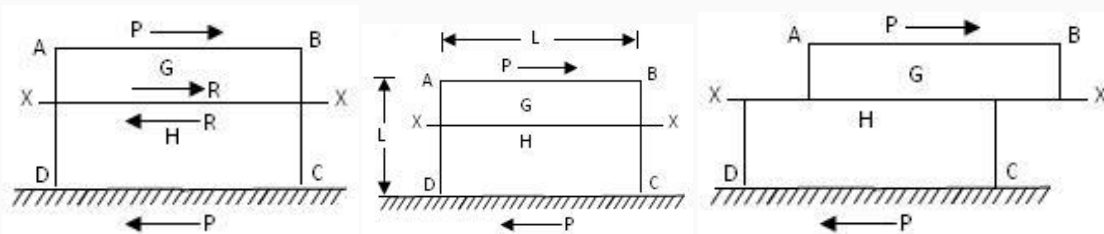

Fig.19.14

In order the part *G* may not move from left to right, the part *H* will offer a resistance *R* along the section *XX* such that $R = P$.

Similarly, considering the equilibrium of the part *H*, we find that the part *G* will offer a resistance *R* along the section *XX* such that the resistance *R* along the section *XX* is called a shear resistance.

Fig.19.15 shows a failure at the section *XX* caused by the tangential forces acting on the top and bottom faces of the block. This type of failure is called a *shear failure*. In shear failure, the two parts into which the block is separated, slide over each other. Hence if such a shear failure should not occur, the section *XX* must be able to offer tangential resistances along the section opposing the force *P* at the top face and the force *P* at the bottom face. For the equilibrium of the system of the shear resistance *R* should be equal to the tangential load *P*. Therefore, $R = P$

Fig.19.15

The intensity of the shear resistance along the section *XX* is called the shear stress.

$$\text{Shear Stress} = q = {R \over P} = {\text{Shear resistance} \over \text{Shear Area}} = {R \over L \times 1}$$

Fig.19.16(a) shows a rectangular block subjected to shear forces *P* on its top and bottom faces.
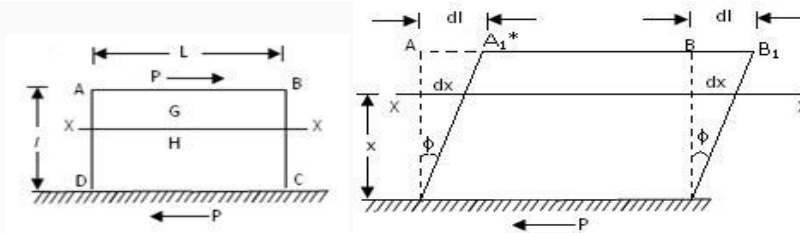
Fig.19.16

When the block does not fail in shear, a shear deformation occurs as shown in Fig.19.16(b). If the bottom face of the block be fixed, it can be realized that the block has deformed to the position A_1B_1CD.

Let us now imagine that the block consists of a number of horizontal layers. These horizontal layers have undergone horizontal displacements by different amounts with respect of the bottom face. We can assume that the horizontal displacement of any horizontal layer is proportional to its distance from the lower face of the block.

Let the horizontal displacement of the upper face of the block be *dl*. Let the height of the block be *l*.

The ratio $\left[{{dl} \over l}\right]$ = $\left[{{Transverse\,displacement} \over {Distance\,from\,the\,lower\,face}}\right]$ is called shear strain.

19.5 ELASTIC LIMIT

When an external force acts on a body, the body tends to undergo some deformation. If the external force is removed and the body comes back to its original shape and size, the body is known as elastic body.

- The body will regain its previous shape and size only when the deformation caused by the external force, is within a certain limit. Thus there is a limiting value of force upto and within which the deformation completely disappears on the removal of forces. The value of stress corresponding to this limiting force is known as the elastic limit.

Hook's Law: It is observed that when a material is loaded such that the intensity of stress is within a certain limit, the ratio of the intensity of stress to the corresponding strain is constant which is a characteristic of that material.

$$\left[{{Intensity\,of\,stress} \over {Strain}}=constant\right]$$

In the case of axial loading, the ration of the intensity of tensile or compressive stress to the corresponding strain is constant and is called *Modulus of Elasticity* or *young's Modulus.*

It is denoted by *E*.

E = $\left[{\sigma \over e}\right]$

LESSON 20.

20.1 CONCEPT OF SHEAR FORCE AND BENDING MOMENT

- Fig.20.1 shows a cantilever AB whose end A is fixed. Let the cantilever carry a vertical load of 20 KN at C.
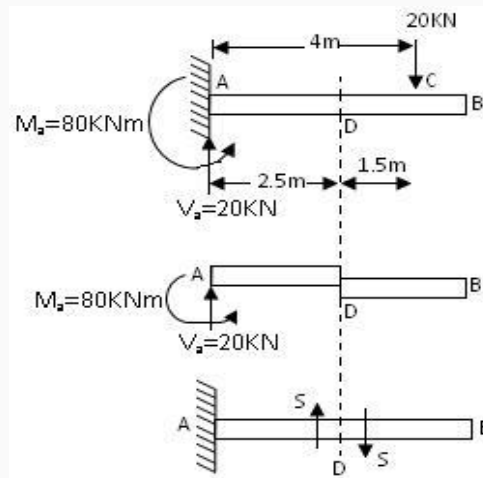

Fig.20.1

For the equilibrium of the cantilever the fixed support at A will provide a vertical reaction vertically upwards, of magnitude $V_a = 20$ KN

Taking moments about A, we have a clockwise moment of $20 \times 4 = 80$ KNm

Hence the equilibrium of the cantilever, the fixed support at A must also provide a reacting moment or fixing moment of 80 KNm of an anticlockwise order.

Now consider a section D. At this section there is a possibility of failure by shear as shown in Fig.20.1 If such a failure will occurs at section D, the cantilever is liable to be sheared off into two parts. It is clear that the force acting normal to the centre line of the member on each part equals $S = 20$ kN.

The force acting on the *right part* of the section D is downward. The resultant force acting on the left part is upward.

The resultant force acting on any one of the parts normal to the axis of the member is called the Shear Force at the section D.

For the case illustrated above the resultant force normal to the axis of the member on the right part of the section is downwards while the resultant force normal to the axis of the member on the left part of the section is upward. Such a shear force is regarded as a positive shear force.

- Let us now study another effect of the load applied on the cantilever. The cantilever is liable to bend due to the load on it.
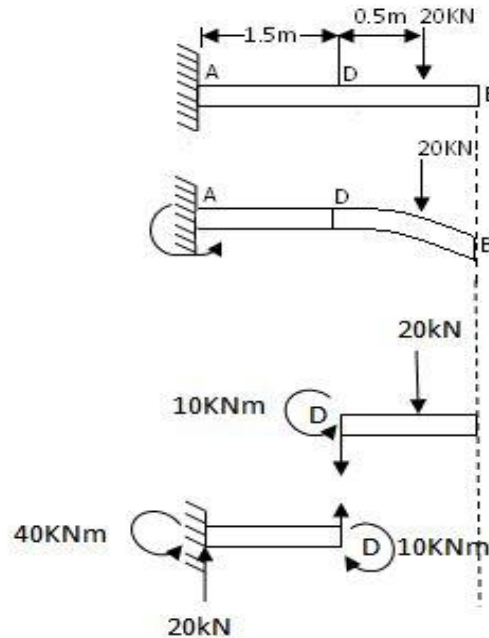
Fig.20.2

For instance, the cantilever has a tendency to rotate in clockwise direction about *A* (Fig.20.1). Hence the fixed support at *A* has to offer a resistance against the rotation.

Taking moments about *A*, we find that the applied load of 20 KN has a clockwise moment of 20 × 2 = 40 kNm. Hence for the equilibrium of the cantilever, the fixed support at *A* will provide a reacting or resisting *anticlockwise* moment of 40 kNm. If the support *A* is not able to provide such a resisting moment, the cantilever will not be in equilibrium and will, therefore, rotate about *A* in the clockwise order.

The magnitude of the reacting moment at *A* depends on the magnitude of the load and the position of the load.

We say that the support *A* provides the necessary fixing or reacting moment at *A*, and that at the section *A* of the beam, there is a bending moment of 20 × 2 = 40 kNm.

Let us no discuss the equilibrium of the part AD (Fig. 20.1) taking moments about D, we get. Moment at fixed end support is equal to 20×1.5=30kNm (clockwise).

Couple = 30kNm

Net moment at D = 40–30=10kNm (anti-clockwise)

Sign conventions for shear force and bending moment

Shear Force: Fig.20.2 shows a simply supported beam *AB*, carrying a load of 100 kN at its middle point. The reaction at the supports will be equal to 50 kN. Hence R_A = R_B = 50 kN.
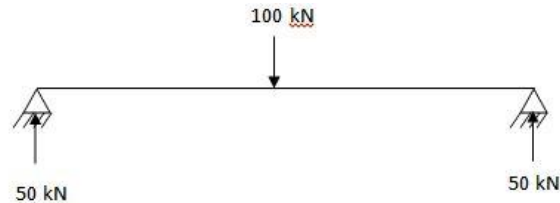
Fig. 20.2

Now imagine the beam to be divided into two positions by the section *XX*. The resultant of the load and reaction to the left of *XX* is 50 kN vertically upwards. And the resultant of the load and the reaction to the right of *XX* is 50 kN downwards.

- The shear force at a section will be considered positive when the resultant of the forces to the left of the section is upward or to the right of the section is downward.

- The shear force at a section will be considered negative when the resultant of the forces to the left of the section is downward or to the right of the section is upward.

Bending Moment:

- The banding moment at a section is considered positive if the bending moment at that section is such that it tends to bend the beam to the curvature having concavity at the top.(Fig.20.3)
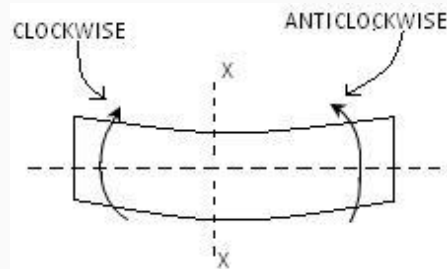

Fig. 20.3

- The bending moment at a section is considered negative if the bending moment at that section is such that it tends to bend the beam to the curvature having convexity at the top.(Fig.20.4)
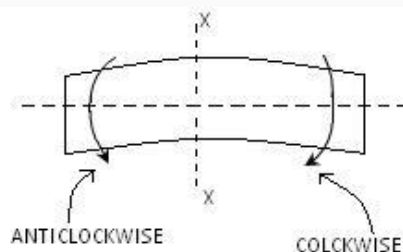

Fig. 20.4

- Positive bending moment is called sagging moment.

- Negative bending moment is called hogging moment.

-

Engineering Mechanics

Some important points to remember

Shear Force: If we have to calculate the shear force at a section the following procedure may be adopted. (see Fig.20.5)

1. Consider the left or the right part of the section.

2. Add the forces normal to the member on one of the parts.

If the right part of the section is chosen, a force on the right part acting downwards is positive while a force on the right part acting upwards is negative. For instance, if the S.F. at a section X of a beam is required and if the right part XB be considered the forces P and Q are positive while the force R is negative.
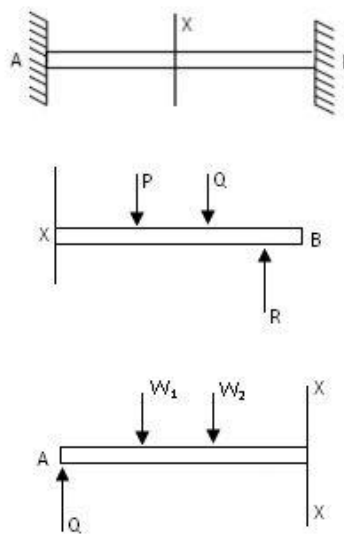
Fig. 20.5

Therefore S.F. at $X = P + Q - R$

If the left part of the section is chosen, a force on the left part acting upwards is positive and a force on the left part acting downwards is negative.

Therefore S.F. at $X = Q - W_1 - W_2$

Bending Moment: To find the bending moment at a section of a beam the following procedure may be adopted.

1. Consider the left or right part of the section.

2. Remove all restraints and all forces on the part selected.

3. Now introduce each force or reacting element one at a time and find its effect at the section.

4. Treat sagging moment as positive and hogging moment as negative.

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- The moment due to every downward force is negative and the moment due to every upward force is positive.

For instance, let the bending moment at the section *A* of the cantilever *AB* (Fig.20.6) be required.

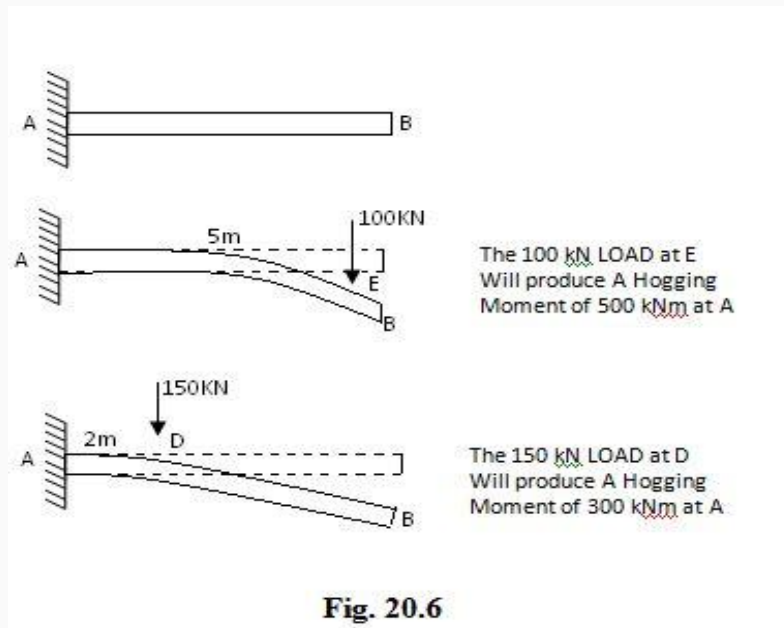**Fig. 20.6**

If the right part of the section be selected.

Remove the restraints on the part *AB*.

Introduce the load of 100 kN at *E.*

The independent effect of the load is to produce a hogging moment of -100 × 5 = -500 kNm

Now consider the independent effect of the 150 kN load at *D*.

Obviously, this will also produce a hogging moment of – 150 × 2 = -300 kNm

Therefore, Resulting bending moment at *A* = -500 – 300 = -800 kNm (hogging)

<div align="center">

LESSON 21.

</div>

21.1 Shear Force and Bending Moment diagrams for Cantilever subjected to different types of loading

(i) Cantilever subjected to concentrated load 'P' at the free end

Fig.21.1 shows a cantilever AB fixed at A and free at B and carrying a load W at the free end B.

<div align="center">

Fig. 21.1

</div>

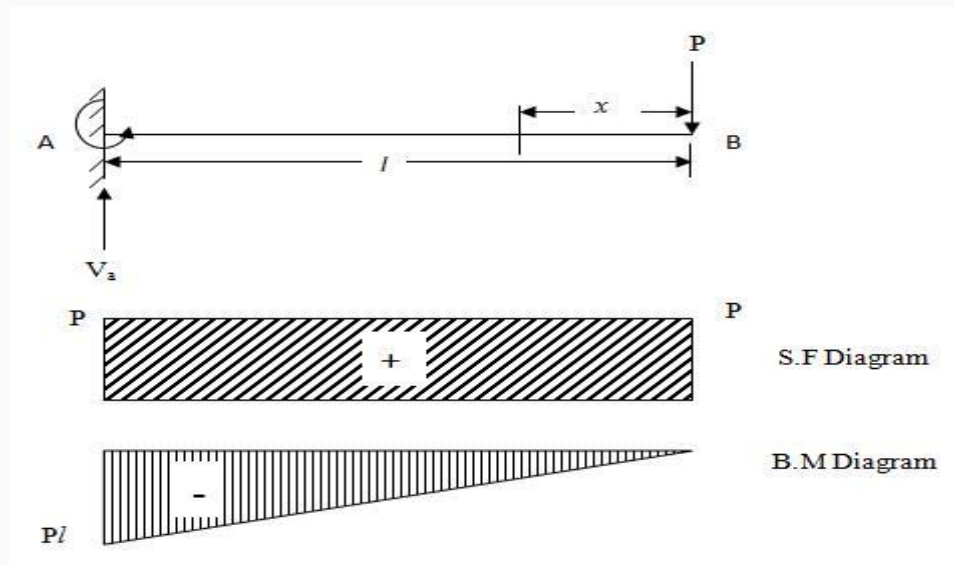Consider a section X at a distance x from the free end

Shear Force (S.F) at $X = S_x = + P$

Bending Moment (B.M) at $X = M_x = - Px$

So, the S.F is constant at all the sections of the member between A and B.

But the B.M at any section is proportional to the section from the free end.

At $x = 0$ (at B), B.M $= 0$

At $x = l$ (at A), B.M $= -Pl$

(ii) Cantilever with more than one concentrated load
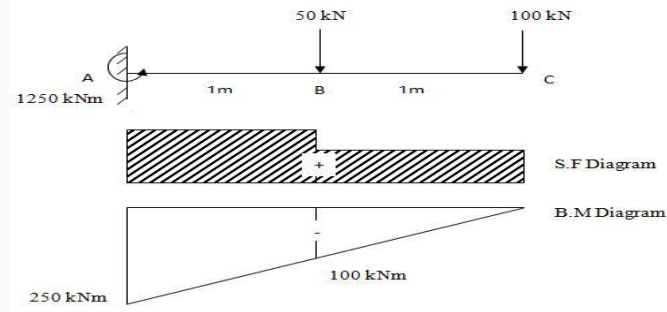
Fig. 21.2

Suppose a cantilever AC is *2 m* long and is subjected to two concentrated loads.

Consider the section between B and C, distance x from C.

S.F = S_x = 100 kN

B.M = M_x = - 100x

At x = 0, M_x = 0

At x = 1, M_x = -100 *kNm*

Now consider section A and B, distance x from C.

S.F = S_x = 100 + 50 = + 150 *kN*

B.M = M_x = - 100x – 50(x-1)

$\qquad$ = -150x + 50

At x = 1, M_x = -100 *KNm*

At x = 2, M_x = -250 *KNm*

(iii) Cantilever subjected to uniformly distributed load of p per unit run over the whole length
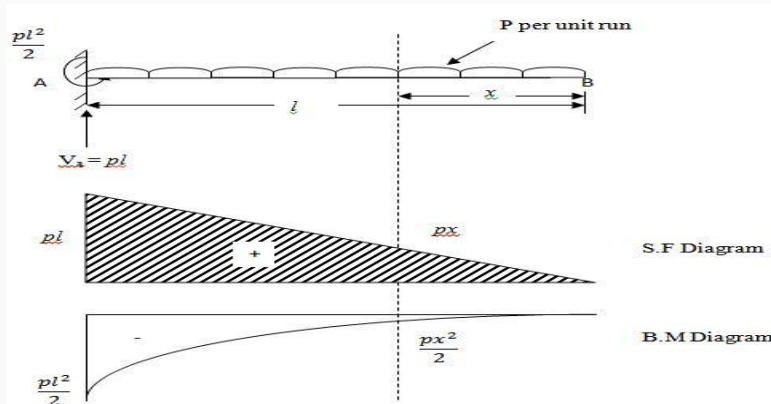
Fig. 21.3

Engineering Mechanics

Fig.21.3 shows a cantilever AB fixed at A and free at B carrying a uniformly distributed load of w per unit run over the whole length.

Consider any section X distant x from the end B.

S.F at $X = S_x = + px$

B.M at $X = M_x = - px \cdot \frac{x}{2}$

Therefore, $M_x = - \frac{p x^2}{2}$

- The variation of S.F is according to a linear law, while the variation of the bending moment is according to a parabolic law.

At $x = 0$, $S_x = 0$ and also $M_x = 0$

At $x = l$, $S_x = + wl$ and $M_x = - \frac{p l^2}{2}$

(iv) Cantilever subjected to a uniformly distributed load of p per unit run for a distance a from the free end
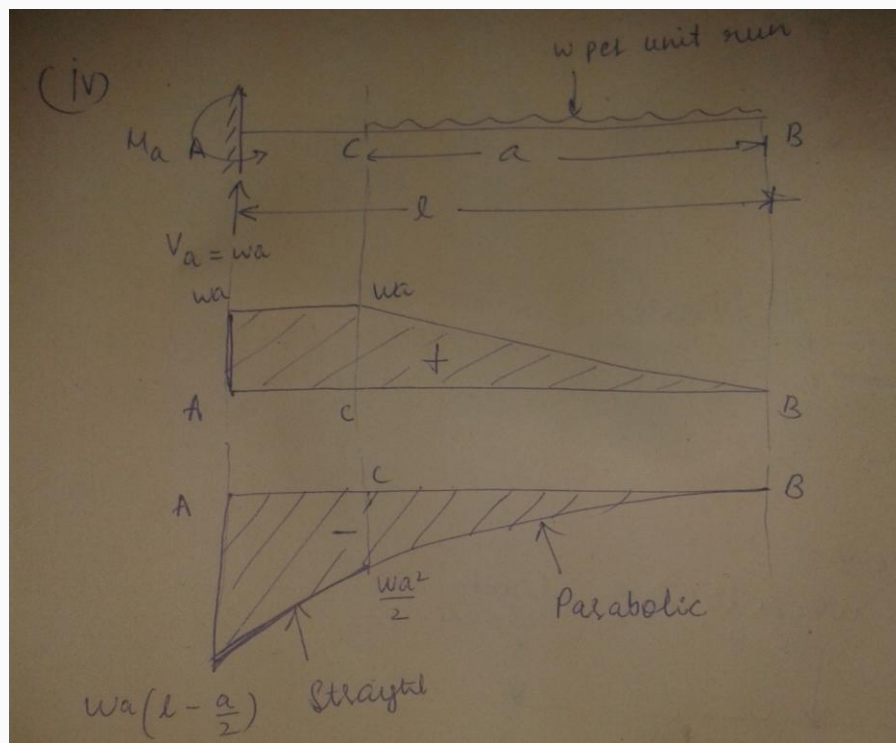
Fig.21.4

Fig.21.4 shows a cantilever AB fixed at A and free at B and carrying *udl* over a distance *a*.

Consider any section between C and B distant x from the free end B.

S.F, $S_x = + px$

B.M, $M_x = -\dfrac{p x^2}{2}$

The above relations are good for all the values of x between $x = 0$ and $x = a$

The variation of S.F will be linear and the variation of B.M for this section will be parabolic.

At $x = 0$, $S_x = 0$ and $M_x = 0$

At $x = a$, $S_x = +pa$ and $M_x = -pa\left(x-\dfrac{a}{2}\right)$

For section A and C, S.F is constant at $+pa$ but the B.M varies according to linear law.

At $x = a$, $M_x = -pa\left(a -\dfrac{a}{2}\right) = \dfrac{w a^2}{2}$

At $x = l$, $M_x = - pa\left(L - \dfrac{a}{2} \right)$

(v) Cantilever subjected to a uniformly distributed load w per unit run over the whole length and a concentrated load P at the free end.
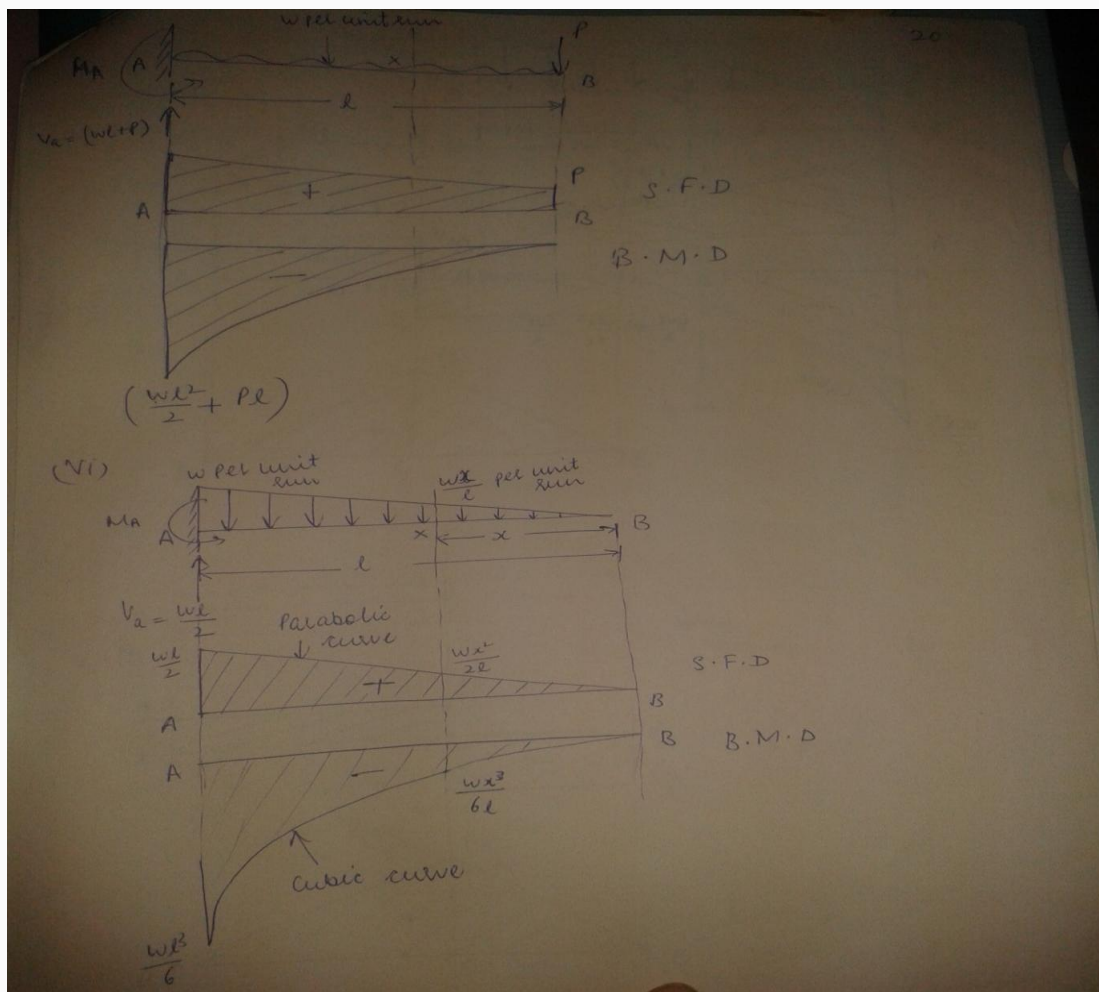

Fig. 21.5 and 21.6

Engineering Mechanics

Fig.21.5 shows a cantilever AB fixed at A and free at B and carrying the load system mentioned above. Consider any section X distant x from the end B. The S.F and the B.M at the section X, are respectively given by.

$S_x = wx + P$

And $M_x = -\left({{w x^2} \over 2} + Px \right)$

At $x = 0$, $S_x = + P$ and $M_x = 0$

At $x = l$, $S_x = + (wl + P)$

$M_x = -\left({{w x^2} \over 2} + Pl \right)$

(vi) Cantilever subjected to a load whose intensity varies uniformly from zero at the free end to w per unit run at the fixed end

Fig.21.6 shows a cantilever AB of length l fixed at A and free at B carrying the load as mentioned above.

Let the intensity of loading at X, at a distance x from the free end B be w_x per unit run.

Therefore $w_x = \left[{x \over l} w \right]$ since the intensity of load increases uniformly from zero at the free end to w at the fixed end.

The total downward load on the free body, equal to the area of the triangular loading diagram (Fig.21.6), is

$$+{1 \over 2}\left({{{w_x} \over l}} \right)\left(x \right) = +{{w x^2} \over {2l}}$$

$S_x \left[+{{w x^2} \over {2l}} \right]$

M_x = Moment of the load acting on XB about X

= area of the load diagram between X and B × distance of the centroid of this diagram from X

$$= -{{w x^2} \over {2l}} . {x \over 3}$$

$M_x = \left[-{{w x^3} \over {6l}} \right]$

At $x = 0$, $S_x = 0$ and $M_x = 0$

$x = l$, $S_x = \left[+ {{wl} \over 2} \right]$ and $M_x = \left[- {{w l^2} \over 6} \right]$

The S.F varies following a parabolic law while the B.M follows cubic law.

(vii) Cantilever subjected to a load whose intensity varies unifrormly from zero at the fixed end to w per unit run at the free end
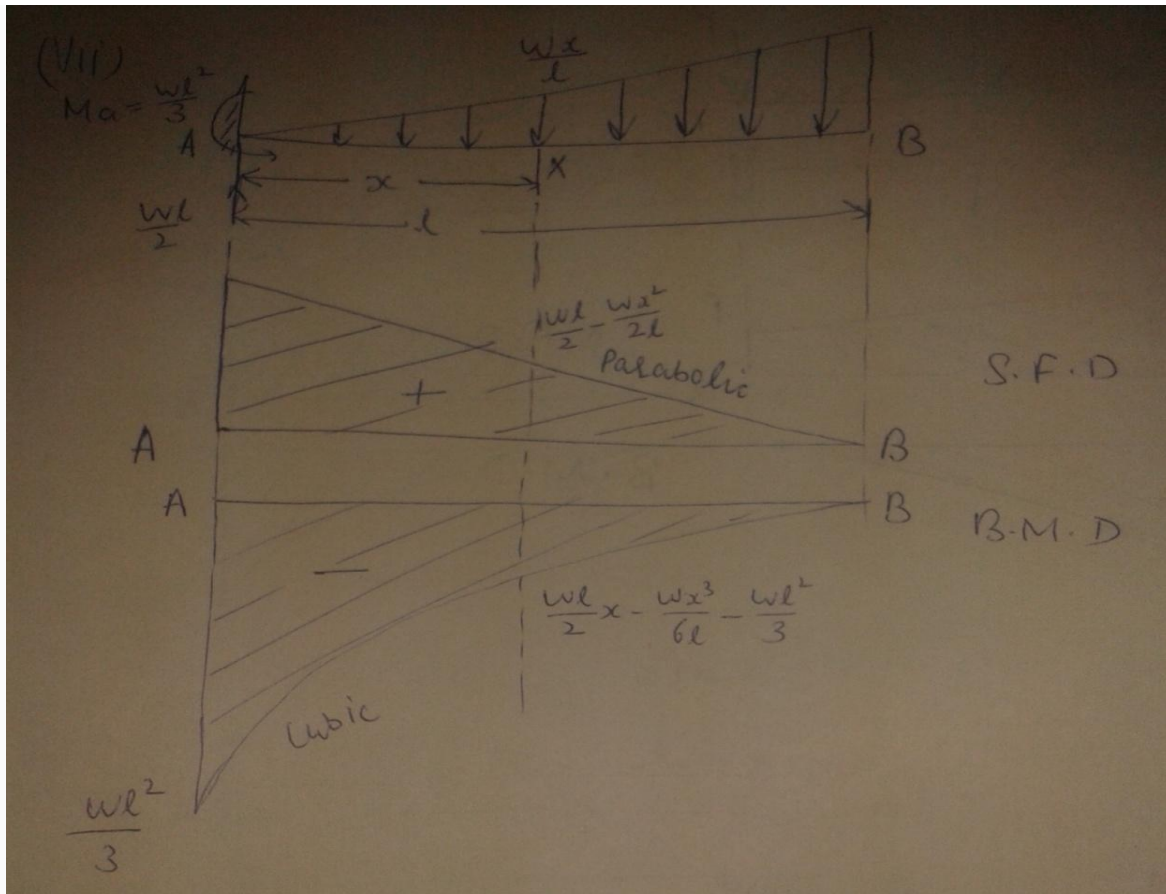
Fig. 21.7

Fig.21.7 shows a cantilever *AB* of length *l* and fixed at *A* and free at *B* and carrying the loading as mentioned above

Let M_a be the reacting moment of fixing moment *A*.

Therefore M_a = moment of the total load amount *A*

$$= {{wl}\over 2}.{{2l} \over 3}={{w{l^2}} \over 3}$$

V_a = vertical reaction at *A*

= Total load on the cantilever

$$V_a = {{wl}\over 2}$$

Consider any section *X* distant *x* from the fixed end *A*

S.F at *X* = algebraic sum of forces on *AX*

$$S_x = {{wl}\over 2} - {x \over 2}.{{wx} \over l}$$

$$S_x = {{wl}\over 2} - {{w{x^2}} \over {2l}}$$

Engineering Mechanics

B.M at X = algebraic sum of forces and reactions on AX above X

$M_x = \left[\frac{wl}{2}x - \frac{w x^2}{2l} . \frac{x}{3} - M_a \right]$

$= \left[\frac{wl}{2}x - \frac{w x^3}{6l} - \frac{w l^3}{3} \right]$

At $x = 0$ i.e at A, $\quad S_x = + \left[\frac{wl}{2} \right]$ and $M_x = - \left[\frac{w l^2}{3} \right]$

At $x = l$ i.e at B, $\quad S_x = \left[\frac{wl}{2} \right] - \left[\frac{w l^2}{2l} \right] = 0$ and $M_x = \left[\frac{wl}{2}l - \frac{w l^3}{6l} - \frac{w l^3}{3} = 0 \right]$

Example: Fig.21.8 shows a cantilever subjected to a system of loads. Draw S.F and B.M diagrams.

Solution: At any section between D and E, distance x from E

S.F = S_x = +500 N

B.M = M_x = -500x

At x = 0, M_x = 0

x = 0.5m, M_x = -250 Nm

At any section between C and D, distance x from E

S.F = S_x = +500 + 500 = + 1000 N

B.M = M_x = -500x – 500(x - 0.5)

= -1000x + 250

At x = 0.5, M_x = -250 Nm

At x = 1m , M_x = -750 Nm

At any section between B and C, distance x from E

S.F = S_x = +500 + 500 + 400 = + 1400 N

B.M = M_x = -500x – 500(x - 0.5) – 400(x – 1)

= -1400x + 650

At x = 1m, M_x = -750 Nm

At x = 1.5m , M_x = -1450 Nm

At any section between A and B, distance x from E

S.F = S_x = +500 + 500 + 400 + 400 = + 1800 N

Engineering Mechanics

B.M = M_x = -500x – 500(x - 0.5) – 400(x – 1) – 400 (x – 1.5)

= -1800x + 1250

At x = 1.5m, M_x = -1450 Nm

At x = 2m , M_x = -2350 Nm

<div align="center">LESSON 22.</div>

22.1 Shear Force and Bending Moment diagrams for Simply Supported Beams subjected to different types of loading

(i) Simply Supported Beam subjected to concentrated load at mid span

<div align="center">Fig. 22.1 and 22.2</div>
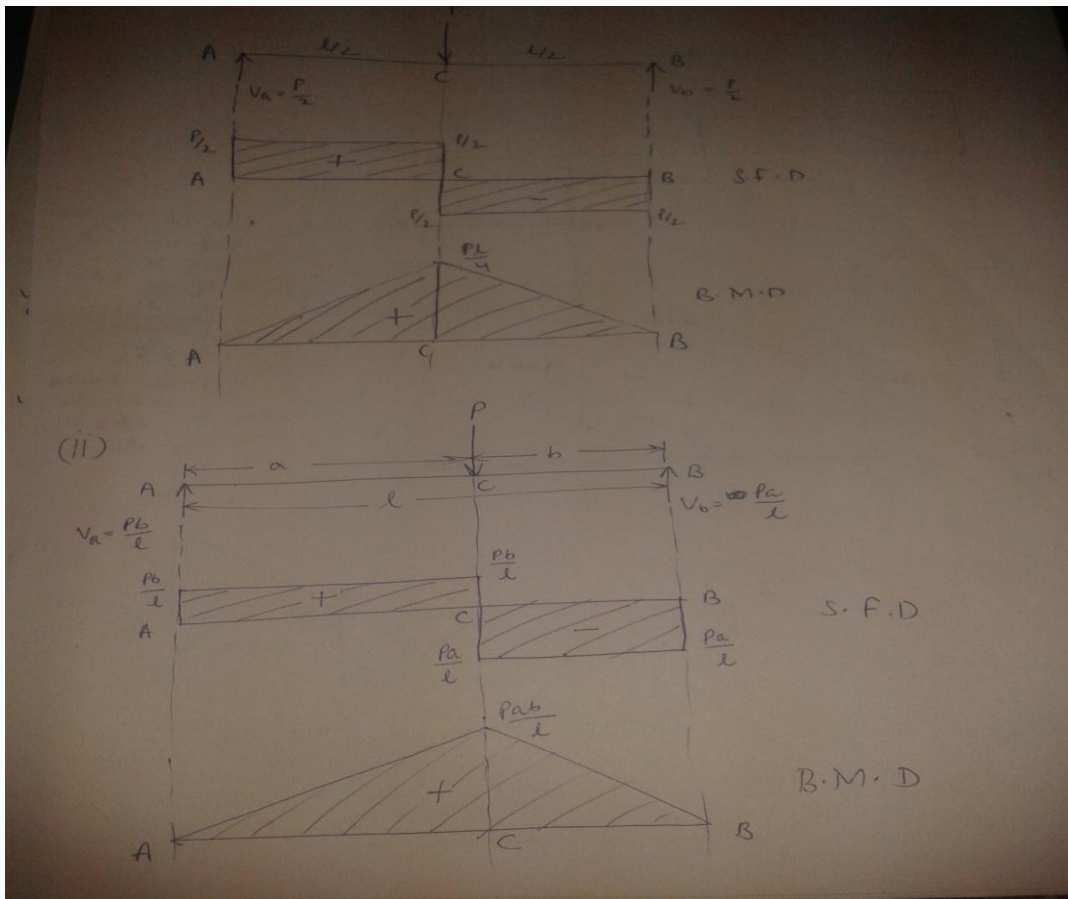
As shown in Fig.22.1, a beam *AB* simply supported at the ends *A* and *B*. The length of the span is *l*. *P* is the concentrated load applied at the centre.

The load is placed symmetrical on the span,

V_a and V_b are the vertical reactions at *A* and *B*.

The reaction at each support will be $\frac{P}{2}$

Therefore, $V_a = V_b = \frac{P}{2}$

For any section between *A* and *C*,

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S.F = S_x = + $\left[\dfrac{P}{2} \right]$

Similarly for section C and B,

S.F = S_x = - $\left[\dfrac{P}{2} \right]$

At the point C, the S.F changes from + $\left[\dfrac{P}{2} \right]$ to - $\left[\dfrac{P}{2} \right]$

For B.M, at any section between A and C distant x from the end A,

M_x = + $\left[\dfrac{P}{2} \right]$ x (sagging moment)

At $x = 0$, $M_x = 0$

At $x = $, $M_x = \left[\dfrac{Pl}{4} \right]$

Hence B.M increases uniformly from zero at A to $\left[\dfrac{Pl}{4} \right]$ at C.

Similarly, B.M decreases uniformly from $\left[\dfrac{Pl}{4} \right]$ at C to zero at B.

Note: It should be noted that maximum bending moment occurs at mid span (at C), where the shear force changes its sign.

(ii) Simply supported beam subjected to a concentrated load placed eccentrically on the span

Fig.22.2 shows a simply supported beam AB of span l carrying a concentrated load P at C eccentrically. Where $AC = a$ and $CB = b$

For the equilibrium of the beam,

Taking moments of the forces on the beam about A,

We have $\quad V_b . l = Pa$

$V_b = \left[\dfrac{Pa}{l} \right]$

$V_a = P - \left[\dfrac{Pa}{l} = \dfrac{P\left(l - a \right)}{l} \right]$

$V_a = \left[\dfrac{Pb}{l} \right]$

We know that $a + b = l$

At any section between A and C,

S.F, S_x = V_a = + $\left[\dfrac{Pb}{l} \right]$

for section C and B,

S.F, S_x = - V_b = - $\left[\dfrac{Pb}{l} \right]$

Similarly, Bending Moment for any section between A and C

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$M_x = + \[{{Pb} \over l}x\]$ (sagging)

At $x = 0$, $M_x = 0$

At $x = a$, $M_x = \[{{Pab} \over l}\]$

Hence the B.M increases uniformly from zero at the left end A to $\[{{Pab} \over l}\]$ at C. Similarly the B.M will decrease uniformly from $\[{{Pab} \over l}\]$ at C to zero at the right end B.

(iii) Simply supported beam subjected to more than one concentrated load
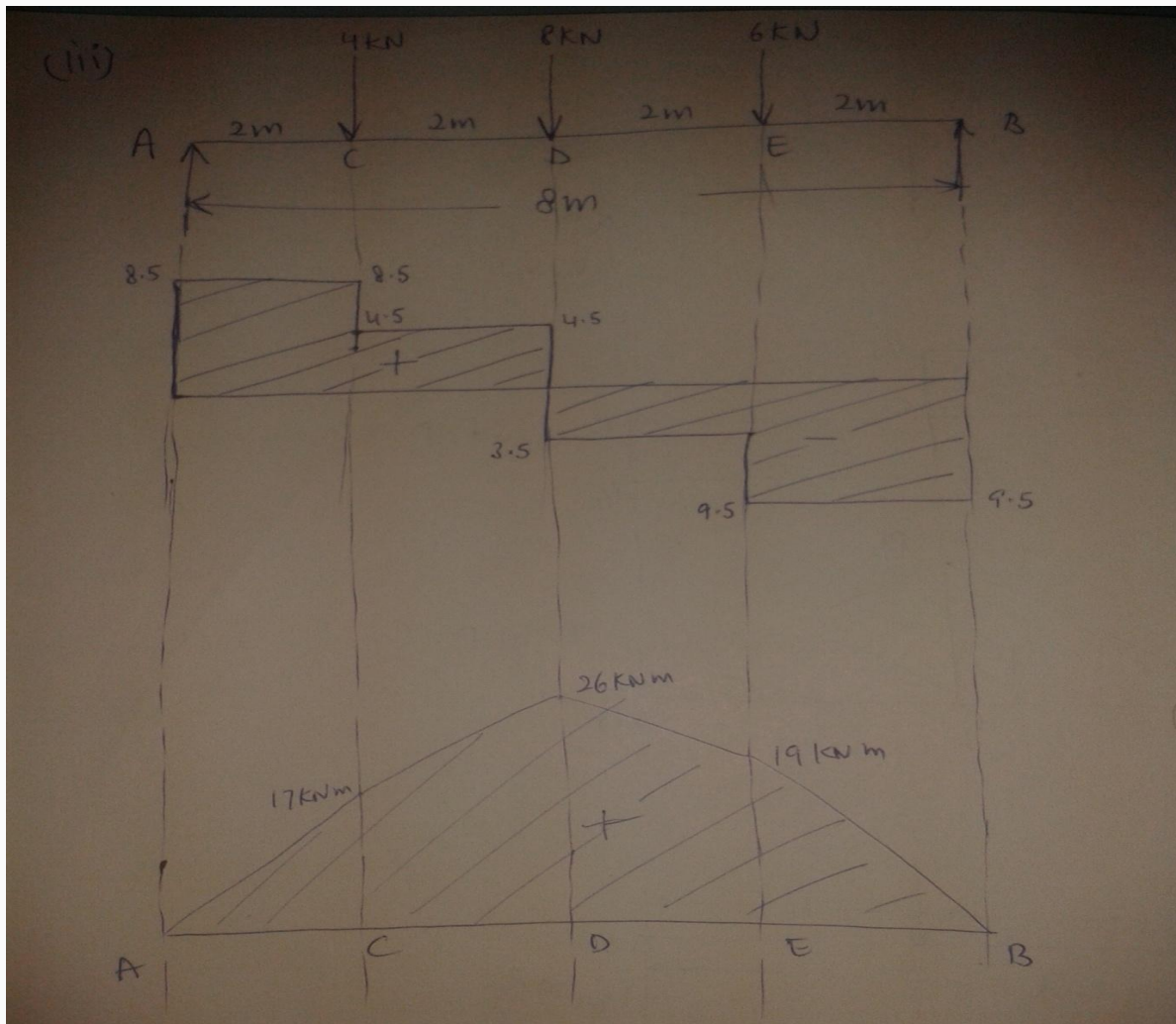
Fig. 22.3

Fig.22.3 shows a simply supported beam AB of span 8m subjected to concentrated loads of 4kN, 8kN and 6kN at distances of 2m, 4m and 6m respectively from support A.

To find the vertical reactions Va and Vb at the supports A and B respectively.

For the equilibrium of beam, Taking moment of the forces on the beam about the left support,

We have,

$V_b \times 8 = 6 \times 6 + 8 \times 4 + 4 \times 2$

$V_b = 9.5 \ kN$

V_a = total load on the beam – V_b

= 18 - 9.5 = 8.5 kN

We will start from the support A,

S.F between A and C = 8.5 kN

S.F between C and D = 8.5 – 4 = 4.5 kN

S.F between D and E = 8.5 – 4 – 8 = -3.5 kN

S.F between E and B = 8.5 – 4 – 8 – 6 = -9.5 kN

B.M at A = 0

B.M at C = 8.5 × 2 = 17 kNm (sagging)

B.M at D = 8.5 × 4 – 4 × 2 = 26kNm (sagging)

B.M at E = 8.5 × 6 – 4 × 4 – 8 × 2 = 19kNm (sagging)

Note: it can be observed from the S.F and B.M diagrams that the maximum B.M occurs at D where the S.F changes its sign.

(iv)Simply supported beam subjected to uniformly distributed load of w per unit run over the whole span
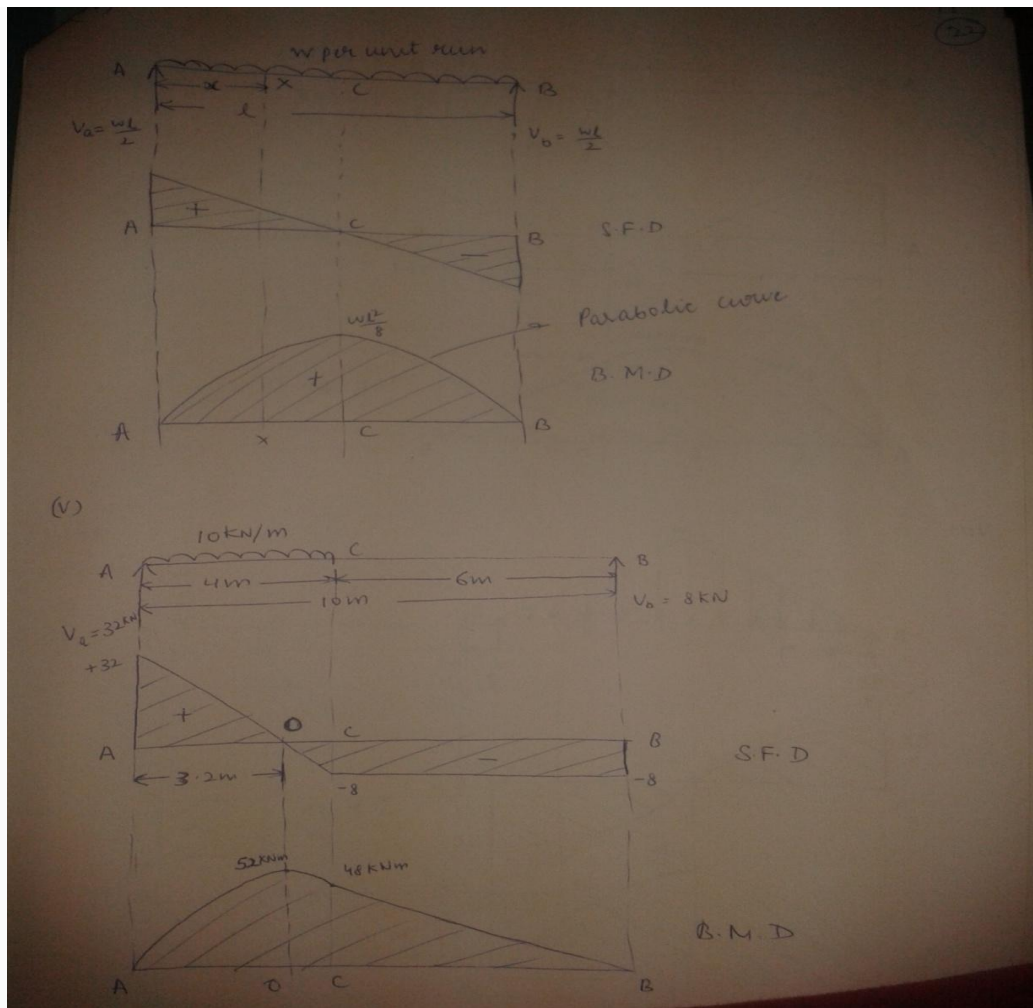
Fig. 22.4 and 22.5

Fig.22.4 shows a simply supported beam AB of span l subjected to uniformly distributed load w per unit run over the whole span.

Since the loading the symmetrical on the span, each vertical reaction equal half the total load on the span.

Therefore, $V_a = V_b = {wl \over 2}$

Consider any section X distant x from the left end A.

S.F and B.M at the section X are given by,

$S_x = + V_a - wx = + {wl \over 2} - wx$

$M_x = + V_a x - {wx^2 \over 2} = {wl \over 2} x - {wx^2 \over 2}$

Therefore, $M_x = + {w \over 2} x (l - x)$

At $x = 0$, $S_x = {wl \over 2}$ and $M_x = 0$

At $x = l$, $S_x = {wl \over 2} - wl = - {wl \over 2}$ and $M_x = 0$

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At $x = \frac{1}{2}$, $S_x = \frac{wl}{2} - \frac{wl}{2} = 0$

And $M_x = + \frac{w}{2} \cdot \frac{l}{2} \left(l - \frac{l}{2} \right) = + \frac{wl^2}{8}$

The S.F diagram is straight line. The S.F uniformly changes from $+ \frac{wl}{2}$ at *A* to $- \frac{wl}{2}$ at *B* and obviously the S.F at mid span is zero.

The B.M diagram is parabola. It increases from zero at *A* to $+ \frac{wl^2}{8}$ at the mid span *C* and from this value it decreases to zero at *B*.

(v)Simply supported beam subjected to a uniformly distributed load over some part of the span

Fig.22.5 shows a simply supported beam as mentioned above.

Taking moments about A,

$V_b \times 10 = 10 \times 4 \times \frac{4}{2}$

Therefore V_b = 8 kN

V_a = 10 × 4 – 8 = 32 kN

Consider any section between A and C distant x from A

S.F at the section is given by, S_x = +32 – 10x

At x = 0, S_x = + 32 kN

At x = 4m, S_x = 32 – 40 = - 8 kN

Let the S.F be zero x meters from A. Equating the S.F to zero, we get,

32 – 10x = 0

x = 3.2 m from A

at any section in AC distant x from A the B.M is given by,

$M_x = + 32x - 10 \frac{x^2}{2} = 32x - 5x^2$

At x = 0, M_x = 0

At x = 4m, M_x = 32 × 4 - 5× 4^2

= 48 kNm

At x = 3.2 m, M_x = 32 × 3.2 – 5 × 3.2^2

= 51.2 kNm

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B.M decreases from 48 kNm at C to zero at B according to the linear law.

Maximum B.M occurs at D where S.F is zero.

(vi)simply supported beam subjected to two couples.

Example 1: Draw the S.F and B.M diagrams for the beam (Fig.-------------).

Sol: By taking moments about A.

$V_b \times 10 + 100 + 80 = 20 \times 10 \times 5$

$V_b = 80$ kN

$V_a = (20 \times 10) – 80 = 120$ kN

S.F Diagram: At any section distant x from A, the shear force is given by,

$S_x = 120 – 20x = 0$

At $x = 0$, S.F = 120 kN

At $x = 10$, S.F = -80 kN

For finding the section of zero shear, equating the general expression for shear force to zero.

$120 – 20x = 0$

$x = 6$ m

B.M Diagram: At any section distant x from the end A, the bending moment is given by,

$M = 120 – 20. x. - 100$

$M = 120 – 10x^2 – 100$

at $x = 0$, B.M = -100 kNm

at $x = 10$, B.M = 100 kNm

at $x = 6m$, B.M = 260 kNm

For finding the point of contraflexure, equating the expression for B.M to zero.

$120x – 10x^2 – 100 = 0$

$x^2 – 12x + 10 = 0$

we get, $x = 0.9m$

Example: Fig.…………shows the simply supported beam. Draw the S.F and B.M diagrams.

Engineering Mechanics

Solution: Let V_a and V_b be the vertical reactions at the supports A and B respectively.

By taking moments about A.

$V_b \times 8 = 5 \times 1 + 10 \times 3.5 + 6 \times 6$

$V_b = 9.5$ kN

$V_a = 5 + 10 + 6 - 9.5 = 11.5$ kN

S.F between A and C = 11.5 kN

S.F between C and D = 11.5 – 5 = 6.5 kN

S.F between D and E = 11.5 – 5 – 10 = - 4.5 kN

B.M at A = 0

B.M at C = 11.5 × 1 = 11.5 kNm

B.M at D = 11.5 × 3.5 – 5 × 2.5 = 27.75 kNm

B.M at E = 9.5 × 2 = 19 kNm

MODULE 7.

LESSON 23.

23.1 INTRODUCTION

When a system of forces acts on a beam perpendicular to the longitudinal axis, they produce not only reactions at the supports, but also cause the beam to bend or deflect.

- Such a system consists of axial forces, shearing forces and bending moments, producing compressive, tensile and shearing stresses on the beam.

- The tensile and compressive stresses are referred as flexural stresses.

- When a segment of beam is in equilibrium, under the action of a moment alone, the state is defined as pure-bending.

- Whereas if in addition twisting and buckling occurs, the combined effect of bending, twisting and buckling will have to be considered. Such a situation becomes complicated one.

- It may be worthwhile mentioning here that bending is quite a common phenomenon.

23.2 PURE BENDING
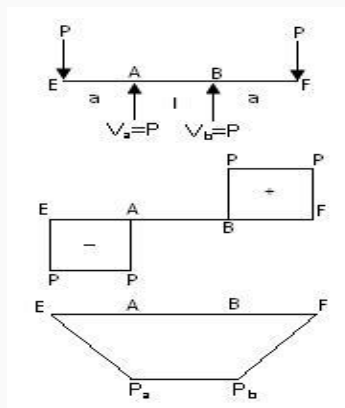
Fig.23.1

Fig.23.1 shows a beam *EABF* of negligible weight with supports at *A* and *B*, *l* units apart. Let the overhangs *EA* = *BE* = *a*.

- A point load *P* be applied at each end of the beam.

- It can be easily seen that between *A* and *B*, the B.M is constant and there is no shear force at this portion.

- We can simply say that this portion (between *A* and *B*) is absolutely free from shear but is subjected to bending moment *Wa*

This condition of the beam between *A* and *B* is called pure bending or simply bending.

In the end we can conclude that A part of a member is said to be in pure bending if no shearing force exists in this part.

23.3 THEORY OF SIMPLE BENDING

In Fig.23.2 *ACDB* is a part of a beam of length Δx subjected to pure bending. The length Δx has deformed to the shape as shown in Fig.23.2(c).

(a)
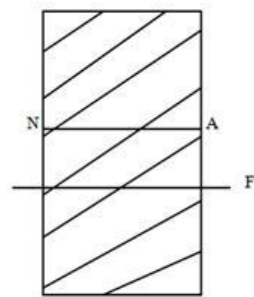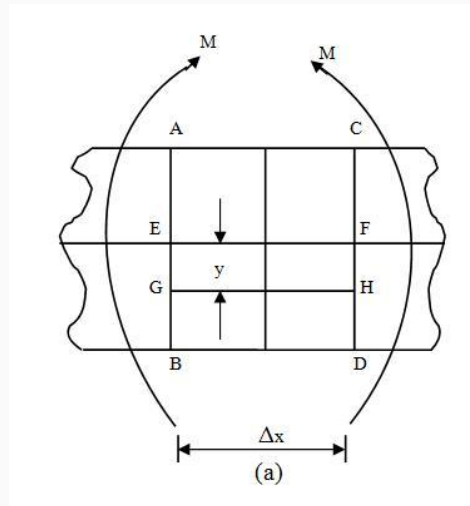
Cross selection of Beam
(b)

STRESS DIAGRAM

(c)
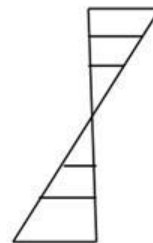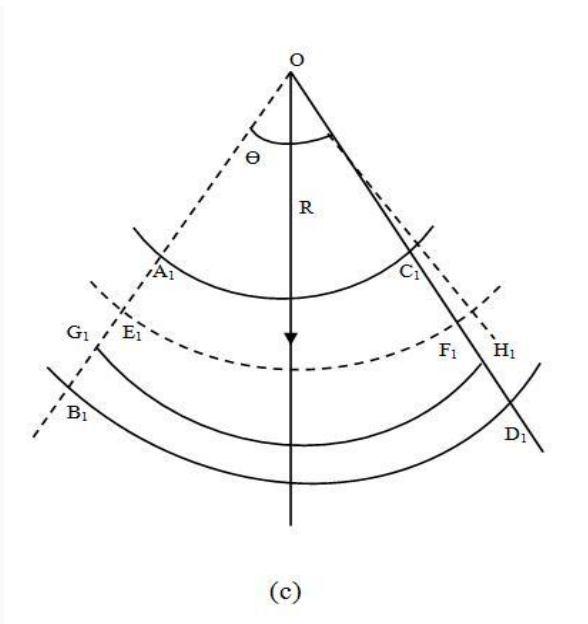
Fig. 23.2

- The side *AC* has deformed to the shape A_1C_1. This fibre has been shortened in length. The fibre *BD* opposite to fibre *AC* has been elongated and has taken the shape B_1D_1. Similarly, the fibre *GH* has been elongated and has taken the shape G_1H_1.

- Hence we can say that the beam of length Δx consists of a large number of fibres, all of them have changed their shapes; some of them have been shortened while some of them are elongated.

1. At a level between the top and bottom of the beam there will be a layer of fibres which are neither shortened nor extended.

Fibres in this layer are not stressed at all.

This layer is called the neutral layer or the neutral surface.

2. The line of intersection of the neutral surface on a cross-section is called the *neutral axis*.

Now if we consider all the fibres between the sections *ABCD*, The extremities of these fibres will remain on the planes A_1B_1 and C_1D_1 after deformation.

Let A_1B_1 and C_1D_1 meet at *O* and the angle between the planes A_1B_1 and C_1D_1 be θ. Let the radius of the neutral surface be *R*.

Consider the fibre *GH* which is at a distance *y* from the neutral layer. Original length of the fibre $GH = \Delta x$

After deformation this fibre will deform and take the position G_1H_1, the new length of the fibre being $(R + y)\,\theta$.

The fibre *EF* in the neutral layer takes the position E_1F_1 without under-going any change in length. Therefore,

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$EF = E_1F_1 = \Delta x$

$\Delta x = R\Theta$

Therefore, change in length of the fibre GH = G_1H_1 - GH

$R\Theta = (R + y)\,\Theta - \Delta x = (R + y)\,\Theta - y\,\Theta$
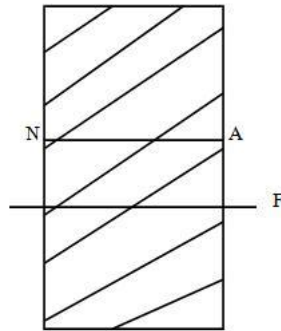
Cross selection of Beam
(b)

Therefore Strain of the fibre GH = ε = $\left[\frac{change\,in\,length}{original\,length}\right]$ = $\left[\frac{y\theta}{R\theta}\right]$

$$\varepsilon = \left[\frac{y}{R}\right]$$

Suppose the stress intensity in the fibre be *f*, we have strain of the fibre = ε = $\left[\frac{f}{E}\right]$

Where *E* is the young's modulus.

Therefore, ε = $\left[\frac{f}{E}\right]$ = $\left[\frac{y}{R}\right]$ or $\left[\frac{f}{y}\right]$ = $\left[\frac{E}{R}\right]$

f = $\left[\frac{E}{R}\right]$ .y

The stress determined from the above flexure formula are called flexural stresses or bending stresses.

- From the above discussion we can conclude that all the fibres which are below the neutral layer are subjected to tensile stresses while those above the neutral layer are subjected to compressive stresses.

23.4 DERIVATION OF THE BENDING EQUATION

The bending equation is $\left[\frac{M}{I}\right]$ = $\left[\frac{f}{y}\right]$ = $\left[\frac{E}{R}\right]$

Assumptions:

In the determination of stress in beam, the following assumptions or conditions will be applied:

1. The beam is initially straight and not curved.

2. All loads applied are steady and delivered to the beam without shock or impact.

3. All beams are stable under the applied loads.

4. Beams with constant area of cross-section with at least one axis of symmetry are to be considered.

5. The modulus of elasticity E is the same in tension and compression.

6. A plane section, taken normal to the axis of the beam, remains plane after the beam is subjected to bending.

7. All longitudinal elements of the beam before bending have the same length.

8. The stresses are within the proportional limit.

9. The applied loads act in a plane containing the axis of symmetry of each cross-section.

10. Material is homogeneous, isotropic and continuous.

Derivation:

To derive the equation of bending, Fig.23.3 shows a simply supported beam, both before bending and after bending. Two transverse sections AB and CD at a small distance dx have been considered.
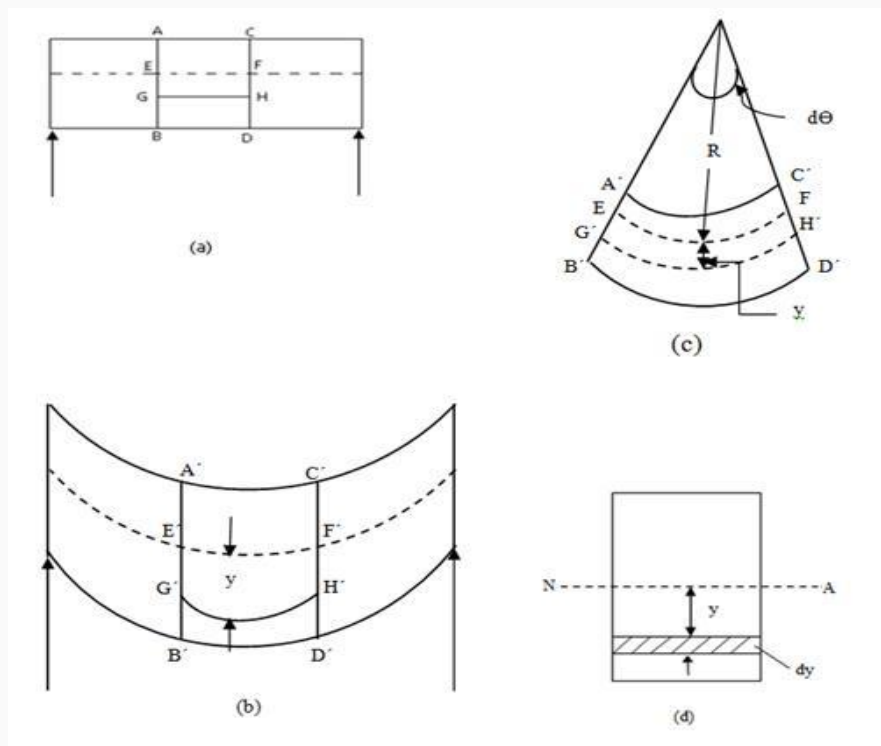

Fig. 23.3

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1. Before bending, AC = BD = EF = GH = dx

2. After bending, Section AB and CD change into A'B' and C'D' respectively.

3. All the fibres above EF are shortened and fibres below EF are lengthened, in proportion to their distance from the fibre EF i.e Upper fibre get into compression and lower fibre get into tension.

- The EF layer which is between these two zones of compression and tension is neither in compression nor in tension, thus creating a neutral layer.

4. Let R be the radius of curvature of neutral layer EF, after bending. Arc EF subtends an angle $d\Theta$ at the centre of curvature, which makes EF = R $d\Theta$ and G'H' = (R + y) $d\Theta$.

$$GH = \varepsilon_{GH} = \frac{G'H' - GH}{GH} = \frac{(R+y)d\Theta - Rd\Theta}{Rd\Theta} = \frac{y}{R}$$

5. Bending tensile strain in

Hence, $\varepsilon_{GH} =$ where f_y is the stress at a distance y from the neutral axis.

f_y, the stress used in the expression is longitudinal and perpendicular to the cross-section. Strain produced will be in longitudinal direction.

This expression can be rewritten as

$$\frac{f_y}{y} = \frac{E}{R}$$

Since E and R constant, For a bent beam = constant, which implies that it can be written for all points at a section depending on the distance y i.e.

$$\frac{f_y}{y} = \frac{f_1}{y_1} = \frac{f_2}{y_2} = \frac{f_{max}}{y_{max}}$$

Or fy = $\frac{f_{max}}{y_{max}}$ .y

It can also be concluded that stress-variation on the compression portion would be similar and the maximum bending stress will occur at extreme fibre, nature of the stress being opposite.

6. Now consider a small area dA at distance y as shown in the cross-section of beam.

Force on this elementary area dA, carrying stress fy is dF = fy.dA = $\frac{f_{max}}{y_{max}}$. y dA

Moment of the force about the N.A will be dM = .y dA.y

Therefore, moments of all such forces for the entire depth will be

$$\int dM = M = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{f_{max}}{y_{max}} . y dA . y = \frac{f_{max}}{y_{max}} \int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2 dA = \frac{f_{max}}{y_{max}} . I$$

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(I = moment of inertia = \[\smallint dA{y^2}\])

Hence,   = \[{M \over {I}} = {{{f_{max}}} \over {{y_{max}}}} = {{{f_y}} \over y}\]

7. Combining equation (7.1) and (7.2), the simple bending formula is obtained i.e.

It is called **Simple Flexure Formula**.

- The word simple signifies that only pure bending is being considered.

- Shear force and bending moment varies on beam from section to section.

- At the point of maximum B.M, S.F is zero.

- The Bending equation is applied for maximum bending moment.

- In case of Cantilever subjected to loading vertically downwards, upper fibres will be in tension while lower fibres will be in compression.

Limitations of the Flexural Formula:

While deriving the Flexural formula, certain assumptions had been made, which limits the extent of its field of application. The formula cannot be applied to cases such as

- When the load on the beam is not applied perpendicularly to its longitudinal axis.

- When the loads are suddenly (or with impact) applied.

- When the cross-sections are unsymmetrical.

23.5 NEUTRAL AXIS

As we have discussed earlier, The neutral axis is defined as an imaginary line in the cross-section of beam along which no stresses occur.

- In other words, when a beam bent downwards, the line of zero stress below which all fibres are in tension and above which they are in compression is called neutral axis.
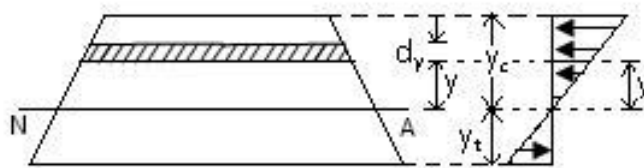

Fig.23.4

Fig.23.4 shows the cross-section of a beam. Let R be the radius of curvature of the neutral layer at this section.

Hence the stress at any point distant y from the neutral axis is given by

$f = \dfrac{E}{R} \cdot y$

where E is the young's modulus. If the section be subjected to pure sagging moment, this stress will be compressive at any point above the neutral axis and tensile below the neutral axis.

Now consider an elemental area da distant y from the neutral axis.

Stress on the elemental area $= f = \dfrac{E}{R} \cdot y$

Therefore, Thrust on an elemental area $= f\,da = \dfrac{E}{R} \cdot y\,da$

Therefore, Total thrust on the beam section $= \dfrac{E}{R}\sum\limits_{y_t}^{y_c} yda$

Since no axial load has been applied, the total thrust on the beam section equals zero.

$\dfrac{E}{R}\sum\limits_{y_t}^{y_c} yda = 0$, Therefore $\sum\limits_{y_t}^{y_c} yda = 0$

This is possible only when the neutral axis is a centroidal axis.

Example 23.1: A steel plate is bent into a circular arc of radius 15 meters. If the plate section be 100 mm wide and 15 mm thick, Find the maximum stresses induced and the bending moment which can produce this stress. Take $E = 2 \times 10^5$ N/mm^2.

Solution: Moment of inertia of the section about the neutral axis.

$I = \dfrac{100 \times 15^3}{12} = 28125$ mm^4

Bending Equation is $\dfrac{M}{I}=\dfrac{f}{y}=\dfrac{E}{R}$

$f = \dfrac{E}{R}y$

$f_{max}=\dfrac{2 \times 10^5}{15 \times 10^3} \times \left(\dfrac{15}{2}\right)$ N/mm^2 = 100 N/mm^2

$M = \dfrac{E}{R} \cdot I = \dfrac{2 \times 10^5}{15 \times 10^3} \times 28125 = 37500$ Nmm

Example 23.2: A thin high-strength steel plate 5 mm thick and 800 mm long is bent by couples M_o into a circular arc which subtends a central angle of 45°. Find the maximum bending stress in the plate. If the central angle is increased will the stress increase or decreases?

Take $E = 2 \times 10^5$ N/mm^2.

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Solution: Let R be the radius of the neutral layer

$L = R\theta$

$R = \frac{L}{\theta} = \frac{800}{\left(\frac{\pi}{4}\right)} = \frac{3200}{\pi}$ mm

$\frac{E}{R} = \frac{f}{y}$, therefore $f = \frac{E}{R} \cdot y$

$f = \frac{2 \times 10^5}{\left(\frac{3200}{\pi}\right)} \times \frac{E}{R} = 490.87$ N/mm²

Example 23.3: Fig. 23.5 shows a rectangular tube section of an aluminium alloy whose ultimate stress is 350 N/mm². Determine the bending moment for which the factor of safety will be 3. Find also the corresponding radius of curvature. Take E = 7 × 10⁴ N/mm².

Solution: Allowable bending stress = f = $\frac{350}{3}$ = 116.67 N/mm²

Moment of Inertia of the section = I = $\frac{60 \times 100^3}{12} - \frac{50 \times 90^3}{12} = 1962500$

Allowable B.M. = M = $\frac{f}{y_{max}} \cdot I = \frac{116.67}{50}\left(1962500\right)$ Nmm

$= 4.579 \times 10^6 = 4.579$ kNm

Radius of Curvature = R = $\frac{E}{f} \cdot y_{max}$

$= \frac{7 \times 10^4}{116.67} \times 50 = 29999.14$ mm

$= 29.999$ m

Example 23.4: A copper wire of 2 mm diameter is bent into a circle and held with its ends just in contact (Fig.23.6). If the maximum permissible strain in copper is 0.0025, find the shortest length of the wire that can be used.

Solution: Let the minimum radius of the circle be R

Let f be the maximum stress in the wire.

$\frac{f}{\left(\frac{d}{2}\right)} = \frac{E}{R}$

Therefore $\frac{d}{2R} = \frac{f}{E} = 0.0025$

$\frac{2}{2R} = 0.0025$

$R = 400$ mm

$= 2\pi R = 2\pi \times 400 = 2513.27$ mm

LESSON 24.

24.1 MOMENT OF RESISTANCE
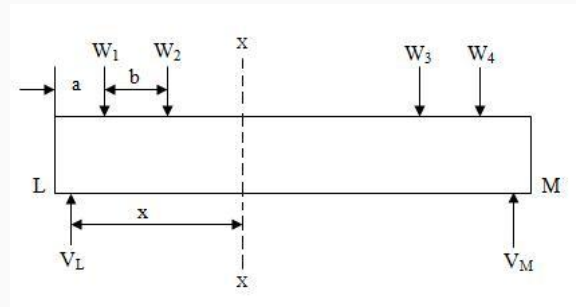
(A)
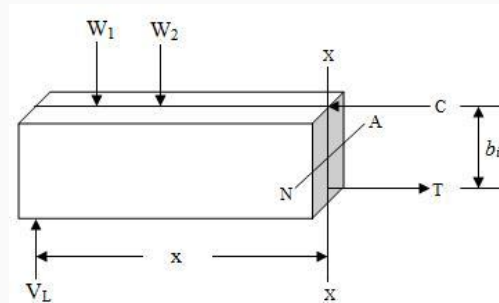
(B)
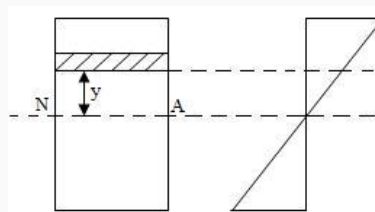
(C)

Fig.24.1

Bending equation gives or \[{M \over {I}} = {{{f_{all}}} \over {{y_{max}}}}\] or M = \[{{I \over {{y_{max}}}}\] fall

$M_R = Z\, f_{max}$

This quantity "ZF" is termed as "Moment of Resistence", which totally depends on allowable stress fall ans section modules Z.

- It does not depend on the system of loading.

- However, it would be necessary that moment of resistence is always greater than or equal to the external bending moment.

24.2 SECTION MODULUS

It is a property of sectional area.

We know that $\dfrac{M}{I} = \dfrac{f}{y}$

Or $f = \dfrac{M}{I/y}$ and $f_{max} = \dfrac{M}{I/y_{max}}$

Where $\dfrac{I}{y_{max}}$ is called section modulus 'Z'.

Therefore $Z = \dfrac{I}{y_{max}}$

$f_{max} = \dfrac{M}{Z}$ which is similar to $f = \dfrac{P}{A}$ (in simple tension and compression)

- For same area of cross-section, but with different shapes, the section modulus will be different.

- In bending it would be desirable to choose sections which give the maximum value of section modulus.

- I sections have been found to give the maximum section modulus for a specific area of cross-section and hence should be preferred for bending loads.

7.8 SECTION MODULUS FOR VARIOUS SHAPES OF BEAM SECTIONS

(i) Rectangular Section

Fig.24.2 shows a rectangular section of width b and depth d. Let the horizontal centroidal axis be the neutral axis.
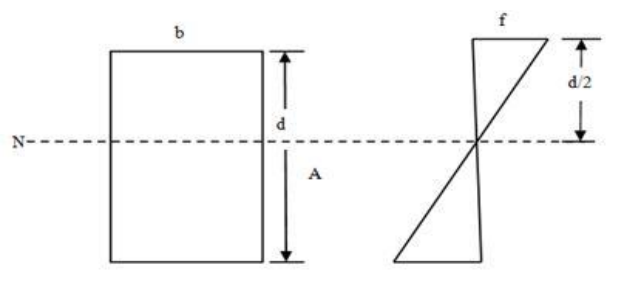
Fig.24.2

Section Modulus = $Z = \dfrac{\text{Moment of inertia about the neutral axis}}{\text{Distance of most distant point from the neutral axis}}$

$= \dfrac{I}{y_{max}}$

I = \[{{b{d^3}} \over {12}}\] and y_{max} = \[{d \over 2}\]

Z = \[{{b{d^3}} \over {12}}\] . \[{2 \over d}\] = \[{{b{d^2}} \over {6}}\]

Let f be the maximum stress offered by the beam section.

Therefore moment of resistance = $M = f Z = f$ \[{{b{d^2}} \over {6}}\]

Or M = \[{1 \over 6}\] $f bd^2$

(ii) Hollow Rectangular Section

Fig.24.3 shows a hollow rectangular section. Let the overall width and depth be B and D. Let the width and depth of the centrally situated rectangular hole be b and d.
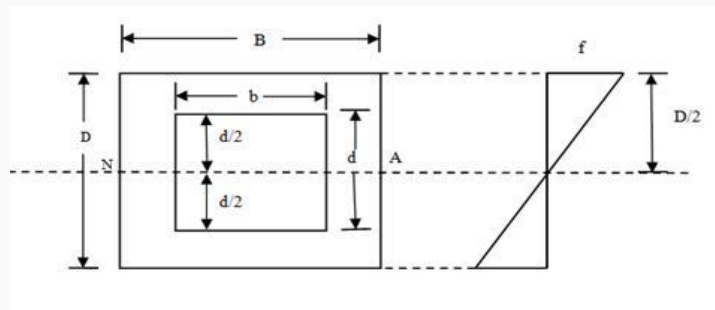
Fig.24.3

Moment of inertia about the neutral axis = I = \[{{B{D^3}} \over {12}}\] - \[{{b{d^3}} \over {12}}\] = \[{1 \over 12}\] [BD³ - bd³]
y_{max} = \[{D \over 2}\]

Therefore, Section Modulus = Z = \[{I \over {{y_{max}}}}={1 \over {12}}(B{D^3}-b{d^3}\]) \[{2 \over D}\]

Z = \[{{B{D^3} - b{d^3}} \over {6D}}\]

If f be the maximum bending stress the moment of resistance = $M = f Z$

M = \[{1 \over 6}f\left({{{B{D^3} - b{d^3}} \over D}} \right)\]

(iii) Circular Section
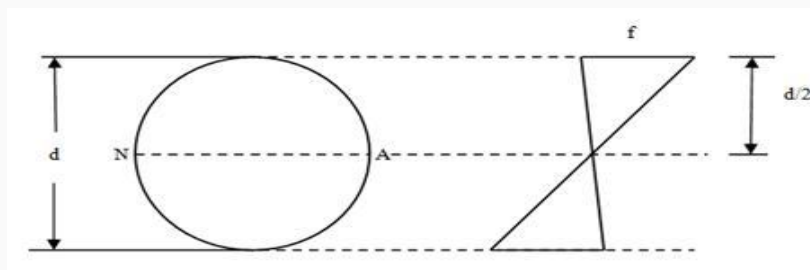
Fig. 24.4

Engineering Mechanics

Let the diameter of the section be d. Moment of inertia of the section about the neutral axis
$= I = \left[\frac{\pi d^4}{64}\right]$

$y_{max} = \left[\frac{d}{2}\right]$

Section Modulus = $Z = \left[\frac{I}{y_{max}}\right] = \left[\frac{\pi d^3}{32}\right]$

If f be the stress offered by the section, the moment of resistance $= M = fZ$

$= f \left[\frac{\pi d^3}{32}\right]$

(iv) Hollow Circular Section
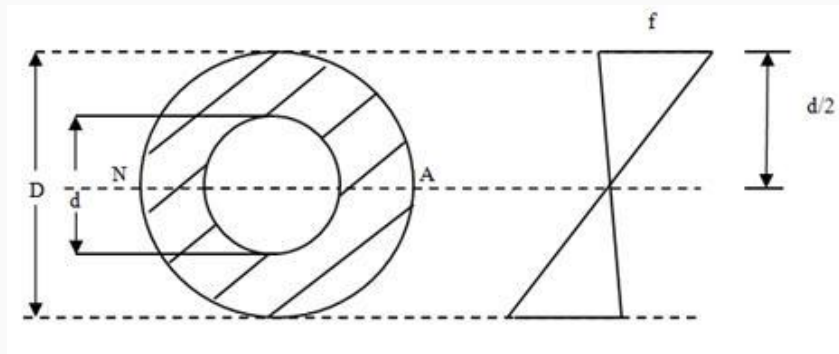
Fig.24.5

Fig.24.5 shows a hollow circular section of external diameter D and internal diameter d.

Moment of inertia about the neutral axis $= I = \left[\frac{\pi}{64}\left[D^4 - d^4 \right]\right]$

$y_{max} = \left[\frac{D}{2}\right]$

Therefore, section modulus $= Z = \left[\frac{I}{y_{max}}\right] = \left[\frac{\pi \left(D^4 - d^4 \right)}{32D}\right]$

If f be the maximum stress offered by the section, the moment of resistance $= M = fZ$

$M = f \left[\frac{\pi \left(D^4 - d^4 \right)}{32D}\right]$

Example 24.1: Fig.24.6 shows the section of a tube of aluminium alloy. Determine the maximum moment that can be applied to the tube if the permissible bending stress is 100 N/mm². Find also the radius of curvature of the tube as it bends. Take E = 72800 N/mm².
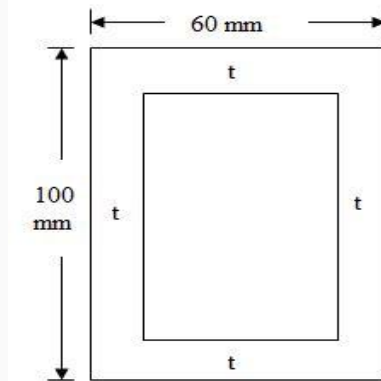
Fig.24.6

Solution: Moment of Inertia of the section,

$$I = \left[\frac{60 \times 100^3}{12} - \frac{50 \times 90^3}{12} = 1962500\right]$$

= 1962500 mm⁴

Maximum moment on the section, $M = \left[\frac{f}{y_{max}}.I = \frac{100 \times 1962500}{50}\right]$ Nmm

= 3925000 Nmm

Radius of Curvature, $R = \left[\frac{EI}{M} = \frac{72800 \times 1962500}{3925000}\right]$ mm

= 36400 mm

Example 24.2: A cast iron test beam 25mm × 25mm in section and 2 m long and supported at the ends fails when a central load of 700N is applied. What uniformly distributed load will break a cantilever of the same material 55mm wide, 110mm deep and 3m long?

Solution: Let us first consider the test beam.

Maximum bending moment = $M = \left[\frac{WL}{4} = \frac{700 \times 2}{4} \times 1000 Nmm\right]$ = 35 × 10⁴ Nmm

Moment of Resistance, $R = \left[\frac{1}{6}.fb{d^2}\right] = \left[\frac{1}{6}.f \times 25 \times 25^2\right] = \left[\frac{15625}{6}f = 2604.17f Nmm\right]$
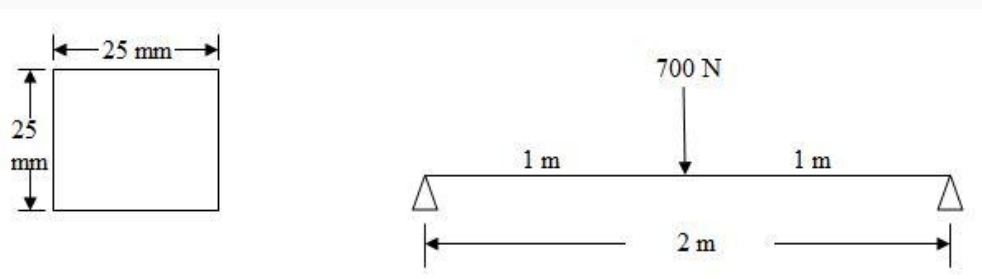
Fig.24.7

Engineering Mechanics

Equating the moment of resistance to the maximum bending moment

$$2604.17f = 35 \times 10^4$$

$f = {35 \times 10^4 \over 2604.17} = 134.39$ N/mm²

Now let us consider the cantilever.

Let the distributed load on the cantilever be w N/m run so as to break it.

Therefore, maximum bending moment = M = ${w l^2 \over 2} = {w \times 3^2 \over 2} \times 1000 \,\text{Nmm}$ = 4500w N mm

Moment of Resistance of the section = ${1 \over 6}.fb d^2 = {1 \over 6} \times 134.39 \times 55 \times 110^2 \,\text{Nmm}$

= 14.91 × 10⁶ N mm
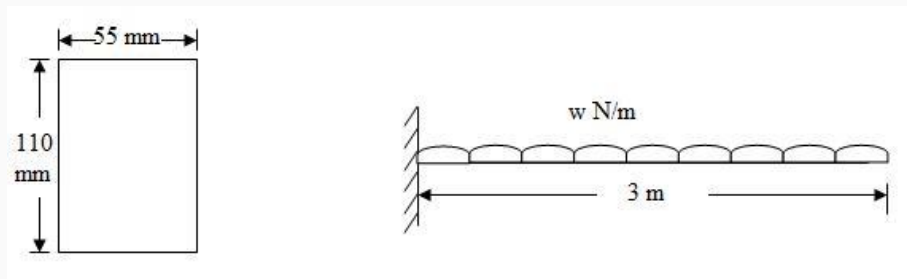
Fig.24.8

Equating the maximum bending moment to moment of resistance we have,

4500 w =14.91 × 10⁶

w= 3313 N/m

Example 24.3: A machine component of semi circular section 300mm diameter acts as a beam of span 1.50m. It is placed with its base horizontal. If it carries a uniformly distributed load of 150kN/m run on the whole span, find the maximum stress induced.

Solution: Moment of inertia of the section about the neutral axis = I = 0.00686 d⁴

Extreme fibre distance from the neutral axis = r - ${4r \over 3\pi} = {3\pi - 4 \over 4\pi} \times r = 0.5756$

r = 0.2878 d

Section modulus, Z = ${0.00686 d^4 \over 0.2878d} = 0.0238 d^3$

Maximum bending moment, M = ${300 \times 1.50^2 \over 8} = 84.375 \,\text{kNm}$

Maximum bending stress $f_{max} = {M \over Z} = {84.375 \times 10^6 \over 0.0238 \times 300^3} = 131.30$ *N/mm²*

Engineering Mechanics

Example 24.4: Find the safe concentrated load that can be applied at the free end of a 2m long cantilever. The section of the cantilever is a hollow square of external side 50mm and internal side 40mm the safe bending stress for the material being 65 N/mm².

Solution: Moment of inertia of the section of the cantilever,

$I = $ $$\frac{{50}^4}{12} - \frac{{40}^4}{12} = 307500$$ mm⁴

Section modulus, $Z = $ $$\frac{I}{y_{max}} = \frac{307500}{30}$$ = 10250 mm³

Let the safe concentrated load at the free end be W Newton

Maximum B.M, $M = W \times 2 \times 1000 = 2000\ W$ *Nmm*

$M = fZ$, $2000\ W = 65 \times 10250$

$W = $ $$\frac{65 \times 10250}{2000} =$$ 333.125 *N*

Example 24.5: The moment of inertia of a beam section 450mm deep is 60.5 × 10⁷ mm⁴. Find the span over which a beam of this section, when simply supported, could carry a uniformly distributed load of 45kN per meter run. The flange stress in the material is not to exceed 110 N/mm².

Solution: Section modulus of section = $Z = $ $$\frac{I}{y_{max}} = \frac{60.5 \times {10}^7}{250}$$

Therefore, $Z = 24.2 \times 10^5$ mm³

Let the maximum span be *l meter*.

Therefore, maximum bending moment = $M = $ $$\frac{w l^2}{8} = \frac{45000 \times l^2}{8} \times 1000 Nmm$$

$= 56.25 \times 10^5\ l^2$ *N mm*

Moment of resistance of the section corresponding to the maximum bending stress of 110 *N/mm²*

$= fZ = 110 \times 24.2 \times 10^5$ *N mm*

Equating the maximum bending moment to the moment of resistance, we get

$56.25 \times 10^5\ l^2 = 110 \times 24.2 \times 10^5$

$l^2 = $ $$\frac{110 \times 24.2 \times {10}^5}{56.25 \times {10}^5} = 47.32$$

Therefore, $l = 6.88$ *m, say 7 m*

LESSON 25.

25.1 SHEAR STRESS DISTRIBUTION FOR BEAM SECTIONS OF VARIOUS SHAPES

Fig……… shows a rectangular section of width *b* and depth *d*. Let the section be subjected to shear force *V*.

Consider a section EF at a distance *y* from the neutral axis.

The intensity of shear stress at this section is given by

$$\tau = {Va\bar y \over Ib}$$

V = Shear Force

y = distance from Neutral axis

b = width of the beam

d = Depth of the beam

I = Moment of Inertia of the beam section about the neutral axis

a $\bar y$ Moment of the area about the neutral axis

$$\tau_{max} = {3V \over 2bd}$$

Example: A timber beam is simply supported at the ends and carries a concentrated load at mid span. The maximum longitudinal stress is *σ* and the maximum shearing stress is *τ*. Find the ratio of the span to the depth of the beam ignoring the self weight of the beam.

If *σ* = 10 N/mm² and *τ* = 1 N/mm²

Solution:

Maximum Shear Force, V = $\dfrac{W}{2}$

Maximum shear stress, *τ*= ${Va\bar y \over Ib}$

τ_{max} = ${3V \over 2bd}$

τ_{max} = ${3W \over 4bd}$ --- -(i)

Maximum Bending Moment = ${WL \over 4}$

Maximum Bending Stress = ${M \over Z}$

$$\frac{WL}{4} = \sigma \times \frac{b\,d^2}{6}$$

$$\sigma = \frac{3WL}{b\,d^2}$$ ---(ii)

Ratio of span to depth = $\frac{L}{D}$

Dividing equation (ii) by (i)

$$\frac{\sigma}{\tau} = \frac{\frac{3WL}{b\,d^2}}{\frac{3W}{4bd}}$$

$$\frac{L}{d} = \frac{\sigma}{2\tau}$$

$= \frac{10}{2 \times 1} = 5$ ($\sigma = 10\,N/mm^2$ and $\tau = 1\,N/mm^2$)

$$\frac{L}{d} = 5$$

Example: A beam of I-Section 550mm deep and 200mm wide has flanges 30mm thick. It carries a shear force of 450 kN at a section. Calculate the maximum intensity of shear stress in the section assuming the moment of inertia to be $6.45 \times 10^8\,mm^4$. Also calculate the total shear force carried by the web and sketch the shear stress distribution across the section.

 Solution: Maximum shear stress occurs at the web (N.A)

$$\tau_{max} = \frac{Va\bar{y}}{Ib}$$

V = 450 kN

$a\,\bar{y}$ Moment above and about the neutral axis

I = $6.45 \times 10^8\,mm^4$

b = breadth at the level of τ_{max} = 20mm

$a\,\bar{y} = a_1\overline{y_1} + a_2\overline{y_2}$

= (200 × 30) × 260 + (20 × 245) × 122.5

= $2.16 \times 10^6\,mm^3$

$$\tau_{max} = \frac{450 \times 10^3 \times 2.16 \times 10^6}{6.45 \times 10^8 \times 20} = 75.35\,N/mm^2$$

For τ_2 , at the bottom and top flange

$a\,\bar{y}$ (200 × 30) × 260 = 1560000 mm³

b = 200mm

$$\tau_1 = \frac{Va\bar{y}}{Ib} = \frac{450 \times 10^3 \times \left(200 \times 30 \right) \times 260}{6.45 \times 10^8 \times 200} = 5.442\,N/mm^2$$

Engineering Mechanics

For τ_2 , $a\,\bar y = 1560000$ mm³

(area is always taken above the point of consideration)

$$\tau_2 = {450 \times {10}^3 \times 1560000 \over 6.45 \times {10}^{8} \times 20}$$
= 54.42 N/mm²

Example: The T-beam section is subjected to vertical shear force of 150 kN. Calculate the shear stress at the neutral axis and at the junction of the web and the flange. Moment of inertia about the horizontal neutral axis is 1.134×10^8 mm⁴.

Solution: (from top flange) = $${a_1\overline{y_1}+a_2\overline{y_2} \over a_1 + a_2}$$

= $${\left(60 \times 250 \right) \times 30+\left(250 \times 60 \right) \times 185 \over \left(250 \times 60 \right)+\left(250 \times 60 \right)}$$ = 107.5 mm

$$\tau_{max}$$ = $${Va\bar y \over Ib}$$

= $${150 \times 1000 \times \left(250 \times 60 \right) \times \left(107.5 - 30 \right) + \left(47.5 \times 60 \right)(47.5/2) \over 1.134 \times {10}^8 \times 60}$$ = 25.77 N/mm²

$$\tau_1$$ = $${\left(150 \times 1000 \right) \times \left(250 \times 60 \right)\left(107.5 - 30 \right) \over 1.134 \times {10}^8 \times 250}$$ = 6.15 N/mm²

$$\tau_2$$ = $${\left(150 \times 1000 \right) \times \left(250 \times 60 \right)\left(107.5 - 30 \right) \over 1.134 \times {10}^8 \times 60}$$ = 25.63 N/mm²

LESSON 26.

26.1 TORSION

Torsion refers to twisting of a straight member under the action of a turning moment or torque which tends to produce a rotation about the longitudinal axis. Some examples of torsion are steering rod, propeller shafts etc.

- A torque is composed of a couple acting in a plane perpendicular to the longitudinal direction of the shaft.

26.2 THEORY OF PURE TORSION

Fig.26.1 shows a solid cylindrical shaft of radius R and length l subjected to a couple or twisting moment T at one end, while its other end is held or fixed by the application of a balancing couple of the same magnitude.

Let LM be a line on the surface of the shaft and parallel to the axis of the shaft before the deformation of the shaft. As an effect of torsion this line, after the deformation of the shaft, takes the form LN.
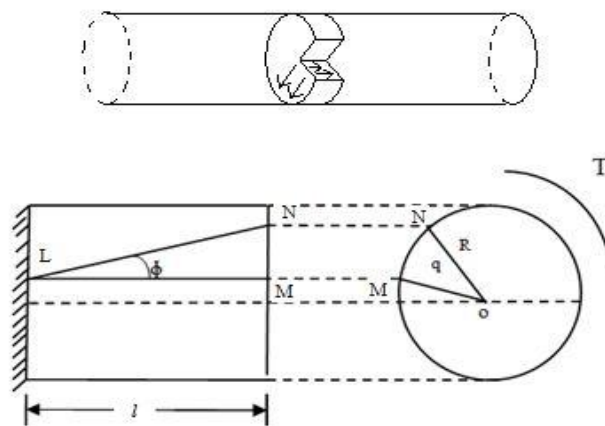
Fig.26.1

The angle $NLM = \phi$ represents the shear strain of the shaft material at the surface. This angle being small, we have ,

$MN = l\,\phi$,   Therefore $\phi = \dfrac{MN}{l}$ -------------------------------(i)

Let the angle MON be the angular movement of the radius OM due to the strain in the length of the shaft. Let $MON = \theta$. Let f_s be the shear stress intensity at the surface of the shaft.

We know, $f_s = \phi\, C$ --------------------------------------(ii)

Where C = Modulus of rigidity of the shaft material

$f_s = \left({{MN} \over l} \right) C$

But $MN = R\theta$, Therefore $f_s = {{R\theta} \over l} . C$

So, $\left[{{f_s} \over R} \right] = \left[{{C\theta} \over l} \right]$ --------------------------------------(iii)

Since C, θ and l are constants, it follows that at any section of the shaft, the shear stress intensity at any point is proportional to the distance of the point from the axis of the shaft. Hence the shear stress is maximum at the surface and shear stress is zero at the axis of the solid shaft. Fig.26.2 shows the shear stress distribution for a solid shaft and a hollow shaft.

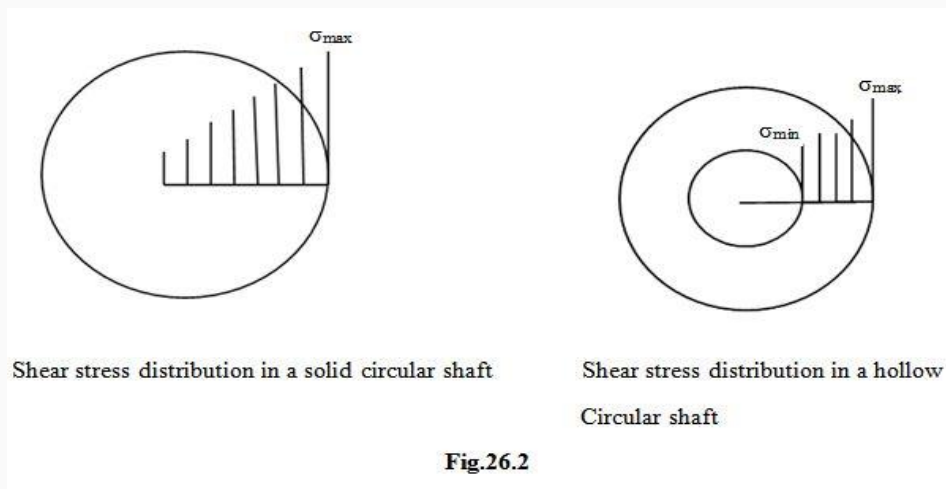Shear stress distribution in a solid circular shaft

Shear stress distribution in a hollow

Circular shaft

Fig.26.2

Longitudinal Shear stresses

It is also important to realize that the shear stresses acting on the cross-sectional planes are accompanied by shear stresses of the same magnitude in the longitudinal planes following the principle of complementary shear stresses. See fig.--------
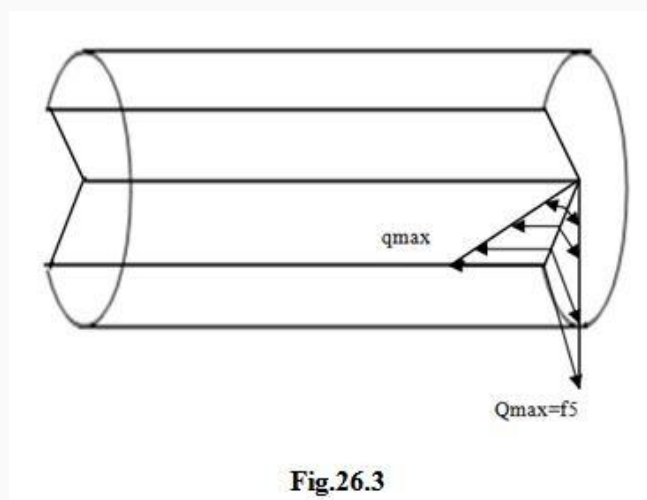
Fig.26.3

Suppose the material is weaker in longitudinal shear than on the cross-sectional planes, the first formation of cracks will appear in the longitudinal direction on the surface.

26.3 TORSIONAL MOMENT OF RESISTANCE

Fig.26.4 shows the section of the shaft of radius R subjected to pure tension. Let f_s be the maximum shear stress which occurs at the surface.

Consider an element area da at a distance r from the axis of the shaft.

Shear stress offered by the elemental area = q = $\left[\frac{r}{R} \right]$ f_s

Therefore, Shear resistance offered by the elemental area $q. \, da$ = $\left[\frac{r}{R} \right]$ $f_s. \, da$

Moment of resistance offered by the elemental area = $\left[\frac{r}{R} \right]$ $f_s. \, da \, . \, r$ = $\left[\frac{f_s}{R} da.r^2 \right]$

Therefore, total moment of resistance offered by the cross-section of the shaft = T

= $\left[\frac{f_s}{R} \right]$ $\sum da. \, r^2$

But $\sum da. \, r^2$ represents the moment of inertia of the shaft section about the axis of the shaft, i.e, the quantity $\sum da. \, r^2$ is the polar moment of inertia I_p of the section of the shaft.

T = $\left[\frac{f_s}{R} \right] . \, I_p$, $\left[\frac{T}{I_p} \right] = \left[\frac{f_s}{R} \right]$ ------------------------------------(iv)

But from eqn. (iii), $\left[\frac{f_s}{R} \right]$ = $\left[\frac{C\theta}{l} \right]$, Therefore $\left[\frac{T}{I_p} \right] = \left[\frac{f_s}{R} \right]$ = $\left[\frac{C\theta}{l} \right]$

Polar Moment of inertia of a circular shaft

Solid shaft : Radius R, Diameter D

I_p = $\left[\frac{\pi R^4}{2} \right]$ = $\left[\frac{\pi D^4}{32} \right]$

Hollow shaft : Outer radius R_1, Inner radius R_2

Outer diameter D_1, Inner diameter D_2

I_p = $\left[\frac{\pi}{2} \left[R_1^4 - R_2^4 \right] \right]$ = $\left[\frac{\pi}{32} \left[D_1^4 - D_2^4 \right] \right]$

Alternatively, if R_m = Mean radius and D_m = Mean diameter and t = wall thickness

I_p = $\left[\frac{\pi R_m t}{2} \left[4R_m^2 + t^2 \right] \right]$ = $\left[\frac{\pi D_m t}{4} \left[D_m^2 + t^2 \right] \right]$

For the case when t is very small

I_p = 2Π $\left[R_m^3 t \right]$ = $\left[\frac{\pi D_3^3 t}{4} \right]$

Assumptions in the Theory of Pure Torsion

(i) The material of the shaft is uniform throughout.

(ii) The twist along the shaft is uniform.

(iii) The shaft is of uniform circular section throughout.

(iv) Cross-section of the shaft, which are plane before twist remain plane after twist.

(v) All radii which are straight before twist remain straight after twist.

LESSON 27.

27.1 POLAR MODULUS

Let T be the torsional moment of resistance of the section of a shaft of radius R and Ip the polar moment of inertia of the shaft section.

The shear stress intensity q at any point on the section distant r from the axis of the shaft is given by

$q = \[{T \over {{I_{p}}}}.r\]$

The maximum shear stress fs occurs at the greatest radius R

Therefore $f_s = \[{T \over {{I_{p}}}}\] .R$

$T = f_s. \[{{{I\_p}} \over R}\]$ or $T = f_s. Zp$

Where $Z_p = \[{{{I\_p}} \over R}\]$ $= \[{{Polarmomentofinertiaoftheshaftsection} \over {Maximumradius}}\]$

This ratio is called the polar modulus of the shaft section. The greatest twisting moment which a given shaft section can resist

 = maximum permissible shear stress × polar modulus

Hence for a shaft of a given material the magnitude of the polar modulus is a measure of its strength in resisting torsion.

Given a number of shafts of the same length and material, the shaft which can resist the greatest twisting moment is the one whose polar modulus is greater.

ü Shafts of the same material and length having the same polar modulus have the same strength.

For a solid shaft of diameter D,

$I_p = \[{\pi{{D^4}} \over {32}}\]$, $R = \[{D \over 2}\]$

$Z_p = \[{\pi{{D^3}} \over {16}}\]$

For a hollow shaft whose outer and inner diameters are D_1 and D_2.

$I_p = \[{\pi \over {32}}\left[{{D_1}^4 - {D_2}^4} \right]\]$

$R_1 = \[{{{D_1}} \over 2}\]$, Therefore $Z_p = \[{\pi \over {16{D_1}}}\left[{{D_1}^4 - {D_2}^4} \right]\]$

Engineering Mechanics

Torsional moment of resistance = $f_s Z_p = f_s$. $$\frac{\pi}{16 D_1}\left[D_1^4 - D_2^4 \right]$$

1. Hollow shafts are more efficient than solid shafts in resisting torsional moment.

2. In a solid shaft the shear stresses are maximum at the outer boundary of the section and zero at the centre. Hence considerable material of the solid shaft is subjected to shear stresses much below the maximum shear stress. This means the shera resisting capacity of the material of the shaft is not utilized.

3. On the other hand in hollow shaft material being present more away from the centre of the section, the shear resistance of the material is higher.

4. Using a hollow shaft results in not only saving of material but also in reduction of weight.

Torsional Rigidity

Let a twisting moment T produce a twist of Θ radians in a length l

We know the relation, $$\frac{T}{I_p} = \frac{C\theta}{l}$$

Therefore Θ = $$\frac{Tl}{C I_p}$$

- For a given shaft the twist is therefore proportional to the twisting moment T.

- In a beam a bending moment produces a bend or deflection, in the same manner, a torque produces a twist in a shaft.

- The quantity CI_p is called Torsional rigidity.

- CI_p stands for the torque required to produce a twist of 1 radian per unit length of the shaft.

- The quantity $$\frac{C I_p}{l}$$ is called torsional stiffness. It is the torque required to produce a twist of 1 radian over the length of the shaft.

- Torsional flexibility is the reciprocal of the torsional stiffness and is equal to $$\frac{l}{C I_p}$$ and is the angle of rotation produced by a unit torque.

Limitations

- The theory presented above is valid for linearly elastic bars of circular sections (solid or hollow).

- The stresses determined from the formulae derived above are acceptable in regions far away from regions of stress concentrations (like holes and abrupt changes in diameter) and also far away from sections subjected to concentrated torques.

- However the angle of twist is not affected by stress concentrations and the formulae can be applied for determining the twist.

- It should also be noted that we derived the torsion formula assuming prismatic circular bars. The formula may however be used, when the changes in diameter are small and gradual.

27.2 POWER TRANSMITTED BY A SHAFT

Let a shaft turning at *N rpm* transmit *P kilo watts*. Let the mean torque to which the shaft is subjected to be *T Nm*.

Therefore, Power Transmitted = *P* = Mean torque × Angle turned per second

$$= T \left[\frac{N}{60}\right] 2\Pi \text{ Watts} = T \left[\frac{N}{60}\right] . \left[\frac{\pi 2}{1000}\right] \text{ Kilowatts}$$

Therefore, $P = \frac{2\pi NT}{6000}$ *kW*

Sometimes angular speed is expressed as the frequency f of rotation which means the number of revolutions per unit of time. The unit of frequency is *Hertz (Hz)*.

1 *Hz* = 1 revolution per second

Power transmitted = $P = \frac{2\pi fT}{1000}$

MODULE 8.

LESSON 28.

28.1 INRODUCTION

28.2 ELEMENT SUBJECTED TO DIRECT STRESS IN TWO PERPENDICULAR DIRECTIONS

At any point in a material where stress is acting (see fig.-----), it is possible to assume that the point consists of a very small triangular block, such that the stress act across the faces of the block. Consider that direct stresses σ_x and σ_y act across the faces *LM* and *MN* and that the block has unit depth perpendicular to *LMN*. Let the stresses τ and σ_n act on the same plane at an angle Θ to *LM*.

Fig.--------------

Resolving normal to *LN*:

$\sigma_n \times LN = \sigma_x \times LM \cos\Theta + \sigma_y\, MN \sin\Theta$

$\sigma_n = \sigma_x \times$ \[{{LM} \over {LN}}\] $\cos\Theta + \sigma_y$ \[{{MN} \over {LN}}\] $\sin\Theta$

$= \sigma_x \cos^2\Theta + \sigma_y \sin^2\Theta = $ \[{{{\sigma _x}} \over 2}\] $\times 2\cos^2\Theta + $ \[{{{\sigma _x}} \over 2}\] $\times 2\sin^2\Theta$

$= $ \[{{{\sigma _x}} \over 2}\] $(1- \sin^2\Theta + \cos^2\Theta) + $ \[{{{\sigma _y}} \over 2}\] $(1- \cos^2\Theta + \sin^2\Theta)$

$= $ \[{{{\sigma _x} + {\sigma _y}} \over 2}\] $+ \sigma_x$ \[\left[{{{co{s^2}\theta-si{n^2}\theta } \over 2}} \right]\] $-$ σ_y \[\left[{{{co{s^2}\theta-si{n^2}\theta } \over 2}} \right]\]

$= $ \[{{{\sigma _x} + {\sigma _y}} \over 2}\] $+ $ \[{{{\sigma _x} - {\sigma _y}} \over 2}\] $\cos2\Theta$----------------------------------(i)

When $\Theta = 0$, then $\sigma_n = $ \[{{{\sigma _x} + {\sigma _y}} \over 2}\] $+ $ \[{{{\sigma _x} - {\sigma _y}} \over 2}\] $= \sigma_x$ ----------------------------------(ii)

and, when $\Theta = \Pi/2$, then $\sigma_n = $ - $= \sigma_y$ ----------------------------(iii)

Resolving parallel to *LN*:

$\tau \times LN = \sigma_x \times LM \sin\Theta - \sigma_y \times MN \cos\Theta$

$\tau = \sigma_x \times$ \[{{LM} \over {LN}}\] $\sin\Theta - \sigma_y \times$ \[{{MN} \over {LN}}\] $\cos\Theta$

$= \sigma_x \cos\Theta \sin\Theta - \sigma_y \sin\Theta \cos\Theta = (\sigma_x - \sigma_y) \sin\Theta \cos\Theta$

$= $ \[{{{\sigma _x} - {\sigma _y}} \over 2}\] $\sin2\Theta$----------------------------(iv)

Engineering Mechanics

The maximum value of τ occurs when $2\theta = \Pi/2$ or $\theta = \Pi/4$ and then

$$\tau_{max} = \frac{\sigma_x - \sigma_y}{2}$$ --------------------------------- (v)

The resultant stress,

$\sigma_r = \sqrt{\sigma_n^2 + \tau^2}$

$\tan\phi = \frac{\tau}{\sigma_n}$ where ϕ is the angle which the resultant stress makes with the normal to the plane is called **obliquity**.

Example: The principal stresses at a point across two perpendicular planes are 100 MN/m² (tensile) and 60 MN/m²(tensile).Find the normal, tangential stresses and the resultant stress and its obliquity on a plane at 30⁰ with the major principal plane.

Solution: Given: σ_x = 100 MN/m² (tensile);

σ_y = 60 MN/m² (tensile); θ = 30⁰

normal stress, $\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$

$$= \frac{100 + 60}{2} + \frac{100 - 60}{2} \cos (2\times30^0)$$

$$= 80 + 20 \cos 60^0$$

$$= 90 \text{ MN/m}^2 \text{ (tensile)}$$

Tangential stress, $\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta = \frac{100 - 60}{2} \sin (2\times30^0)$

$$= 20 \sin 60^0$$

$$= 17.32 \text{ MN/m}^2$$

Hence, τ = 17.32 MN/m²

Resultant stress, $\sigma_r = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{{90}^2 + {17.32}^2}$ = 91.65 MN/m²

Hence, σ_r = 91.65 MN/m²

Obliquity ϕ: $\tan\phi = \frac{\tau}{\sigma_n} = \frac{17.32}{90}$ = 0.1924
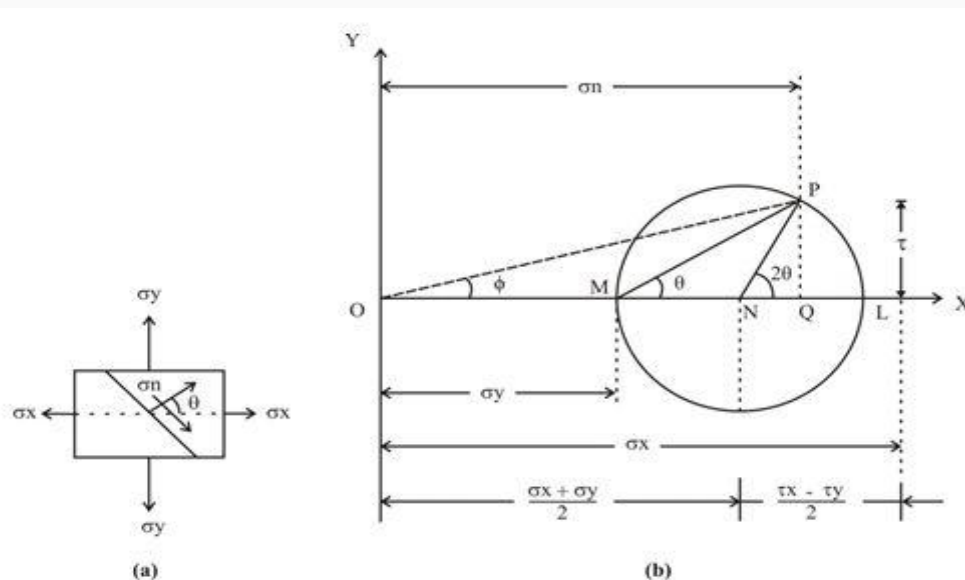
          Φ = 10.89⁰

LESSON 29.

29.1 MOHR'S CIRCLE

A German scientist Otto Mohr devised a graphical method for finding the normal and shear stresses on any interface of an element when it is subjected to two perpendicular stresses. This method is explained as follows:

29.1.1 Mohr's circle construction for 'like stresses'

Steps of construction:

MOHR'S CIRCLE CONSTRUCTION FOR LIKE STRESSES

Fig.29.1

Using some suitable scales, measure OL and OM equal to σ_x and σ_y respectively on the axis OX.

1. Bisect LM at N.

2. With N as centre and NL or NM radius, draw a circle.

3. At the centre N draw a line NP at an angle 2Θ, in the same direction as the normal to the plane makes with the direction of σ_x. In Fig.1 which represents the stress system, the normal to the plane makes an angle Θ with the direction of σ_x in the anticlockwise direction. The line NP therefore, is drawn in the anticlockwise direction.

4. From P, draw a perpendicular PQ on the axis OX. PQ will represent τ and OQ σ_n.

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Now, from stress diagram

$NP = NL = \dfrac{\sigma_x - \sigma_y}{2}$

$PQ = NP \sin2\Theta = \dfrac{\sigma_x - \sigma_y}{2} \sin2\Theta = \tau$ ------------------------------------(8.1)

Similarly, $OQ = ON + NQ = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2} \cos2\Theta = \sigma_n$--------------(8.2)

Also, from stress circle, τ is maximum when

$2\Theta = 90°$, or $\Theta = 45°$

And, $\tau_{max} = \dfrac{\sigma_x - \sigma_y}{2}$ ---(8.3)

Sign conventions used:

(i) In order to mark τ in stress system, we will take the clockwise shear as positive and anticlockwise shear as negative.

(ii) Positive value of τ will be above the axis and negative values below the axis.

(iii) If Θ is in the anticlockwise direction, the radius vector will be above the axis and Θ will be reckoned positive. If Θ is in the clockwise direction, it will be negative and the radius vector will be below the axis.

(iv) Tensile stress will be reckoned positive and will be plotted to the right of the origin O. Compressive stress will be reckoned negative and will be plotted to the left of the origin O.

29.1.2 Mohr's circle construction for 'Unlike stresses'

In case σ_x and σ_y are not like, the same procedure will be followed except that σ_x and σ_y will be measured to the opposite sides of the origin. The construction is given in Fig.2.It may be noted that the direction of σ_n will depend upon its position with respect to the point O. If it is to the right of O, the direction of σ_n will be the same as that of σ_x.
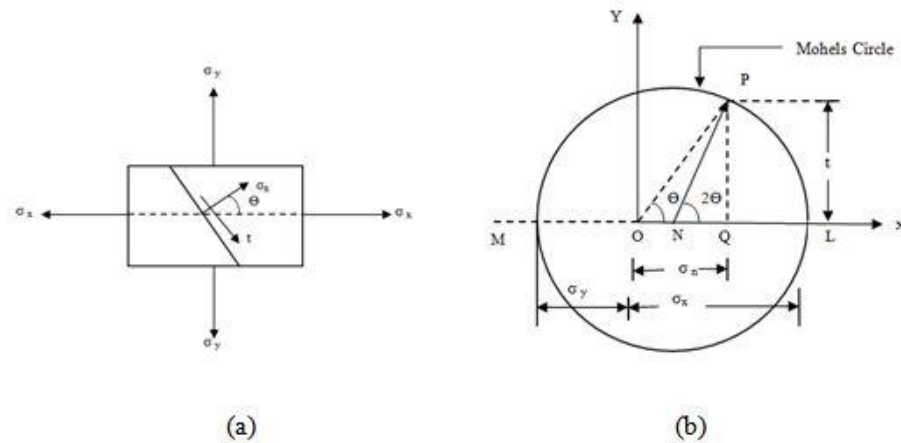
(a)   (b)

Fig.29.2

29.1.3 Mohr's circle construction for two perpendicular direct stresses with state of simple shear

Refer to Fig.3. Following steps of construction are followed if the material is subjected to direct stresses σ_x and σ_y along with a state of simple shear.
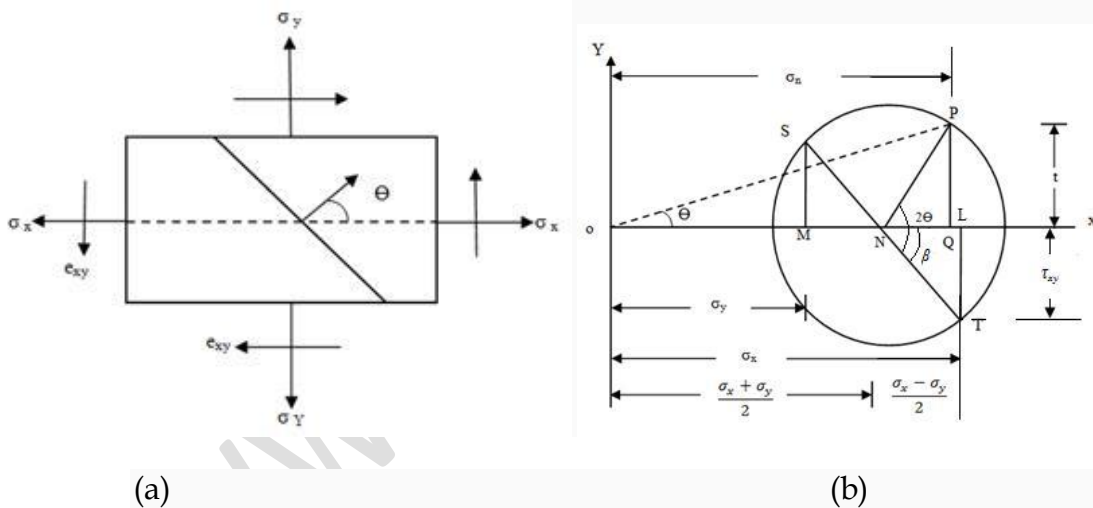

(a)   (b)

Fig.29.3

1. Using some suitable scale, measure $OL = \sigma_x$ and $OM = \sigma_y$ along the axis OX.

2. At L draw LT perpendicular to OX and equal to τ_{xy}. LT has been drawn downward (as per sign conventions adopted) because τ_{xy} is acting up with respect to the plane across which σ_x is acting, tending to rotate it in the anticlockwise direction and is negative.

3. Similarly, make MS perpendicular to OX and equal to τ_{xy}, but above OX.

4. Join ST to cut the axis in N.

5. With N as centre and NS or NT as radius, draw a circle.

6. At N make NP at angle 2θ with NT in the anticlockwise direction.

7. Draw PQ perpendicular to the axis. PQ will give τ while OQ will give σ_n and OP will give σ_r.

Proof: Let the radius of the stress circle be R.

Then, $\quad R = \sqrt{NL^2 + LT^2} = \sqrt{{\left[{\sigma_x - \sigma_y \over 2} \right]}^2 + {\tau_{xy}}^2}$

Also, $\quad R\cos\beta = NL = {\sigma_{x - \sigma_y} \over 2}$

$\quad\quad R\sin\beta = LT = \tau_{xy}$

Now, $\quad OQ = ON + NQ = ON + R\cos(2\theta - \beta)$

$\quad\quad = ON + R\cos 2\theta \cos\beta + R\sin 2\theta \sin\beta$

$\quad\quad = {\sigma_{x + \sigma_y} \over 2} + {\sigma_{x - \sigma_y} \over 2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad$ (as per eqn-------)
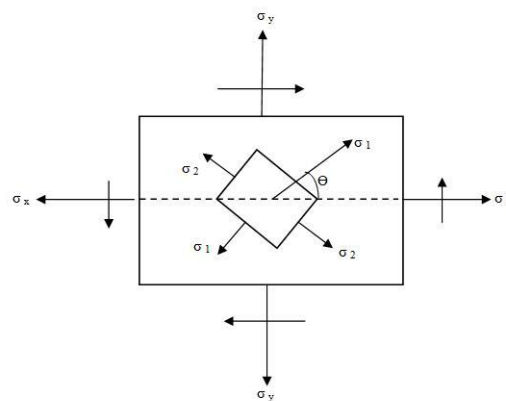
$\quad\quad = \sigma_n$

Similarly,

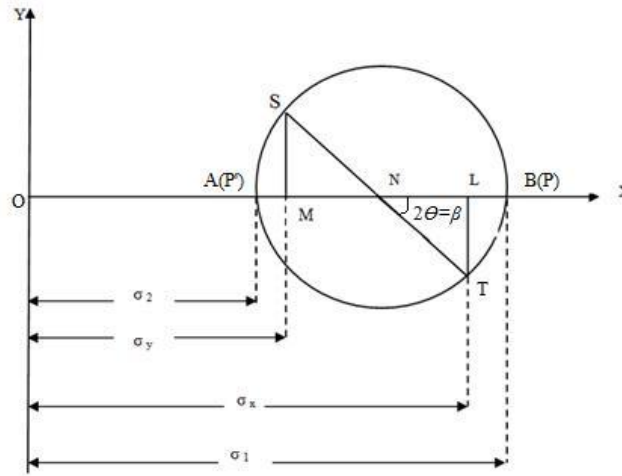$PQ = R\sin(2\theta - \beta) = R\sin 2\theta \cos\beta - R\cos 2\theta \sin\beta$

$\quad = {\sigma_{x - \sigma_y} \over 2}\sin 2\theta - \tau_{xy}\cos 2\theta \quad\quad\quad\quad\quad\quad\quad$ (as per eqn-------)

29.1.4 Mohr's circle construction for principal stresses

Refer to Fig.4.The following are the steps of construction:

(a)

(b)

Fig. 29.4

1. Mark *OL* and *OM* proportional to σ_x and σ_y.

2. At *L* and *M*, erect perpendiculars *LT* = *MS* proportional to in appropriate directions.

3. Join *ST*, intersecting the axis in *N*.

Since τ = o, *NV* represents the major principal plane, *P* coinciding with *B*. Similarly *NP'* represents minor principal plane, *P'* coinciding with *A*.

OV = *ON* + *NV* = $\left[{{{\sigma _{x + {\sigma _y}}}} \over 2} \right]$ + *R*,   where *R* is the radius of the circle.

= $\left[{{{\sigma _{x - {\sigma _y}}}} \over 2} \right]$ + $\sqrt {{{\left[{{{{\sigma _x} - {\sigma _y}} \over 2}} \right]}^2} + {\tau _{xy}}^2}$ = σ_1

Similarly,   *OU* = *ON* – *NU*

= $\left[{{{\sigma _{x + {\sigma _y}}}} \over 2} \right]$ - $\sqrt {{{\left[{{{{\sigma _x} - {\sigma _y}} \over 2}} \right]}^2} + {\tau _{xy}}^2}$ = σ_2

$\tan \beta$ = $\frac{{LT}}{{LN}}$ = $\frac{{\tau xy}}{{\frac{{{\sigma _x} - {\sigma _y}}}{2}}}$

= $\left[{{2\tau xy} \over {{\sigma _x} - {\sigma _y}}} \right]$ = $\tan 2\theta$ (where $\beta = 2\theta$)

LESSON 30.

Example: Draw the mohr's stress circle for direct stresses of 70 MN/m² (tensile) and 40 MN/m² (compressive) and estimate the magnitude and direction of the resultant stresses and planes making angles of 30⁰ and 70⁰ with the plane of the first principal stress. Find also the normal and tangential stresses on these planes.

Solution:
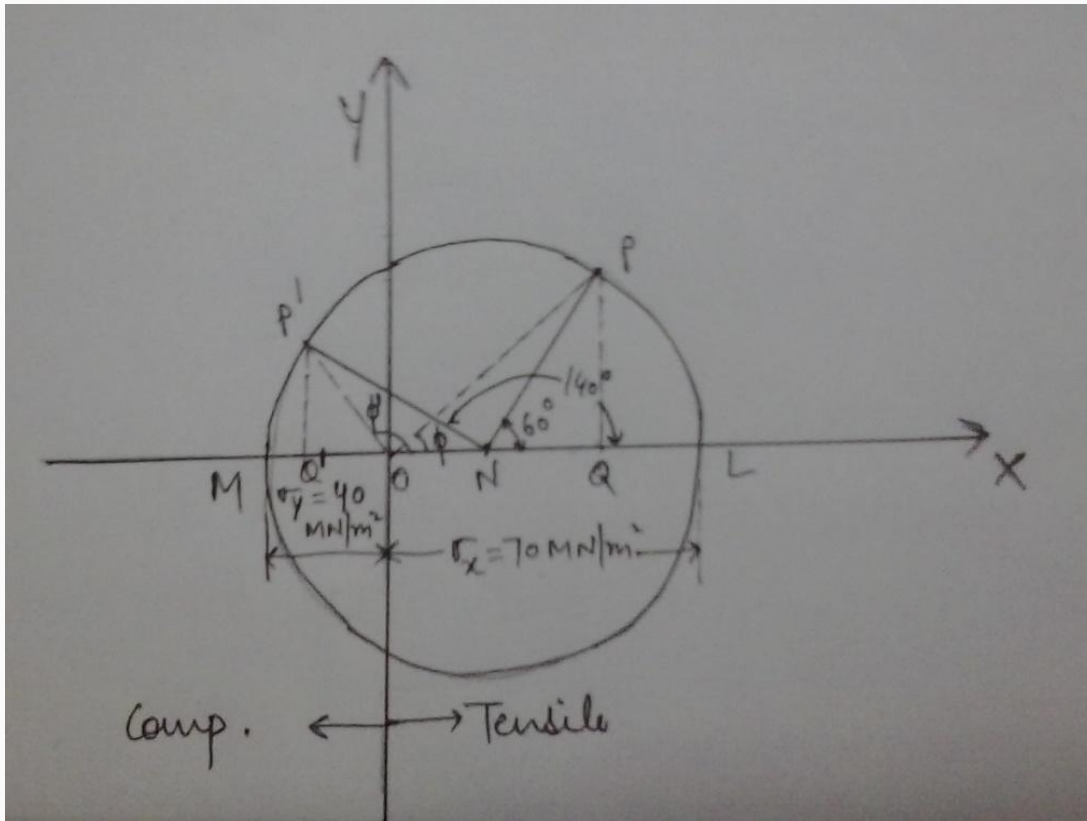
- Plot OL = 70 MN/m² (tensile) and OM = 40 MN/m² (compressive)

- Draw NP at 60⁰ and NP' at 140⁰

Case 1 :

- σ_n = OP = 42.5 MN/m² (tensile)

- τ = PQ = 47.63 MN/m² (shear)

- σ_r = 63.83 MN/m²

- φ = 48.26⁰

   Case 2 :

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- σ_n = OP' = 27.13 MN/m^2 (compressive)

- τ = P'Q' = 35.35 MN/m^2 (shear)

- σ_r = 44.56 MN/m^2

- φ = 127.5^0

LESSON 31.

Example: At a point in a bracket the stresses on two mutually perpendicular planes are 45 MN/m² (tensile) and 25 MN/m² (tensile). The shear stress across these planes is 12 MN/m². Find the magnitude and direction of the resultant stress on a plane making an angle of 45⁰ with the plane of first stress. Find also the normal and tangential stresses on the planes.
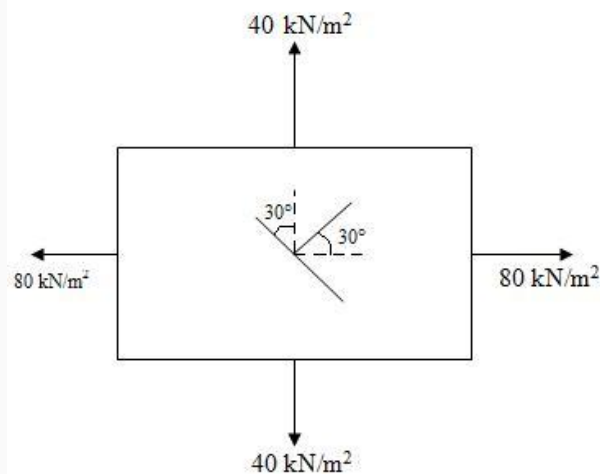
Solution:
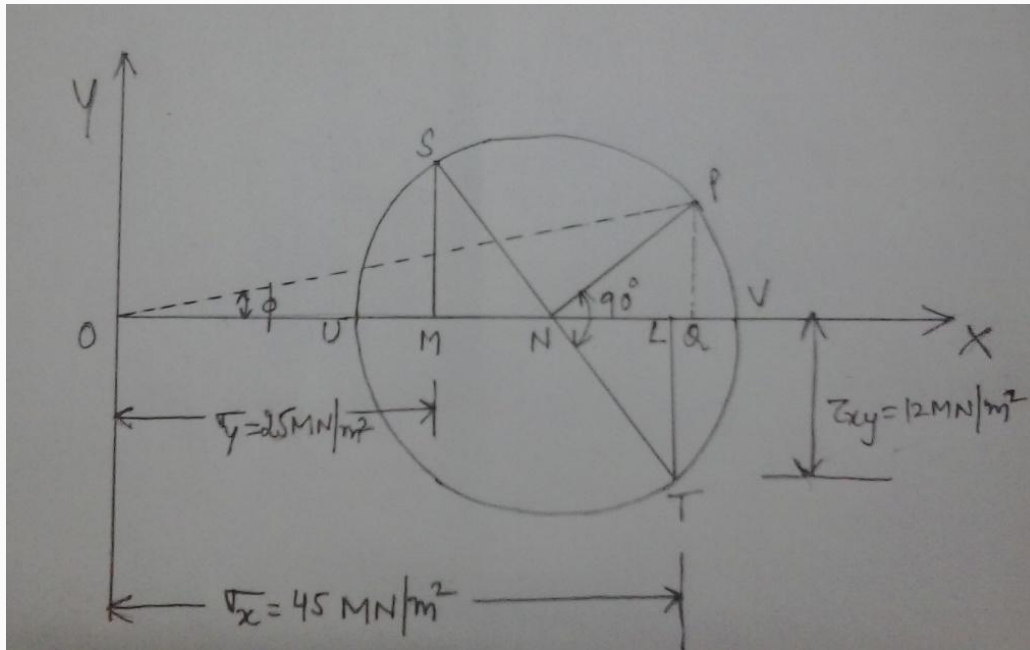
- Plot OL = 45 MN/m² and OM = 25 MN/m²

- Drop perpendicular LT and MS , each 12 MN/m² as shown in figure.

- Join ST to get N and draw the mohr's circle to pass through S and T.

- Draw NP at 90⁰ to NT.

- Draw perpendicular to OQ.

- σ_n = OQ = 47 MN/m² (tensile)

- τ = PQ = 10 MN/m² (shear)

- σ_r = 48.05 MN/m²

- φ = 12⁰

LESSON 32.

32.1 PRINCIPAL PLANES AND PRINCIPAL STRESSES

- A body may be subjected to stresses in one plane or in different planes.

- There are always three mutually perpendicular planes along which the stresses at a certain point (in a body) can be resolved completely into stresses normal to these planes.

- These planes which pass through the point in such a manner that the resultant stress across them is totally a normal stress are known as "Principal planes" and normal stresses across these planes are termed as "principal stresses".

- The plane carrying the maximum normal stress is called the major principal planes and the corresponding stress the major principal stress. The plane carrying the minimum normal stress is known as major principal plane and the corresponding stress as major principal stress.

Example: The principal stresses in the wall of container (Fig.32.1) are 40 kN/m² and 80 kN/m². Determine the normal, shear and resultant stresses in magnitude and direction in a plane, the normal of which makes an angle of 30° with the direction of maximum principal stresses.

Solution: Given: σ_x = 80 kN/m² (tensile);

σ_y = 40 kN/m² (tensile);

$\Theta = 30°$

(i) Normal stress, σ_n:

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots(2.1)$$

$$= \frac{80 + 40}{2} + \frac{80 - 40}{2}\cos 60°$$

$$= 60 + 10 = 70 \text{ kN/m}^2$$

(ii) Shear stress, τ :

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta = \frac{80 + 40}{2}\sin 60° \quad\ldots\ldots\ldots\ldots\ldots(2.2)$$

$$= 17.32 \text{ kN/m}^2$$

(iii) Resultant stress, σ_r , ϕ:

$$\sigma_r = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{70^2 + 17.32^2}$$

i.e, σ_r = 72.11 kN/m^2

If ϕ is the angle that the resultant makes with the normal to the plane, then

$$\tan\phi = \frac{\tau}{\sigma_n} = \frac{17.32}{70} = 0.2474$$

ϕ = 13°54'

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