OPERATIONS RESEARCH

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COURSE OUTLINE

Lesson 1

ELEMENTARY CONCEPTS AND OBJECTIVES OF OPERATIONS RESEARCH

1.1 Introduction

Operations Research (OR) is relatively a new discipline. The first formal activities of OR were initiated in England during the Second World War, when a team of British scientists set out to make decisions regarding the best utilization of war material. OR begins when some mathematical and quantitative technique is used to verify the decision being taken. OR provides a quantitative technique or a scientific approach to the executives for making better decisions for operations under their control.

1.2 Historical Background of Operations Research

The term Operations Research was first coined in 1940 by McClosky and Trefthen in a small town Bowdsey of U.K. This new science came into existence in military context. As the name implies, 'Operations Research' was apparently invented by the team dealing with research on (military) operations. The work of the team of science was named as Operations Research in U.K. During the Second World War, military management called upon scientists from various disciplines, and organized them into teams to assist in solving strategic and tactical problems i.e. to discuss, evolve and suggest ways and means to improve the execution of various military projects. By their joint efforts, experience and deliberations, they suggested certain approaches that showed remarkable progress. This new approach to systematic and scientific study of the operations of the system is called the 'Operations Research' or 'Operational Research'.

The encouraging results obtained by the British OR team quickly motivated the military management of the United States to start on similar activities. The successful applications of the U.S. teams included the invention of new flight patterns, planning sea mining and effective utilization of electronic equipment. This work of OR team was given various names in the United States: Operational Analysis, Operations Evaluation, Operations Research, Systems Analysis, Systems Evaluation, System Research and Management Science.

Following the end of the war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their complex managerial type problems. The most common problem was to seek methods so as to minimize the total cost and maximize the total profit. The first mathematical technique in the field, called the Simplex Method of linear programming, was developed in 1947 by an American Mathematician George B. Dantzig. Since then, new techniques and applications have been developed through the efforts and cooperation of interested individuals in both academic institutions and industry.

In India, Operations Research came into existence in 1949 with the opening of an OR unit at the Regional Research Laboratory at Hyderabad. At the same time, another group was set up in the Defence Science Laboratory which devoted itself to the problems of stores, purchase and planning. In 1953, OR unit was established in Indian Statistical Institute, Calcutta for the application of OR methods in national planning and survey. OR Society of India was formed in 1955. In India, Prof. P. C. Mahalanobis made the first important application of OR in formulating the Second Five Year Plan in order to forecast the trends of demand, availability of resources and for scheduling the complex schemes necessary for developing the economy of the country. In the industrial sector, in spite of the fact that opportunities of OR work at present are very much limited, organized industries in India are gradually becoming conscious of the role of Operations Research and a good number of them have well trained OR teams. Most popular practical application of OR in India is linear programming.

1.3 Definitions of Operations Research

Literally the word 'operation' may be defined as some action that we apply to some problems or hypotheses and the word 'Research' means an organized process of seeking out facts about the same. In fact, it is difficult to precisely define OR mainly because of the fact that its boundaries are not clearly marked. OR has been variously described as 'Science of Use', 'Quantitative Common Sense', 'Scientific approach to decision making problems', etc. Some of the commonly used and widely accepted definitions of OR are given as under:

1. "OR is a scientific method of providing executive departments with a quantitative basis for decisions under their control".

   \- P.M. Morse and G.E. Kimball

2. "OR is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problems".

   \- Churchman, Ackoff and Arnoff

3. "OR is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough going rationality in dealing with his decision problems".

   \- D.W. Miller and M.K. Starr

4. "OR is the attack of modern science on problems of likelihood that arise in the management and control of men and machines, materials and money in their natural environment, its special technique is to invent a strategy of control by measuring, comparing and predicting probable behavior through a scientific model of a situation".

   \- Beerr

5. "OR is a scientific method of providing the executive with an analytical and objective basis for decisions".

   \- P.M.S. Blackett

6. The term OR connotes various attempts to study operations of war by scientific methods. From a more general point of view OR can be considered to be an attempt to study these operations of modern society which involve organizations of men or of men and machines.

   \- P.M. Morse

7. OR is a management activity pursued in two complementary ways – one half by the free and bold

exercise of commonsense untrammeled by any routine and other half by the application of a repertoire of well established pre-created methods and techniques.

- Jagjit Singh

8. OR is the attack of modern science on complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business and defence. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management to determine its policies and actions scientifically.

- Operational Research Quarterly

9. "OR is the art of giving bad answers to problems which otherwise have worse answers".

- T.L. Saaty

From all the above opinions, it may be concluded that whatever else OR may be, it is certainly concerned with optimization theory.

1.4 Scope of Operations Research

OR is mainly concerned with the techniques of applying scientific knowledge, besides the development of science. It provides an understanding which gives the expert/manager new insights and capabilities to determine better solutions in his decision making problems, with great speed, competence and confidence. OR has been found to be used in the following five major areas of research:

1. OR is useful to the Directing Authority in deciding optimum allocation of various limited resources such as men, machines, material, time, money, etc., for achieving the optimum goal.
2. OR is useful to Production Specialist in
   i. Designing, selecting and locating sites.
   ii. Determining the number and size.
   iii. Scheduling and sequencing the production runs by proper allocation of machines; and
   iv. Calculating the optimum product mix.
3. OR is useful to the Marketing Manager (executive) in determining:
   i. How to buy, how often to buy, when to buy and what to buy at the minimum possible cost.
   ii. Distribution points to sell the products and the choice of the customers.
   iii. Minimum per unit sale price.
   iv. The customer's preference relating to the size, colour, packaging etc., for various products and the size of the stock to meet the future demand; and
   v. The choice of different media of advertising.

4. OR is useful to the Personnel Administrator in finding out:
   i. Skilled persons at a minimum cost.
   ii. The number of persons to be maintained on full time basis in a variable work load like freight handling etc.; and
   iii. The optimum manner of sequencing personnel to a variety of jobs.
5. OR is useful to the Financial Controller to
   i. Find out a profit plan for the company.
   ii. Determine the optimum replacement policies.
   iii. Find out the long-range capital requirements as well as the ways and means to generate these requirements.

Lesson 2

APPLICATIONS OF OPERATIONS RESEARCH IN DECISION MAKING

2.1 Introduction

The Operations Research may be regarded as a tool which is utilized to increase the effectiveness of management decisions. Scientific method of OR is used to understand and describe the phenomena of operating system. Mathematical and logical means of Operations Research provides the executive with quantitative basis for decision making and enhance ability to make long range plans and to solve everyday problems of industry with greater efficiency and competence.

2.2 Features (Characteristics) of OR

OR is a tool employed to increase the effectiveness of managerial decisions as an objective supplement to the subjective feeling of the decision makers. There are five salient features of OR.

2.2.1 Decision making

Primarily OR is addressed to managerial decision making or problem solving.

2.2.2 Scientific approach

OR employs scientific methods for the purpose of solving problems. It is a formalized process of reasoning.

2.2.3 Objective

OR attempts to locate the best or optimal solution to the problem under consideration. For this purpose, it is necessary that a measures of effectiveness is defined which is based on the goals of the organization. This measure is then used as the basis to compare the alternative courses of action. The examples are profit, net returns, cost of production etc.

2.2.4 Inter-disciplinary team approach

OR is inter-disciplinary in nature and requires a team approach to a solution of the problem. Managerial problems have economic, physical, psychological, biological, sociological and engineering aspects. This requires a blend of people with expertise in the areas of Mathematics, Statistics, Engineering, Economics, Management, Computer Science and so on.

2.2.5 Digital computer

Use of a digital computer has become an integral part of the OR approach to decision making. The computer may be required due to the complexity of the model, volume of data required and the computations to be made.

2.3 Modeling in Operations Research

A model as used in OR is defined as idealized representation of a real-life system. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect. A model in OR is a simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation are considered. The construction of a model helps in putting the complexities and possible uncertainties in a decision making problem, into a logical framework amenable to comprehensive analysis. With such a model we can have several decision alternatives, their anticipated effects indicating the need for relevant data for analyzing the alternatives that leads to informative conclusions.Models can be broadly classified according to following characteristics:

2.3.1 Classification by structure

2.3.1.1 Iconic models

These models are pictorial representation of real systems and have the appearance of the real thing. They represent the system as it is by scaling it up or down (i.e., by enlarging or reducing the size). In other words it is an image. Examples of such models are child's toy, photograph, a schedule of operations, histogram, pictogram, cartogram, a physical model such as small scale model of a dairy plant, an engine etc. These kinds of models are called 'Iconic' because they are look alike items to understand and interpret the real things. An iconic model is said to be 'scaled down' or 'scaled up' according to the dimensions of the model and are smaller or greater than those of the real item. Iconic models are easy to observe, build and describe, but are difficult to manipulate and not very useful for the purpose of prediction. Commonly these models represent a static event. These models can be constructed up to three dimensions (e.g., atom, globe, small aeroplane, cube, etc.) while it is not possible to construct them physically for higher dimensions. Further these models do not include those aspects of the real system that are irrelevant for the analysis.

2.3.1.2 Analogue models

2.3.1.2 Analogue models

These models are more abstract than the iconic ones as there is no 'look alike' correspondence between these models and real items. These models are the one in which one set of properties is used to represent another set of properties. After the problem is solved, the solution is re-interpreted in terms of the original system. For example, graphs and maps in various colours are analogue models in which different colours correspond to different characteristics e.g., blue representing water, brown representing land, yellow representing production etc. Further, graphs are very simple analogues because distance is used to represent such properties as time, number, per cent, age, weight, and many other properties. Contour lines on the map represent the rise and fall of the heights. Demand curves, flow charts in production control and frequency curves in statistics are analogue models of the behaviour of events. Graphs of time series, bar diagrams, stock market changes are other examples of analogue models. Analogue models are less specific, less concrete but easier to manipulate and can represent dynamic situations. These are, generally, more useful than the iconic ones because of their vast capacity to represent the characteristics of the real system under investigation.

2.3.1.3 Mathematical models (Symbolic Model)

The Symbolic or mathematical model is one which employs a set of mathematical symbols (i.e. letters, numbers etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or set of equations to describe the behavior (or properties) of the system. These models are most general and precise. The solution to these models is then obtained by applying well developed mathematical techniques to the model. The symbolic model is usually the easiest to manipulate experimentally and the most general and abstract. Its function is more often explanatory rather than descriptive. Explanatory model is that which contains controlled variables, while the descriptive model does not contain controlled variables.

2.3.2 Classification by purpose

2.3.2.1 Descriptive model

It describes the some aspects of a situation based on observation, survey, questionnaire results or other available data.

2.3.2.2 Predictive models

Such models can answer 'what if' type of questions i.e. they can make predictions regarding certain events.

2.3.2.3 Prescriptive models

When a predictive model has been repeatedly successful, then it can be used to prescribe a source of action. For example, linear programming is a prescriptive model because it prescribes what the managers ought to do.

2.3.3 Classification by nature of environment

2.3.3.1 Deterministic model

Such models assume conditions of complete certainty and perfect knowledge for example; linear programming, transportation and assignment models are deterministic models.

2.3.3.2 Probabilistic (or Stochastic) model

These models handle those situations in which consequences or payoff of managerial actions cannot be predicted with certainty.

2.3.4 Classification by Behavior

2.3.4.1 Static models

These models do not consider the impact of changes that take place during the planning horizon i.e. they are independent of time.

2.3.4.2 Dynamic models

These models consider time as one of the important variables and depict the impact of changes generated over time. In this instead of one decision, a series of independent decisions are required during the planning horizons.

2.4 Characteristics of a Good Model

- A good model should be capable of taking into account new formulations without having any significant change in its frame.
- Assumptions made in the model should be as few as possible.
- It should be simple and coherent having less number of variables.

- It should be open to parametric type treatments.

2.5 Advantages of a Model

- Through the model, the problem under consideration becomes controllable.
- It provides some logical and systematic approach to the problem.
- It indicates the limitations and scope of an activity
- Models help in incorporating useful tools that eliminate duplication of methods applied to solve any specific problem.
- Models help in finding avenues for new research and improvements in a system.

2.6 Methodology of Operations Research

The major phases of an O.R. study are as follows:

Phase I: Formulation of the Problem

In this phase, the problem is formulated in an appropriate form. This phase should give a statement of the problem's elements that include the controllable (decision) variables, the uncontrollable parameters, the restrictions or constraints on the variables and the objectives for defining a good or improved solution.

Phase II: Constructing a mathematical model

The second phase of the investigation is concerned with the choice of proper data inputs and the design of the appropriate information output. In this phase, both static and dynamic structural elements and the representation of inter-relationship among the elements in terms of mathematical formulae need to be specified. A mathematical model should include mainly the following three basic sets of elements:

(i) Decision Variables and Parameters

(ii) Constraints or Restrictions

(iii) Objective Function

Phase III: Deriving the solution from the model

This phase of the study deals with the mathematical calculations for obtaining the solution to the model. A solution of the model means those values of the decision variables that optimize one of the objectives and give permissible levels of performance on any other of the objectives.

Phase IV: Testing the model and its solution

This phase of the study involves checking the validity of the model used. A model may be said to be valid if it can give a reliable prediction of the system's performance.

Phase V: Controlling the solution

This phase of the study establishes control over the solution by proper feedback of the information on variables which deviated significantly. As soon as one or more of the controlled variables change significantly, the solution goes out of control. In such a situation the model may accordingly be modified.

Phase VI: Implementing the solution

This phase of the study deals with the implementation of the tested results of the model. This would basically involve a careful explanation of the solution to be adopted and its relationship with the operating realities.

2.7 General Methods of Deriving the Solution

There are three methods to derive the solution to an OR model.

2.7.1 Analytical method

Analytical methods involve expressions of the model by graphic solutions or by mathematical computations. For example, area indicated by mathematical function may be evaluated through the use of integral calculus. Solutions of various inventory models are obtained by using the analytical procedure. The analytical methods involve all the tools of classical mathematics, such as calculus, finite differences, etc. The kind of mathematics required for a particular OR study depends upon the nature of the model.

2.7.2 Numerical or iterative method

Numerical methods are concerned with the iterative or trial and error procedures, through the use of numerical computations at each step. These numerical methods are used when some analytical methods fail to derive the solution. The algorithm is started with a trial (initial) solution and continued with the set procedure to improve the solution towards optimality. The trial solution is then replaced by the improved one and the process is repeated until the solution converges or no further improvement is possible. Thus the numerical methods are hit and trial methods that end at a certain step after which no further improvement to the solution is possible.

2.7.3 Monte Carlo method

This method is essentially a simulation technique, in which statistical distribution functions are created by generating a series of random numbers. Various steps associated with a Monte Carlo method are:

(a) For appropriate model of the system, sample observations are made and then the probability distributions for the variables of interest are determined.

(b) Convert the probability distribution to a cumulative distribution.

(c) Select the sequence of random numbers with the help of random number tables.

(d) Determine the sequence of values of variables of interest with the sequence of random numbers obtained in the previous step

(e) Fit an appropriate mathematical function to the values obtained in step (d)

2.8 Advantages/ Merits of OR Techniques

2.8.1 Optimum use of production factors

Linear programming techniques indicate how a manager can utilize most effectively his inputs/ factors and by more efficiently selecting and distributing these elements.

2.8.2 Improved quality of decision

The effect on the profitability due to changes in the production pattern will be clearly indicated in the simplex table. These tables give a clear picture of the happenings within the basic restrictions and possibilities of behaviour of compound elements involved in the problem.

2.8.3 Preparation of future managers

These methods substitute a means for improving the knowledge and skill of young managers.

2.8.4 Modification of mathematical solution

OR presents a possible practical solution when one exists, but it is always a responsibility of the manager to accept or modify the solution before its use. The effect of these modifications may be evaluated from the computation steps and tables.

2.8.5 Alternative solutions

OR techniques suggest all the alternative solutions available for the same profit so that the management may decide on the basis of its strategies.

2.9 Limitations of OR

2.9.1 Practical application

Formulation of an industrial problem to an OR set programme is a difficult task.

2.9.2 Reliability of the proposed solution

A non-linear relationship is changed to linear for fitting the problem to linear programming. This may disturb the solution.

2.9.3 Money and time cost

When the basic data is subject to frequent changes, the cost of changing programmes manually is a costly affair.

2.9.4 Combining two or more objective functions

Very frequently maximum profit does not come from manufacturing the maximum quantity of the most profitable product at the most convenient machine and at the minimum cost, since this may lead to underutilization of certain lines of production.

The aim is not to optimize individual objective function. It is, therefore, necessary to have a single objective function which can cover several objective functions at the same time. Despite all the above limitations, OR is a powerful tool and an analytical process that offers the presentation of an optimal solution.

Lesson 3

MATHEMATICAL FORMULATION OF THE LINEAR PROGRAMMING PROBLEM AND ITS GRAPHICAL SOLUTION

3.1 Introduction

A large number of business and economic situations are concerned with problems of planning and allocation of resources to various activities. In each case there are limited resources at our disposal and our problem is to make such a use of these resources so as to maximize production or to derive the maximum profit, or to minimize the cost of production etc. Such problems are referred to as the problems of constrained optimization. Linear programming (LP) is one of the most versatile, popular and widely used quantitative techniques. Linear Programming is a technique for determining an optimum schedule chosen from a large number of possible decisions. The technique is applicable to problem characterized by the presence of a number of decision variables, each of which can assume values within a certain range and affect their decision variables. The variables represent some physical or economic quantities which are of interest to the decision maker and whose domain are governed by a number of practical limitations or constraints which may be due to availability of resources like men, machine, material or money or may be due quality constraint or may arise from a variety of other reasons. The most important feature of linear programming is presence of linearity in the problem. The word Linear stands for indicating that all relationships involved in a particular problem are linear. Programming is just another word for "planning" and refers to the process of determining a particular plan of action from amongst several alternatives. The problem thus reduces to maximizing or minimizing a linear function subject to a number of linear inequalities

3.2 Definitions of Various Terms Involved in Linear Programming

3.2.1 Linear

The word linear is used to describe the relationship among two or more variables which are directly proportional. For example, if the production of a product is proportionately increased, the profit also increase proportionately, then it is a linear relationship. A linear form is meant a mathematical expression of the type,

$a_1X_1 + a_2X_2 + -- +a_nX_n$ *where* $a_1, a_2, ---, a_n$ *are constant and* $X_1, X_2, ---, X_n$ *are variables.*

3.2.2 Programming

The term "Programming" refers to planning of activities in a manner that achieves some optimal result with resource restrictions. A programme is optimal if it maximizes or minimizes some measure or criterion of effectiveness, such as profit, cost or sales.

3.2.3 Decision variables and their relationship

The decision (activity) variables refer to candidates (products, services, projects etc.) that are competing with one another for sharing the given limited resources. These variables are usually inter-related in terms of utilisation of resources and need simultaneous solutions. The relationship among these variables should be linear.

3.2.4 Objective function

The Linear Programming problem must have a well defined objective function for optimization. For example, maximization of profits or minimization of costs or total elapsed time of the system being studied. It should be expressed as linear function of decision variables.

3.2.5 Constraints

There are always limitations on the resources which are to be allocated among various competing activities. These resources may be production capacity, manpower, time, space or machinery. These must be capable of being expressed as linear equalities or inequalities in terms of decision variables.

3.2.6 Alternative courses of action

There must be alternative courses of action. For example, there may be many processes open to a firm for producing a commodity and one process can be substituted for another.

3.2.7 Non-negativity restriction

All the variables must assume non-negative values, that is, all variables must take on values equal to or greater than zero. Therefore, the problem should not result in negative values for the variables.

3.2.8 Linearity and divisibility

All relationships (objective functions and constraints) must exhibit linearity, that is, relationships among decision variables must be directly proportional. For example, if our resources increase by some percentage, then it should increase the outcome by the same percentage. Divisibility means that the variables are not limited to integers. It is assumed that decision variables are continuous, i.e., fractional values of these variables must be permissible in obtaining an optimal solution.

3.2.9 Deterministic

In LP model (objective functions and constraints), it is assumed that the entire model coefficients are completely known (deterministic), e.g. profit per unit of each product, and amount of resources available are assumed to be fixed during the planning period.

3.3 Formulation of a Linear Programming Problem

The formulation of the Linear Programming Problem (LPP) as mathematical model involves the following key steps:

Step 1. Identify the decision variables to be determined and express them in terms of algebraic symbols as $X_1, X_2, ---, X_n$.

Step 2. Identify the objective which is to be optimized (maximized or minimized) and express it as a linear function of the above defined decision variables.

Step 3. Identify all the constraints in the given problem and then express them as linear equations or inequalities in terms of above defined decision variables.

Step 4. Non-negativity restrictions on decision variables.

The formulation of a linear programming problem can be illustrated through the following examples:

Example 1:

A milk plant manufactures two types of products A and B and sells them at a profit of Rs. 5 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires one minute of processing time on G and two minutes on H; type B requires one minute on G and one minute on H. The machine G is available for not more than 6 hours 40 minutes, while machine B is available for 8 hours 20 minutes during any working day; formulate the problem as LP problem.

Solution :

Let X_1 be the number of products of type A and X_2 the number of products of type B. Since the profit on type A is Rs. 5 per product, $5X_1$ will be the profit on selling X_1units of type A. Similarly, $3X_2$ will be the profit on selling X_2 units of type B. Therefore, total profit on selling X_1 units of A and X_2 units of B is given by $Z = 5X_1+3X_2$. Since this has to be maximized, hence the objective function can be expressed as

$$\text{Max } Z = 5X_1+3X_2 \quad \text{(Objective function)}$$

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total time in minutes required on machine G is given by X_1+X_2. Similarly the total time in minutes required on machine H is given by $2X_1+X_2$. But machine G is not available for more than 6 hour 40 minutes (400 minutes), therefore

$$X_1+X_2 \leq 400 \qquad \text{(first constraint on machine G)}$$

Also the machine H is available for 8 hours 20 minutes only, therefore

$$2X_1+X_2 \leq 500 \qquad \text{(second constraint on machine H)}$$

Since it is not possible to produce negative quantities,

$$X_1 \geq 0,\ X_2 \geq 0 \qquad \text{(non-negativity restrictions)}$$

Example 2 :

A dairy plant packs two types of milk in pouches viz., full cream and single toned. There are sufficient ingredients to make 20,000 pouches of full cream & 40,000 pouches of single toned. But there are only 45,000 pouches into which either of the products can be put. Further it takes three hours to prepare enough material to fill 1000 pouches of full cream milk & one hour for 1000 pouches of single toned milk and there are 66 hours available for this operation. Profit is Rs. 8 per pouch for full cream milk and Rs. 7 per pouch for single toned milk. Formulate it as a linear programming problem.

Solution

Let number of full cream and single toned milk pouches to be packed is X_1 and X_2.

Let profit be Z

Objective function: Max. $Z = 8X_1+7X_2$

Subject to constraints: $X_1 + X_2 \leq 45000$ (1)

$X_1 \leq 20000$ (2)

$X_2 \leq 40000$ (3)

$3\frac{X_1}{1000} + \frac{X_2}{1000} \leq 66 \Rightarrow 3X_1 + X_2 \leq 66000$ (4)

Non-negativity restrictions $X_1, X_2 \geq 0$

Example 3 :

Consider two different types of food stuffs say F_1 and F_2. Assume that these food stuffs contain vitamin A and B. Minimum daily requirements of vitamin A and B are 40mg and 50mg respectively. Suppose food stuff F_1 contains 2mg of vitamin A and 5mg of vitamin B while F_2 contains 4mg of vitamin A and 2mg of vitamin B. Cost per unit of F_1 is Rs. 3 and that of F_2 is Rs. 2.5. Formulate the minimum cost diet that would supply the body at least the minimum requirements of each vitamin.

Solution :

Let number of units needed for food stuffs F_1 and F_2 to meet the daily requirements of vitamins A and B be respectively X_1 and X_2.

Objective function: Minimize $Z = 3X_1+2.5X_2$

Subject to constraints:

$2X_1+4X_2 \geq 40$

$5X_1+2X_2 \geq 50$

Non-negativity restrictions $X_1, X_2 \geq 0$

3.4 Mathematical Formulation of a General Linear Programming Problem

The general formulation of LP problem can be stated as follows. If we have n-decision variables $X_1, X_2, \ldots, X_n$ and m constraints in the problem , then we would have the following type of mathematical formulation of LP problem

Optimize (Maximize or Minimize) the objective function:

$$Z = C_1X_1 + C_2X_2 + \cdots + C_nX_n \quad (3.4.1)$$

subject to satisfaction of m- constraints :

$$\left.\begin{array}{l} a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n\ (\le = \ge)\ b_1 \\ a_{21}X_1 + a_{22}X_2 + \dots + a_{2n}X_n\ (\le = \ge)\ b_2 \\ a_{i1}X_1 + a_{i2}X_2 + \dots + a_{in}X_n\ (\le = \ge)\ b_i \\ a_{m1}X_1 + a_{m2}X_2 + \dots + a_{mn}X_n\ (\le = \ge)\ b_m \end{array}\right\} \quad (3.4.1)$$

Where the constraint may be in the form of an inequality (≤ or ≥) or even in the form of an equality (=) and finally satisfy the non-negativity restrictions

$$X_1, X_2, \dots, X_n \ge 0 \qquad (3.4.3)$$

where C_j (j=1, 2,….,n); b_i (i=1, 2,…,m) and a_{ij} are all constants and m<n, and the decision variables $X_j \ge 0$, j=1, 2,…,n. If b_i is the available amount of resource i then a_{ij} is amount of resource i that must be allocated (technical coefficient) to each unit of activity j.

NOTE: By convention, the values of RHS parameters b_i (i=1, 2, 3,…,m) are restricted to non-negative values only. If any value of b_i is negative then it is to be changed to a positive value by multiplying both sides of the constraint by -1. This not only changes the sign of all LHS Coefficients and of RHS parameters but also changes the direction of inequality sign.

3.4.1 Matrix Form of LP problem

The linear programming problem (3.4.1), (3.4.2) and (3.4.3) can be expressed in matrix form as

$$\text{Maximize } Z = CX \qquad \text{(objective function)}$$

$$\text{subject to } AX = b, b \ge 0 \qquad \text{(constraint equation)}$$

$$X \ge 0 \qquad \text{(non-negativity restriction)}$$

where $X = (X_1, X_2, \dots, X_n)$, $C = (C_1, C_2, \dots, C_n)$ and $b = (b_1, b_2, \dots, b_m)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

3.5 Graphical Solution of Linear Programming Problem

LP problems which involve only two variables can be solved graphically. Such a solution by geometric method involving only two variables is important as it gives insight into more general case with any number of variables.

3.5.1 Feasible solution

A set values of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called feasible solution of the problem.

3.5.2 Feasible region

The collection of all feasible solutions is known as the feasible region. Any point which does not lie in the feasible region cannot be a feasible solution to the LP problems. The feasible region does not depend on the form of the objective function in any way. If we can represent the relations of the general LP problem on an n dimensional space, we will obtain a shaded solid figure (known as Convex-polyhedron) representing the domain of the feasible solution.

3.5.3 Optimal solution

A feasible solution of a linear programming problem which optimizes its objective function is called the optimal solution of the problem. Theoretically, it can be shown that objective function of a LP problem assumes its optimal value at one of the vertices (called extreme points) of this solid figure.

Steps to find graphical solution of the linear programming problem

Step 1: Formulate the linear programming problem.

Step 2: Draw the constraint equations on XY-plane.

Step 3: Identify the feasible region which satisfies all the constraints simultaneously. For less than or equal to constraints the region is generally below the lines and for greater than or equal to constraints, the region is above the lines.

Step 4: Locate the solution points on the feasible region. These points always occur at the vertices of the feasible region.

Step 5: Evaluate the optimum value of the objective function.

The geometric interpretation and solution for the LP problem using graphical method is illustrated with the help of following examples

Example 4 Find the graphical solution of problem formulated in example 1.

$$\text{Max } Z = 5X_1+3X_2$$

Subject to

$$X_1+X_2 \leq 400$$

$$2X_1+X_2 \leq 500$$

$$X_1 \geq 0 \text{ , } X_2 \geq 0$$

Solution

Any point lying in the first quadrant has X_1, $X_2 \geq 0$ and hence satisfies the non-negativity restrictions. Therefore, any point which is a feasible solution must lie in the first quadrant. In order to find the set of points in the first quadrant which satisfy the constraints, we must interpret geometrically inequalities such as $X_1+X_2 \leq 400$. If equality holds then we have the equation $X_1+X_2= 400$ i.e., any point on the straight line satisfies the equation. Any point lying on or below the line $X_1+X_2= 400$ satisfies the constraint $X_1+X_2 \leq 400$. However, no point lying above the line satisfies the inequality. In a similar manner, we can find the set of points satisfying $2X_1+X_2 \leq 500$ and the non-negativity restrictions are all the points in the first quadrant. The set of points satisfying the constraints and the non-negativity restrictions is the set of points in the shaded region of the figure OABC as shown in Fig 3.1 Any point in this region is a feasible solution, and only the points in this region are feasible solutions. To solve the LP problem, we must find the point or points in the region of feasible solutions which give the largest value of the objective function. Now for any fixed value of Z, Z =

$5X_1+3X_2$ is a straight line, for each different values of Z, we obtain a different line. Again, all the lines corresponding to different values of Z are parallel; clearly our interest is to find the line with the largest value of Z which has at least one point in common with the region of feasible solutions. The line $Z= 5X_1+3X_2$ is drawn for various values of Z. It can be seen that the point farthest from $Z = 5X_1+3X_2$ and yet in the feasible region is B (100, 300) and thus point B maximizes Z while satisfying all the constraints. In other words, the corner B of the region of feasible solutions is the optimal solution of the LP problem. Since B is the intersection of the lines $X_1+X_2=400$ and $2X_1+X_2=500$, it can be seen that at vertex B, $X_1=100$ and $X_2=300$. The maximum value of Z = 5(100)+3(300)=1400.The fact that the optimum occurred at the vertex B of the feasible region is not a coincidence but on the other hand represents a significant property of optimal solutions of all LP problems. To find the optimal solution find the values of objective function at the various extreme points as shown in the following Table 3.1

Table 3.1 Computation of maximum value of objective function

Extreme Point	Coordinates	Profit Function $Z = 5X_1+3X_2$
O	$X_1=0$, $X_2=0$	Z=5(0)+3(0)=0
A	$X_1=0$, $X_2=400$	Z=5(0)+3(400)=1200
B	$X_1=100$, $X_2=300$	Z=5(100)+3(300)=1400
C	$X_1=250$, $X_2=0$	Z=5(250)+3(0)=1250

Note: A fundamental theorem in LP states that the feasible region of any LP is a convex polygon (that is the n dimensional version of two dimensional polygon), with a finite number of vertices, and further for any LP problem, there is at least one vertex which provides an optimal solution. Whenever a LP problem has more than one optimal solution, we say that there are alternative optimal solutions. Physically, this means that the resources can be combined in more than one way to maximize profit.

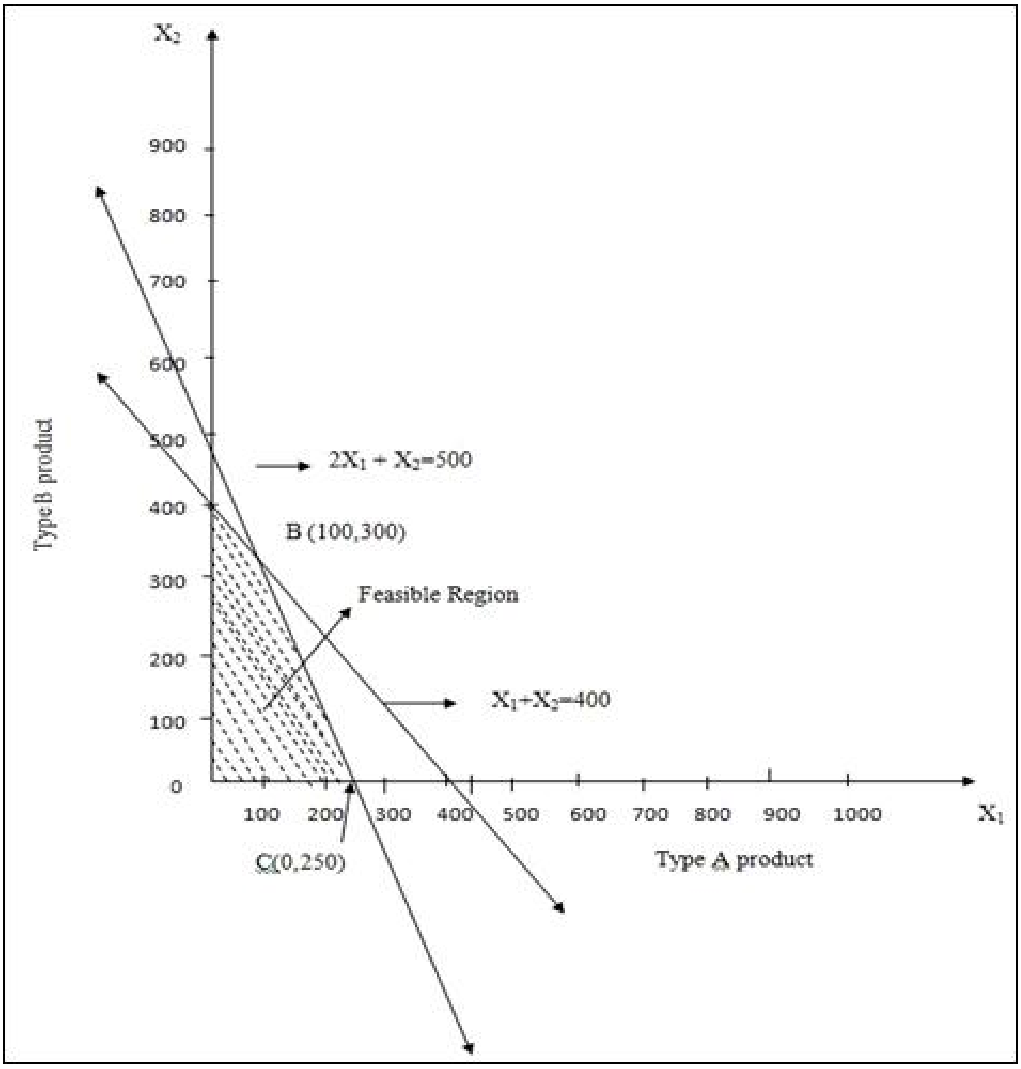


Fig. 3.1 Feasible region

Example 5: Find the graphical solution of problem formulated in Example 2.

Solution :

Let number of full cream and single toned milk pouches to be produced is $X_1$and X_2 and Let profit be Z

Objective function: Max. $Z = 8X_1+7X_2$

Subject to constraints:

$X_1 + X_2 \leq 45000$ (1)

$X_1 \leq 20000$ (2)

$X_2 \leq 40000$ (3)

$3X_1 + X_2 \leq 66000$ (4)

Non-negativity restrictions $X_1, X_2 \geq 0$

First of all draw the graphs of these inequalities (as discussed in Example 5) which is shown in Fig. 3.2.

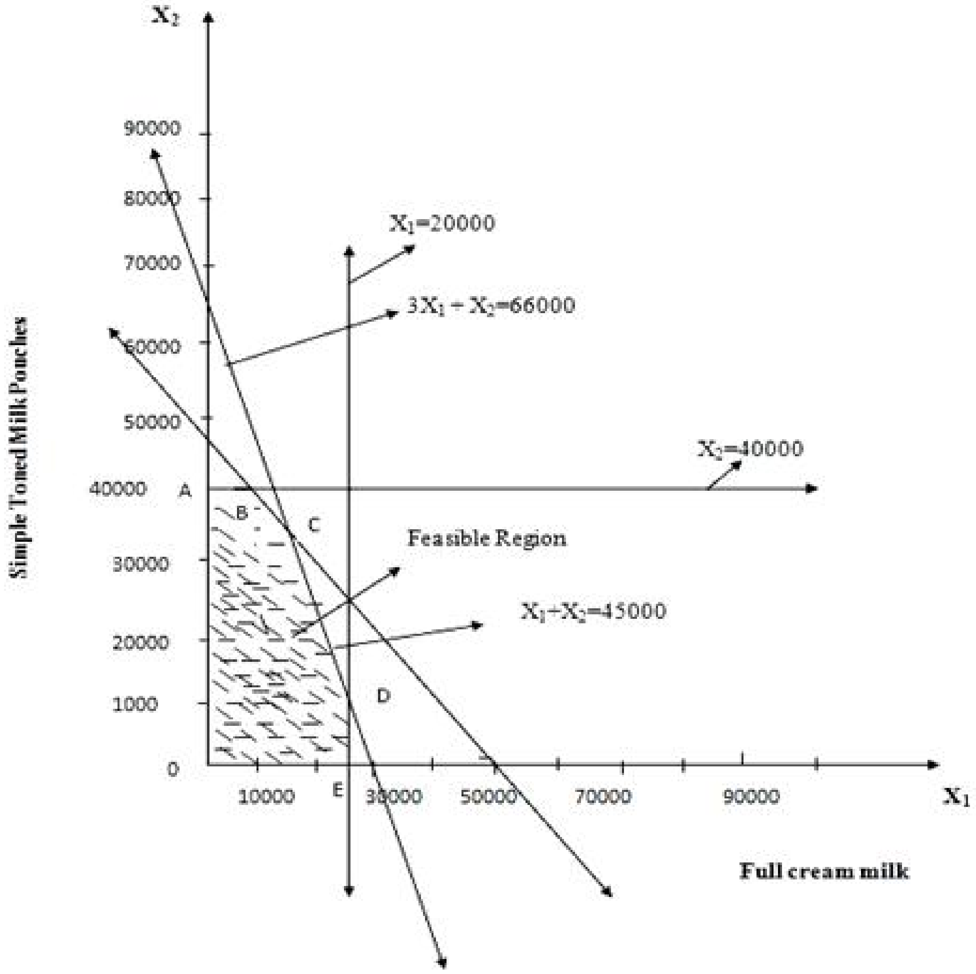


Fig. 3.2 Feasible region

To find the optimal solution find the values of objective function at the various extreme points as shown in the following Table 3.2

Table 3.2 Computation of maximum value of objective function

Extreme Point	Coordinates	Profit Function $Z = 8X_1+7X_2$
O	$X_1=0$, $X_2=0$	Z=8(0)+7(0)=0
A	$X_1=0$, $X_2=40000$	Z=8(0)+7(40000)=280000
B	$X_1=5000$, $X_2=40000$	Z=8(5000)+7(40000)=320000
C	$X_1=10500$, $X_2=34500$	Z=8(10500)+7(34500)=325500
D	$X_1=20000$, $X_2=6000$	Z=8(20000)+7(6000)=202000
E	$X_1=20000$, $X_2=0$	Z=8(20000)+7(0)=160000

So maximum value of Z occurs at point C (10500, 34500) so it is the optimal solution. It can be concluded that Dairy Plant must produce 10500 pouches of full cream milk and 34500 pouches of single toned milk. Let us consider the following example to illustrate graphical solution for minimization problem

3.5.4 Special cases in Linear Programming

Up till now we have discussed those problems which have a unique optimal solution. However, it is possible for LP problem to have following special cases.

3.5.4.1 Unbounded solution: A linear programming problem is considered to have an unbounded solution if it has no limits on the constraints and further, the common feasible region is not bounded in any respect.

Example 6: Find the graphical solution of problem formulated in example 3.

Solution

Let number of units needed for food stuffs F_1 and F_2 to meet the daily requirements of vitamins A and B be respectively X_1 and X_2.

Objective function: Minimize $Z = 3X_1 + 2.5X_2$

Subject to constraints: $2X_1 + 4X_2 \geq 40$

$5X_1 + 2X_2 \geq 50$

Non-negativity restrictions $X_1, X_2 \geq 0$

First of all draw the graphs of these inequalities (as discussed in example 5) .Since the inequalities are of the greater than or equal to type, the feasible region is formed by considering the area to the upper right side of each equation i.e. away from origin .The shaded area which is shown above CBD is satisfied by the two constraints as shown in Fig. 3.3 is feasible region. The feasible region for minimization problem is unbounded and unlimited. Since the optimal solution corresponds to one of the corner (extreme) points, we will calculate the values of objective function for each corner point viz. D (20, 0); B (7.5, 6.25) and C (0, 25). The calculations are shown in Table 3.3

Table 3.3 Table showing the computation of minimum value of objective function

Extreme Point	Coordinates	Cost Function Min. $Z = 3X_1+2.5X_2$
D	$X_1=20$, $X_2=0$	Z=3(20)+2.5(0)=60
B	$X_1=7.5$, $X_2=6.25$	Z=3(7.5)+2.5(6.25)=38.125
C	$X_1=0$, $X_2=25$	Z=3(0)+2.5(25)=62.5

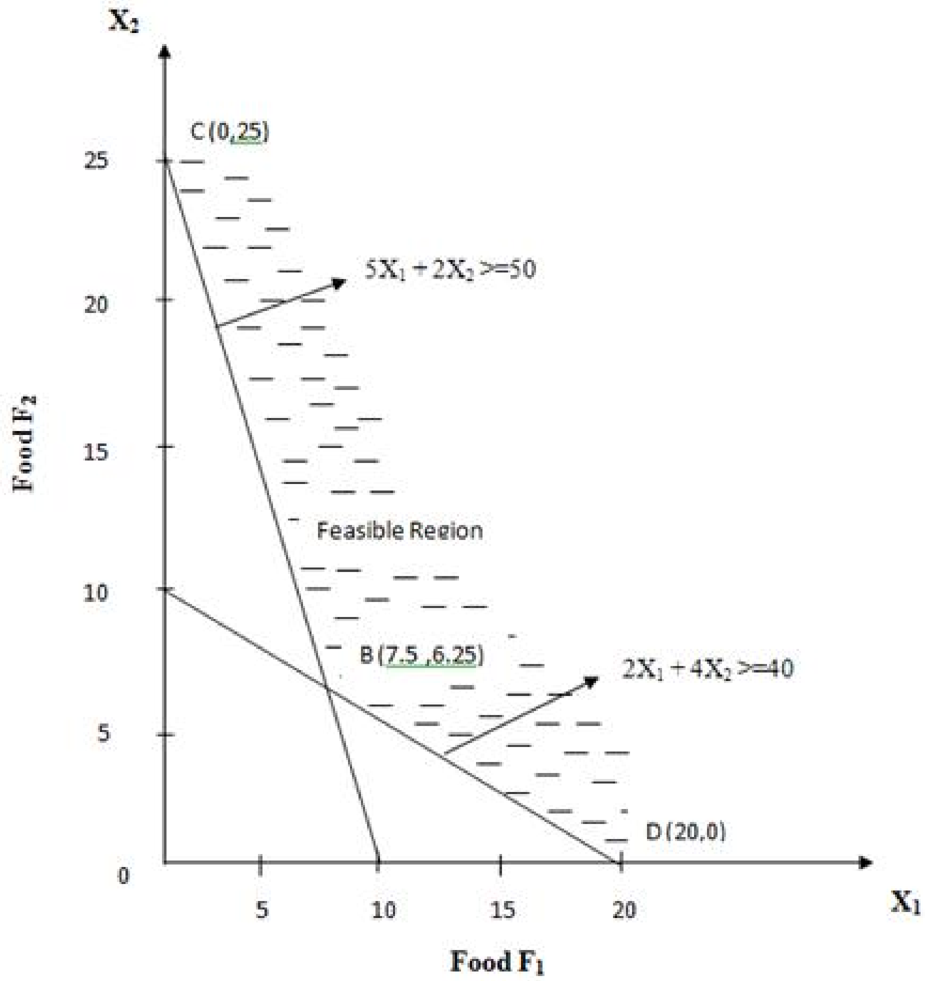


Fig. 3.3 Feasible region for minimization problem

The minimum cost is obtained at the corner point B(7.5,6.25) i.e. X_1=7.5 and X_2=6.25. Hence to minimize the cost and to meet the daily requirements of vitamin A and B, number of units needed of food stuffs F_1 and F_2 be 7.5 and 6.25 respectively.

3.5.4 Infeasible solution

In this there is no solution to an LP Problem that satisfies all the constraints. Graphically, it means that no feasible solution region exists. Such a condition indicates that the LP problem has been wrongly formulated.

3.5.5 Redundant constraint

In a properly formulated LP problem, each of the constraints will define a portion of the boundary of the feasible solution region. Whenever, a constraint does not define a portion of the boundary of the feasible solution region, it is called a redundant constraint. Let us consider the following example to illustrate this.

Example 7: Maximize $Z=1170X_1+1110X_2$

Subject to:

$$9X_1+5X_2 \geq 500$$
$$7X_1+9X_2 \geq 300$$
$$5X_1+3X_2 \leq 1500$$
$$7X_1+9X_2 \leq 1900$$
$$2X_1+4X_2 \leq 1000$$

$$X_1, X_2 \geq 0$$

The feasible region of this LP problem is indicated in Fig. 2.4. In this the feasible region has been formulated by two constraints.

$$9X_1+5X_2 \geq 500$$
$$7X_1+9X_2 \leq 1900$$
$$X_1, X_2 \geq 0$$

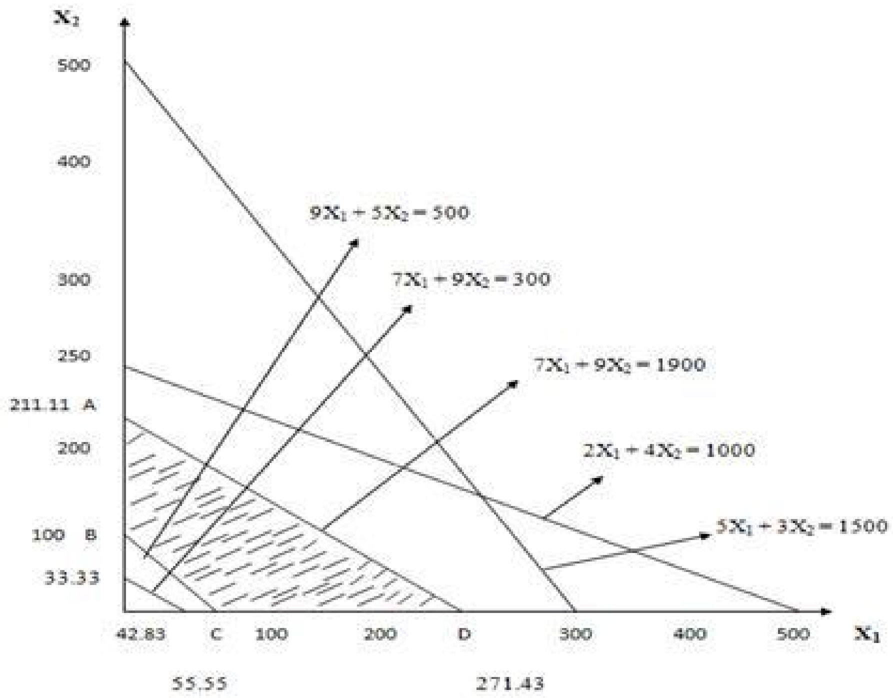


Fig 3.4 Feasible Region and Redundant Constraints

The remaining three constraints, although present, is not affecting the feasible region in any manner. Such constraints are known as redundant constraints.

Lesson 4

SIMPLEX METHOD

4.1 Introduction

In the previous chapter we considered the formulation of linear programming problems and the graphic method of solving them. Although the graphical approach to the solution of such problems is an invaluable aid to understand its basic structure, the method is of limited application in industrial problems as the number of variables occurring there is often considerably large. The Simplex Method provides an efficient technique which can be applied for solving linear programming problems of any magnitude-involving two or more decision variables. The Simplex Method is the name given to the solution algorithm for solving LP problems developed by George B. *Dantzig* in 1947. It is iterative procedure having fixed computational rules that leads to a solution to the problem in a finite number of steps. By using this one is capable of solving large LP problems efficiently.

4.2 Principle of Simplex Method

As the fundamental theorem of LP problem tells us that at least one basic feasible solution of any LP problem must be optimal, provided the optimal solution of the LP problem exists. Also, the number of basic feasible solutions of the LP problem is finite and at the most ${}^{n}C_{m}$ (where, n is number of decision variables and m is the number of constraints in the problem). On the other hand, the feasible solution may be infinite in number. So it is rather impossible to search for optimal solutions from amongst all feasible solutions. Furthermore, a great labour will also be required in finding out all the basic feasible solutions and select that one which optimizes the objective function. In order to remove this difficulty; a simple method was developed by Dantzig (1947) which is known as Simplex Algorithm. Simplex Algorithm is a systematic and efficient procedure for finding corner point solutions and taking them for optimality. The evaluation of corner points always starts from the point of origin. This solution is then tested for optimality i.e. it tests whether an improvement in the objective function is possible by moving to adjacent corner point of the feasible function space. This iterative search for a better corner point is repeated until an optimal solution if it exists, is determined.

4.3 Basic Terms Involved in Simplex Procedure

The following terms relevant for solving a linear programming problem through simplex procedure are given below:

4.3.1 Standard form of linear programming problem

The standard form of LP problem is to develop the procedure for solving general LP problem. The optimal solution of the standard form of a LP problem is the same as original LP problem. The characteristics of standard form are given in following steps:

Step 1. All the constraints should be converted to equations except for the non-negativity restrictions which remain as inequalities (≥ 0).

Step 2. The right side element of each constraint should be made non-negative.

Step 3. All variables must have non-negative values.

Step 4. The objective function should be of maximization form.

4.3.2 Slack variables

If a constraint has less than or equal sign, then in order to make it on equality we have to add something positive to the left hand side. The non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable. For example, consider the constraints.

$$3X_1 + 5X_2 \leq 2,\ 7X_1 + 4X_2 \leq 5,\ X_1, X_2 \geq 0$$

We add the slack variables $S_1 \geq 0$, $S_2 \geq 0$ on the left hand sides of above inequalities respectively to obtain

$$3X_1+5X_2+S_1 = 2$$

$$7X_1+4X_2+S_2 = 5$$

$$X_1, X_2, S_1, S_2 \geq 0$$

4.3.3 Surplus variables

If a constraint has greater than or equal to sign, then in order to make it an equality we have to subtract something non-negative from its left hand side. The positive variable which is subtracted from the left hand side of the constraint to convert it into equation is called the surplus variable.

For example, consider the constraints.

$$3X_1 + 5X_2 \geq 2,\ 2X_1 + 4X_2 \geq 5,\ X_1, X_2 \geq 0$$

We subtract the surplus variables $S_3 \geq 0$, $S_4 \geq 0$ on the left hand sides of above inequalities respectively to obtain

$$3X_1+5X_2 - S_3 = 2$$

$$2X_1 + 4X_2 - S_4 = 5$$

$$X_1, X_2, X_3, X_4 \geq 0$$

4.3.4 Solution to LPP

Any set $X = \{X_1, X_2, X_3, \ldots, X_{n+m}\}$ of variables is called a solution to LP problem, if it satisfies the set of constraints only.

4.3.5 Feasible solution (FS)

Any set $X = \{X_1, X_2, X_3, \ldots\ldots, X_{n+m}\}$ of variables is called a feasible solution of L.P. problem, if it satisfies the set of constraints as well as non-negativity restrictions.

4.3.6 Basic solution (BS)

For a system of m simultaneous linear equations in n variables(n>m), a solution obtained by setting (n-m) variables equal to zero and solving for the remaining variables is called a basic solution. Such m variables (of course, some of them may be zero) are called basic variables and remaining (n-m) zero-valued variables are called non-basic variables.

4.3.7 Basic feasible solution (BFS)

A basic feasible solution is a basic solution which also satisfies the non-negativity restrictions, that is all basic variables are non-negative. Basic solutions are of two types

(a) **Non-degenerate BFS**: A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive X_j (j=1, 2,…,m). In other words, all basic m variables are positive, and the remaining (n –m) variables will be all zero.

(b) **Degenerate BFS**: A basic feasible solution is called degenerate, if one or more basic variables are zero-valued.

4.3.8 Optimum basic feasible solution

A basic feasible solution is said to be optimum, if it also optimizes (maximizes or minimizes) the objective function.

4.3.9 Unbound solution

If the value of the objective function Z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

4.4 Computational Aspect of Simplex Method for Maximization Problem

Step 1: Formulate the linear programming model. If we have n-decision variables $X_1, X_2, \ldots, X_n$ and m constraints in the problem , then mathematical formulation of L P problem is

Maximize $Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n$

Subject to the constraints:

$$a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n \le b_1$$
$$a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n \le b_2$$
$$\vdots$$
$$a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n \le b_m$$

$$X_1, X_2, \ldots X_n \ge 0$$

Step 2: Express the mathematical model of LP problem in the standard form by adding slack variables in the left–hand side of the constraints and assign zero coefficient to these variables in the objective function.Thus we can restate the problem in terms of equations as follows:

Maximize $Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n \quad +0X_{n+1} \quad +0X_{n+2} + \ldots + 0X_{n+m}$

Subject to the constraints:

$$a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + X_{n+1} = b_1$$
$$a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n + X_{n+2} = b_2$$
$$\cdot$$

$$a_{m1} X_1 + a_{m2} X_2 + \ldots + a_{mn} X_n + X_{n+m} = b_m$$

$$X_1, X_2, \ldots, X_n, \quad X_{n+1}, X_{n+2}, \ldots, X_{n+m} \geq 0$$

Step 3: Design the initial feasible solution .An initial basic feasible solution is obtained by setting $X_1 = X_2 = \ldots = X_n = 0$. Thus, we get $X_{n+1} = b_1$, $X_{n+2} = b_2, \ldots, X_{n+m} = b_m$.

Step 4: Construct the starting (initial) simplex tableau. For computational efficiency and simplicity, the initial basic feasible solution, the constraints of the standard LP problem as well as the objective function can be displayed in a tabular form, called the *Simplex Tableau* as shown below.

Table 4.1 Initial simplex tableau

C_j (contribution per unit) →			c_1	c_2	…	c_n	0	0	…	0	
C_b	Basic Variables B	Value of Basic Variables $b(=X_B)$	Coefficient Matrix				Identify Matrix				Minimum Ratio*
			X_1	X_2	…	X_n	X_{n+1}	X_{n+2}	…	X_{n+m}	
0	s_1	b_1	a_{11}	a_{12}	…	a_{1n}	1	0	…	0	
0	s_2	b_2	a_{21}	a_{22}	…	a_{2n}	0	1	…	0	
.	.	.	.	.	…	.					
.	.	.	.	.	…	.					
.	.	.	.	.	…	.					
0	s_m	b_m	a_{m1}	a_{m2}	…	a_{mn}	0	0	…	1	
Contribution loss per unit:	$Z = \sum C_{Bj} a_{ij}$		0	0	…	0	0	0	…	0	
Net contribution per units:	$\Delta_j = Z_j - C_j$		c_1	c_2	…	c_n	0	0	…	0	

* Negative ratio is not to be considered.

The interpretation of the data in the above '*Tableau*' is given as under. Other simplex tableau will have similar interpretations.

(i) The first row, called the *objective row* of the simplex table indicates the values of C_j (j subscript refer to the column number) which are the coefficients of the (m + n) variables in the objective function. These coefficients are obtained directly from the objective function and the value C_j would remain the same in the succeeding tables. The second row of the table provides the major column headings for the table and these column headings remain unchanged in the succeeding tables of the Simplex Method.

(ii) The first column labelled C_B, also known as *objective column*, lists the coefficient of the current basic variables in the objective function. The second column labelled *'Basic variables'* points out the basic variables in the basis, and in the initial simplex tableau these basic variables are the slack variables. The third column labelled '*Solution values*' (= x_B), indicates the resources or the solution values of the basic variables.

(iii) The *body matrix* (under non-basic variables) in the initial simplex tableau consists of the coefficients of the decision variables in the constraint set.

(iv) The *identity matrix* in the initial simplex tableau represents the coefficient of the slack variables that have been added to the original inequalities to make them equation. The matrix under non-basic variables in the simplex tableau is called *coefficient matrix*. Each simplex tableau contains an identity matrix under the basic variables.

(v) To find an entry in the Z_j row under a column, we multiply the entries of that column by the corresponding entries of C_B – column and add the products, *i.e.*, $Z = \sum C_B \, a_{ij}$.

The Z_j entry under the "*Solution column*" gives the current value of the objective function.

(vi) The final row labeled $\Delta_j = Z_j - C_j$ called the *index* (or net evaluation) row, is used to determine whether or not the current solution is optimal. The calculation of $Z_j - C_j$ row simply involves subtracting each Z_j value from the corresponding C_j value for that column, which is written at the top of that column. Each entry in the Δ_j row represents the net contribution (or net marginal improvement) to the objective function that results by introducing one unit of each of the respective column variables.

Step 5: *Test the Solution for Optimality*. Examine the index row of the above simplex tableau. If all the elements in the index row are positive then the current solution is optimal. If there exist some negative values, the current solution can further be improved by removing one basic variable from the basis and replacing it by some non-basic one.

Step 6: *Revision of the Current Simplex Tableau*. At each iteration, the Simplex Method moves from the current basic feasible solution to a better basic feasible solution. This involves replacing one current basic variable (called the departing variable) by a new non-basic variable (called the entering variable).

(a) Determine which variable to enter into the solution-mix net. One way of doing this is by identifying the column (and hence the variable) with the most negative number in the Δ_j row of the previous table.

(b) Determine the departing variable or variable to be replaced. Next we proceed to determine which variable must be removed from the basis to pave way for the entering variable. This is accomplished by dividing each number in the quantity (or solution values) column by the corresponding number in the pivot column selected in (a), *i.e.,* we compute the respective ratios b_1/a_{1j}, b_2/a_{2j}, …, b_m/a_{mj} (only for those a_{ij}'s; i=1,2,…, m which are strictly positive). These quotients are written in the last column labelled 'Minimum Ratio' of the simplex tableau. The row corresponding to smallest of these non-negative ratios is called the pivot (or key) row and the corresponding basic variable will leave the basis. Let the minimum of { b_1/a_{1j}, b_2/a_{2j}, …, b_m/a_{mj} ;

$a_{ij} > 0\}$ be b_k/a_{kj}, then corresponding variables in the pivot row s_k will be termed as outgoing (or departing) variable in the next tableau to be constructed just after we put an arrow → of type to right of k^{th} row of the simplex tableau 1.

(c) *Identify the pivot number.* The non-zero positive number that lies at the intersection of the pivot column and pivot row of the given table is called the pivot (or key) number. We place a circle around the number.

Step 7: *Evaluate (update) the new solution.* After identifying the entering and departing variable, find the new basic feasible solution by constructing a new simplex tableau from the current one by using the following steps:

(a) Compute new values for the pivot row by simply dividing every element of the pivot row by the pivot number.

(b) New entries in the C_B column and X_B column are entered in the new table of the current solution

(c) Compute new values for each of the remaining rows by using the following formula
New row numbers=Number in old rows-{(corresponding number above or below pivot number)x(corresponding number in the row replaced in (a))}

(d) Test for optimality. Compute the Z_j and index rows as previously demonstrated in the initial simplex tableau. If all numbers in the index row are either zero or positive, an optimal solution has been made attained. *i.e.,* there is no variable which can be introduced in the solution to cause the objective function value to increase.

4. *Revise the solution.* If any of the numbers in the index ($\Delta_j = Z_j - C_j$) row are negative, repeat the entire steps 5 & 6 again until an optimal solution has been obtained.

The above procedure is illustrated through the following example.

Example 1

A firm produces three products A, B, and C each of which passes through three different departments fabrication, finishing, packaging. Each unit of product A requires 3, 4 and 2 hours respectively, B requires 5, 4 and 4 hours respectively and C requires 2, 4 and 5 hours respectively in 3 departments respectively. Every day 60 hours are available in fabrication department, 72 hours in finishing and 100 hours in packaging department. If unit contribution of unit A is Rs. 5, Rs. 10 for B and Rs. 3 for C. Then determine number of units of each product so that total contribution to cost is maximized and also determine if any capacity would remain unutilized.

Solution:

Step 1 : Formulate this as LPP. Let X_1, X_2 and X_3 be the number of units produced of the products A, B and C respectively.

Objective function: $\quad$ $\text{Max } Z = 5X_1 + 10X_2 + 3X_3$

Subject to constraints: $\quad$ $3X_1 + 5X_2 + 2X_3 \leq 60$

$$4X_1 + 4X_2 + 4X_3 \leq 72$$

$$2X_1 + 4X_2 + 5X_3 \leq 100 \qquad X_1, X_2, X_3 \geq 0$$

Step 2 :Now converting into standard form of LPP

$$\text{Max } Z = 5X_1 + 10X_2 + 3X_3 + 0S_1 + 0S_2 + 0S_3$$

$$3X_1 + 5X_2 + 2X_3 + S_1 = 60$$

$$4X_1 + 4X_2 + 4X_3 + S_2 = 72$$

$$2X_1 + 4X_2 + 5X_3 + S_3 = 100 \qquad X_1, X_2, X_3, S_1, S_2, S_3 \geq 0$$

where S_1 ,S_2 and S_3 are slack variables.

Step 3: Find the initial feasible solution .An initial basic feasible solution is obtained by setting $X_1 = 0$, $X_2 = 0$ and $X_3 = 0$. Thus, we get $S_1 = 60$, $S_2 = 72$ and $S_3 = 100$.

Step 4: Construct the starting (initial) simplex tableau.

			Cj →	5	10	8	0	0	0	
	B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
R_1	S_1	0	60	3	**(5)**	2	1	0	0	60/5=12→
R_2	S_2	0	72	4	4	4	0	1	0	72/4=18
R_3	S_3	0	100	2	4	5	0	0	1	100/4=25
			Z_j	0	0	0	0	0	0	
			Δ_j	−5	−10	−8	0	0	0	

Step 5: The most negative value of Δ_j is −10 hence X_2 is the incoming variable (↑) and the least positive minimum ratio is 12 hence S_1 is the outgoing variable (→). The element under column X_2 and row R_1 is the key element i.e. 5 so divide each element of row R_1 by 5 (i.e. $R_1^a \rightarrow \frac{R_1}{5}$). Subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeros in the remaining positions. Performing the row operations

$R_2^b - R_2^a \rightarrow 4R_1^a \rightarrow$ and $R_3^b - R_3^a \rightarrow 4R_1^a$

we get the second **Simplex tableau as**

			Cj →	5	10	8	0	0	0	
	B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	M.R.
R_1^b	X_2	10	12	3/5	1	2/5	1/5	0	0	12/2/5=30
R_2^b	S_2	0	24	8/5	0	12/5	−4/5	1	0	24/12/5=10⟶
R_3^b	S_3	0	52	-2/5	0	17/5	−4/5	0	1	52/17/5=15.294
			Z_j	6	10	4	2	0	0	
			Δ_j	1	0	−4	2	0	0	

Step 6: The most negative value of Δ_j is -4 hence X_3 is the incoming variable (↑) and the least positive minimum ratio is 10 hence S_2 is the outgoing variable (→). The element under column X_2 and row

R_1 is the key element i.e. 5 so divide each element of row R_2^b by 12/5 (i.e. $R^c \rightarrow R^b * 5/12$). Subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeros in the remaining

positions. Performing the row operations $R_1^d \rightarrow R_1^c - \frac{2}{5}R_2^c$ and $R_3^d \rightarrow R_3^c - \frac{17}{5}R_2^c$

we get the third **Simplex tableau as**

		Cj →	5	10	8	0	0	0
B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
X_2	10	8	1/3	1	0	1/3	−1/6	0
X_3	8	10	2/3	0	1	−1/3	5/12	0
S_3	0	18	−8/3	0	0	1/3	−17/12	1
		Z_j	26/3	10	8	2/3	5/3	0
		Δ_j	11/3	0	0	2/3	5/3	0

It is apparent from this table that all $\Delta_j = Z_j - C_j$ are positive and therefore an optimum solutions is reached . So $X_1 = 0$, $X_2 = 8$, $X_3 = 10$

$$Z = 5X_1 + 10X_2 + 8X_3 = 160$$

And also as S_3 is coming out to be 18 so there are 18 hours unutilized in finishing department.

In case the objective function of the given LP problem is to be minimized, then we convert it into a problem of maximization by using

Min. Z* = − Max. (−Z). The procedure of finding optimal solution using Simplex Method is illustrated through the following example:

Example 4.2

Minimize the objective function Z: $X_1 - 3X_2 + 2X_3$

Subject to constraints $3X_1 - X_2 + 3X_3 \leq 7$

$-2X_1 + 4X_2 \leq 12$

$-4X_1 + 3X_2 + 8X_3 \leq 10$

$X_1, X_2, X_3 \geq 0$

Solution

Converting this minimization problem into maximization problem

Objective function Max $Z^{\#}: -X_1 - 3X_2 + 2X_3 + 0S_1 + 0S_2 + 0S_3$

Constraints $3X_1 - X_2 + 3X_3 + S_1 = 7$

$-2X_1 + 4X_2 + S_2 = 12$

$-4X_1 + 3X_2 + 8X_3 + S_3 = 10$

$X_1, X_2, X_3, S_1, S_2, S_3 \geq 0$

Starting Simplex Table

		Cj	−1	3	−2	0	0	0	
B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	Min. Ratio
S_1	0	7	3	−1	3	1	0	0	-
S_2	0	12	−2	4	1	0	1	0	3 →
S_3	0	10	−4	3	8	0	0	1	10/3
		Δ_j	1	−3	2	0	0	0	

Now performing the row operations $R_2 \to \frac{R_2}{4}$, $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 3 \times R_2$

		Cj	−1	3	−2	0	0	0	Min. Ratio
B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	
S_1	0	10	5/2	0	3	1	1/4	0	4 →
X_2	3	3	−1/2	1	0	0	1/4	0	-
X_3	0	1	−5/2	0	8	0	−3/4	1	-
		Δ_j	-1/2	0	2	0	3/4	0	

Now performing the row operations $R_1^1 \to R_1^1 \times \frac{2}{5}$, $R_2^1 \to R_2^1 + \frac{1}{2} \times R_1^1$ and $R_3^1 \to R_3^1 + \frac{5}{2} \times R_1^1$

and

↑ B.V.	C_B	Cj X_B	−1 X_1	3 X_2	−2 X_3	0 S_1	0 S_2	0 S_3
X_1	−1	4	1	0	6/5	2/5	1/10	0
X_2	3	5	0	1	3/5	1/5	3/10	0
S_3	0	11	0	0	11	1	−1/2	1
		Δ_j	0	0	13/5	1/5	4/5	0

Now all Δ_j are positive. Therefore, Optimal Solution is reached

Therefore, $X_1 = 4$, $X_2 = 5$ & $X_3 = 0$

& Max $Z^{\#} = -1 \times 4 + 3 \times 5 = 11$

Or Minimum $Z = -Z^{\#} = -11$

Lesson 5

DEFINITION AND MATHEMATICAL FORMULATION

5.1 Introduction

The Transportation Problem (TP) is a special type of LP problem where the objective is to minimize the cost of distributing a single commodity from a number of supply sources (e.g. factories) to a number of demand destinations (e.g. warehouses). The objective of the problem is to determine the amount to be transported from each source to each destination so as to maintain the supply and demand requirements at the lowest transportation cost. The characteristic of a TP are such that it is usually solved by a specialized method rather than by simplex method. A key problem in many projects is the allocation of scarce resources among various activities. TP refers to a planning model that allocates resources, machines, materials, capital etc. in the best possible manner so that the costs are minimized or profits are maximized.

5.2 Formulation of Transportation Problem

Transportation problem is an important class of the linear programming problem in which, the objective is to transport various quantities of a single homogenous commodity that are stored at various origins to different destinations in such a way that the transportation cost is minimum. Transportation problem arises in situations involving physical movements of goods e.g. milk and milk products from plants to cold storages, cold storages to wholesalers, wholesalers to retailers and retailers to customers. The solution of a TP is to determine the quantity to be shifted from each plant to each cold storage so as to maintain the supply and demand requirements at the lowest transportation cost.

Let us suppose that there are **m** origins (dairy plants) supplying a certain homogeneous dairy product to **n** destinations (cold storages). Let plant P_i (i = 1, 2, …,m) produce a_i units and cold storage W_j (j = 1, 2,. . ., n) requires b_j units. Suppose that the cost of transporting from plant Pi to cold storage W_j is directly proportional to the amount/quantity transported and let C_{ij} be the cost of transporting one unit of product from i^{th} origin to j^{th} destination and X_{ij} be the amount/quantity transported from i^{th} origin to j^{th} destination. The objective is to determine the number of units to be transported from i^{th} origin to j^{th} destination such that the transportation cost $\left(\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}X_{ij}\right)$ is minimized. For balanced transportation problem, it is assumed that the total supply equals to the total demand. The transportation problem in a tabular form can be represented as follows:

Table 5.1 Tabular representation of transportation problem

Dairy Plants/ Sources	Warehouses (Cold storage)/Destinations							Dairy Plant Capacity
	W_1	W_2	…..	…..	W_j	…..	W_n	
P_1	$X_{11}(C_{11})$	$X_{12}(C_{12})$	….	….	$X_{1j}(C_{1j})$	……	$X_{1n}(C_{1n})$	a_1

P_2	$X_{21}(C_{21})$	$X_{22}(C_{22})$	....	......	$X_{2j}(C_{2j})$	.....	$X_{2n}(C_{2n})$	a_2
:	:	:	:	:	:	:		:
P_i	$X_{i1}(C_{i1})$	$X_{i2}(C_{i2})$	:	:	$X_{ij}(C_{ij})$	:	$X_{in}(C_{in})$	a_i
:	:	:	:	:	:	:		:
P_m	$X_{m1}(C_{m1})$	$X_{m2}(C_{m2})$	....	....	$X_{mj}(C_{mj})$	......	$X_{mn}(C_{mn})$	a_m
Cold storage Requirements	b_1	b_2	....	.....	b_j	.....	bn	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Where C_{ij} and X_{ij} are unit cost of transportation and quantity transported respectively in the cell (i, j). Then the sum of the product of X_{ij} and C_{ij} of allocated cells $\left(\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}X_{ij}\right)$

gives us the net cost in transporting X_{ij} units from plant Pi to cold storage W_j.

The following example will help to demonstrate the formulation of the transportation problem:

Example 1 :

Milk in a milk shed area is collected on three routes A, B and C. There are four chilling centers P, Q, R and S where milk is kept before transporting it to a milk plant. Each route is able to supply on an average one thousand litres of milk per day. The supply of milk on routes A, B and C are 150, 160 and 90 thousand litres respectively. Daily capacity in thousand litres of chilling centers is 140, 120, 90 and 50 respectively. The cost of transporting 1000 litres of milk from each route (source) to each chilling center (destination) differs according to the distance. These costs (in Rs.) are shown in the following table:

Routes	Chilling centers			
	P	Q	R	S
A	16	18	21	12
B	17	19	14	13
C	32	11	15	10

The problem is to determine how many thousand liters of milk is to be transported from each route on daily basis in order to minimize the total cost of transportation.

Solution

A transportation problem can be formulated as a linear programming problem. To illustrate this, let us see how this example can be formulated as a linear programming problem. Let X_{ij} represent the quantity transported from i^{th} route to j^{th} chilling center. These are the decision variables. In this example there are twelve decision variables. Let C_{ij} represents the cost of transported thousand litres of milk from i^{th} route to j^{th} chilling center. The objective is to find the values for the X_{ij} so as to minimize total transportation cost. Thus the LP objective function is:

$$Minimise\ \ Z = C_{11}X_{11} + C_{12}X_{12} + C_{13}X_{13} + C_{14}X_{14} + C_{21}X_{21} + C_{22}X_{22} + C_{23}X_{23} + C_{24}X_{24} + C_{31}X_{31} + C_{32}X_{32} + C_{33}X_{33} + C_{34}X_{34}$$

The supply of milk on routes A, B and C and daily capacity in thousand litres of chilling centers P, Q, R and S impose constraints. The total quantity transported from route A must be equal to its capacity i.e. 150 thousand litres milk. Thus, for route A, the constraint is:

$$X_{11} + X_{12} + X_{13} + X_{14} = 150$$

Similarly, the constraint for other two routes B and C can be expressed as under:

$$X_{21} + X_{22} + X_{23} + X_{24} = 160$$

$$X_{31} + X_{32} + X_{33} + X_{34} = 90$$

Similarly, we must satisfy the demand for each of the four chilling centers. The units can be transported through route A, B and C. Thus, for Chilling centre P, the constraint is:

$$X_{11} + X_{21} + X_{31} = 140$$

Similarly, constraints for the other three chilling centers Q, R and S

$$X_{12} + X_{22} + X_{32} = 120\ ;$$

$$X_{13} + X_{23} + X_{33} = 90$$

$$X_{14} + X_{24} + X_{34} = 50$$

Finally, all values of X_{ij} must be greater than or equal to zero, as negative units cannot be transported. Thus,

$X_{ij} \geq 0$ for i=1,2,3 ; j=1,2,3,4.

Thus the linear programming formulation of the transportation problem becomes:

$$Minimize\ Z = C_{11}X_{11} + C_{12}X_{12} + C_{13}X_{13} + C_{14}X_{14} + C_{21}X_{21} + C_{22}X_{22} + C_{23}X_{23} + C_{24}X_{24} + C_{31}X_{31} + C_{32}X_{32} + C_{33}X_{33} + C_{34}X_{34}$$

Subject to

Supply constraints	**Demand Constraints**
$X_{11} + X_{12} + X_{13} + X_{14} = 150$	$X_{11} + X_{21} + X_{31} = 140$
$X_{21} + X_{22} + X_{23} + X_{24} = 160$	$X_{12} + X_{22} + X_{32} = 120$
$X_{31} + X_{32} + X_{33} + X_{34} = 90$	$X_{13} + X_{23} + X_{33} = 90$
	$X_{14} + X_{24} + X_{34} = 50$

and $X_{ij} \geq 0$ for i=1,2,3 ; j=1,2,3,4.

5.3 Mathematical Formulation of the Transportation Problem

Using the notations described in the previous section, the transportation problem consist of finding X_{ij} (i =1, 2, . . ., m; j=1, 2, . . . , n) in order to minimize the total transportation cost Z = $\left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}X_{ij}\right)$

It is assumed that the total availabilities $\sum_{i=1}^{m} a_i$ equals to the total requirements $\sum_{j=1}^{n} b_j$ i.e. $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ (rim condition). The problem now is to determine non–negative values of X_{ij} (≥ 0) which satisfy both the availability constraints $\sum_{i=1}^{m} X_{ij} = a_i \ \forall i$ and requirement constraints $\sum_{i=1}^{m} X_{ij} = b_i \ \forall i$

Such types of problems where supply and demand are exactly equal are known as *Balanced Transportation Problem***.** Supply (from various sources) is written in the rows, while a column is an expression for the demand of different warehouses. In general, if a transportation problem has **m** rows and **n** columns, then the problem is solvable if there are exactly (m + n –1) basic variables. However, a transportation problem is *unbalanced* if the total supply and the total demand are not equal.

Lesson 6

INITIAL BASIC FEASIBLE SOLUTION

6.1 Introduction

In the last lesson we have learnt about Transportation Problem (TP) and its formulation. Transportation problem can be solved by simplex method and transportation method. In simplex method the solution is very lengthy and cumbersome process because of the involvement of a large number of decision and artificial variables. In this lesson we will look for an alternate solution procedure called transportation method in which initial basic feasible solution of a TP can be obtained in a better way by exploiting the special structure of the problem.

6.2 Some Definitions

The following terms are to be defined with reference to Transportation Problem

6.2.1 Feasible solution (FS)

By feasible solution we mean a set of non-negative individual allocations ($X_{ij} \geq 0$) which satisfies the row and column conditions (rim condition).

6.2.2 Basic feasible solution (BFS)

A feasible solution is said to be basic if the number of positive allocations equals m+n-1; that is one less than the number of rows and columns in a transportation problem.

6.2.3 Optimal solution

A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.

6.3 Solution for Transportation Problem

The solution algorithm to a transportation problem can be summarized into following steps:

Step 1: Formulate the problem. The formulation of transportation problem is similar to a LP problem formulation. Here the objective function is to minimize the total transportation cost and the constraints are the supply and demand available at each source and destination, respectively.

Step 2: Obtain an initial basic feasible solution. This initial basic feasible solution can be obtained by using any of the following five methods:

a) North West Corner Rule
b) Minimum cost method
c) Row Minimum Method
d) Column Minimum Method
e) Vogel's Approximation Method

The solution obtained by any of the above methods must fulfill the following conditions:

i. The solution must be feasible, i.e., it must satisfy all the supply and demand constraints. This is called **rim condition**.

ii. The number of positive allocations must be equal to m+n–1, where, m is number of rows and n is number of columns.

The solution that satisfies both the above mentioned conditions is called a non-degenerate basic feasible solution.

Step 3: Test the initial solution for optimality. Using any of the following methods one can test the optimality of an initial basic solution:

i. Stepping Stone Method

ii. Modified Distribution Method **(MODI)**

If the solution is optimal then stop, otherwise, find a new improved solution.

Step 4: Updating the solution. Repeat Step 3 until the optimal solution is obtained.

6.4 Methods of Obtaining an Initial Basic Feasible Solution

Five methods are described to obtain the initial basic feasible solution of the transportation problem. These methods can be explained by considering the following example

Example 1 :

Let us consider the formulation of TP as given in example 1 of the previous lesson which can be represented by the following e transportation table:

Routes	Chilling centers				Route Capacity
	P	Q	R	S	
A	16	18	21	12	150
B	17	19	14	13	160
C	32	11	15	10	90
Chilling Centre Capacity	140	120	90	50	400

Total supply and demand

Each cell in the table represents the amount transported from one route to one chilling center. The amount placed in each cell is, therefore, the value of a decision. The smaller box within each cell contains the unit transportation cost for that route.

6.4.1 North west corner rule (NWC) method

Step I: The first assignment is made in the cell occupying the upper left-hand (North West) corner of the transportation table. The maximum feasible amount is allocated there. That is X_{11} = Min (a_1, b_1) and this value of X_{11} is then entered in the cell (1, 1) of the transportation table.

Step II: a) If $b_1 > a_1$, we move down vertically to the second row and make the second allocation of magnitude X_{21}=Min. (a_2, b_1- X_{11}) in the cell (2, 1).

b) If $b_1 < a_1$ we move horizontally to the second column and make the second allocation of magnitude X_{12} = Min. (a_1–X_{11}, b_2) in the cell (1, 2).

c) If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude X_{12} = Min. $(a_1–a_1, b1) = 0$ in the cell (1, 2) or X_{21}=Min. $(a_2, b_1–b_1) = 0$ in the cell (2, 1)

Step III: Repeat steps I & II by moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

6.4.1.1 Illustration of North –West Corner Rule:

Let us illustrate the above method with the help of example 1given above. Following North-West corner rule, the first allocation is made in the cell (1,1), the magnitude being X_{11}=Min.(150,140)=140. The second allocation is made in cell (1,2) and the magnitude of allocation is given by X_{12}=Min. (150-140,120)=10. Third allocation is made in the cell (2,2), the magnitude being X_{22}=Min. (160,120-10)=110. The magnitude of fourth allocation in the cell (2,3) is given by X_{23}=Min. (160-110, 90)=50. The fifth allocation is made in the cell (3,3), the magnitude being X_{33}= Min. (90-50, 90)=40. The sixth allocation in the cell (3,4) is given by X_{34}=Min. (90-40, 50) = 50. Now all requirements have been satisfied and hence an initial basic feasible solution to T.P. has been obtained.

Routes	Chilling centers								Route Capacity
	P		Q		R		S		
A	140	16	10	18		21		12	150
B		17	110	19	50	14		13	160
C		32		11	40	15	50	10	90
Chilling Centre Capacity	140		120		90		50		400

The transportation cost according to the above allocation is given by $z = 16 \times 140 + 18 \times 10 + 19 \times 110 + 14 \times 50 + 15 \times 40 + 10 \times 50 = 6310$

6.4.2 Matrix Minima (Least Cost Entry) Method

Various steps of Matrix Minima method are given below:

Step I: Determine the smallest cost in the cost matrix of the transportation table. Let it be C_{ij}. Allocate X_{ij} = Min. (a_i, b_j) in the cell (i, j).

Step II: a) If $X_{ij} = a_i$, cross off the i^{th} row of the transportation table and decrease b_j by a_i and go to Step III.

b) If $X_{ij} = b_j$, cross off the j^{th} column of the transportation table and decrease a_i by b_j and go to Step III.

c) If $X_{ij} = a_i = b_j$, cross of either the i^{th} row or the j^{th} column but not the both.

Step III: Repeat steps I & II for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

6.4.2.1 Illustration of Matrix Minima (Least Cost Entry) Method:

Let us illustrate this method by considering example 1given earlier in this lesson . The first allocation is made in the cell (3, 4) where the cost of transportation is minimum, the magnitude being X_{34}=50. This satisfies the requirement of S chilling centre. Therefore cross off the fourth column. The second allocation is made in the cell (3,2) having minimum cost among remaining cells and its magnitude being X_{32}=Min.(120,90-50)=40. This satisfies the supply of route C. Cross off the third row. Among the remaining cells, the minimum cost is found in cell (2, 3) so the third allocation is done in cell (2, 3) and its magnitude being X_{23}=Min.(90,160)=90. The requirement of the chilling center R is fulfilled, hence third column is crossed off. Out of the remaining cells the minimum cost is found in cell (1,1) and its magnitude is X_{11}=Min(140,150)=140, as the requirement of chilling center P is exhausted hence column first is deleted. Among remaining two cells minimum cost is found in cell (1,2) and its magnitude X_{12}=Min(80,150-140)=10 and the last allocation is done in the cell (2,2) and its magnitude is X_{22}=Min(80-10,70)=70. Now all requirements have been satisfied and hence an initial basic feasible solution to T.P. has been obtained and given in the following table.

Routes	Chilling centers								Route Capacity
	P		Q		R		S		
A	140	16	10	18		21		12	150
B		17	70	19	90	14		13	160
C		32	40	11		15	50	10	90
Chilling Centre Capacity	140		120		90		50		400

The transportation cost according to the above allocation is given by $z = 16 \times 140 + 18 \times 10 + 19 \times 70 + 11 \times 40 + 14 \times 90 + 10 \times 50 = 5950$

6.4.3 Row Minima Method

Various steps of Row Minima method are given below:

Step I: The smallest cost in the first row of the transportation table is determined, let it be C_{ij}. Allocate the maximum feasible amount X_{ij} = Min. (a_i, b_j) in the cell (i, j), so that either the capacity of origin O_i is exhausted or the requirement at destination D_j is satisfied or both.

Step II: a) If $X_{ij} = a_1$, so that the availability at origin O_1 is completely exhausted, cross off the first row of the table and move down to the second row.

b) If $X_{ij}=b_j$ so that the requirement at destination D_j is satisfied. Cross off the j^{th} column and reconsider the first row with the remaining availability of origin O_1.

c) If $X_{ij}=a_1=b_j$, the origin capacity of O_1 is completely exhausted as well as the requirement at destination D_1 is completely satisfied. An arbitrary choice is made. Cross off the first row and make the second allocation $X_{1k} = 0$ in the cell (1, k) with C_{1k} being the new column cost in the first row. Cross off the first row and move down to the second row.

Step III: Repeat steps I & II for the resulting reduced transportation problem until the requirements are satisfied.

6.4.3.1 Illustration of Row Minima Method:

Let us illustrate this method by considering Example 1given earlier in this lesson. In the first row smallest cost is in the first row which is in the cell (1,4) allocate minimum possible amount X_{14}=Min(50,150)=50. This exhausts requirement of S chilling centre and thus we cross off the fourth column from the transportation table. In the resulting transportation table, the minimum cost in the first row is 16 in the cell (1, 1), the second allocation is made in the cell (1, 1) with X_{11}=Min (140,150-50) =100. This satisfies the supply of Route A, hence first row is deleted. In the second row of the remaining transportation table, the minimum cost is present in cell (2,3) so third allocation is done X_{23}=Min(90,160)=90.This satisfies the requirement of R chilling centre, hence third column is deleted. The next allocation is made in the cell (2,1) with X_{21}=Min(140-100,160-90)=40.This exhausts the requirement of Route P and so we cross-out the first column .The fifth allocation is made in cell X_{22}=Min(120,70-40)=30.This exhausts the supply of route B and hence it is crossed out. In third row the cell left is (3, 2). Hence the last allocation of amount X_{32}=Min (120-30, 90) =90 is obviously made in the cell (3,2). Now all requirements have been satisfied and hence an initial basic feasible solution of TP has been obtained and given in the following table.

Routes	Chilling centers								Route Capacity
	P		Q		R		S		
A	140	16	10	18		21		12	150
B		17	70	19	90	14		13	160
C		32	40	11		15	50	10	90
Chilling Centre Capacity	140		120		90		50		400

The transportation cost according to the above allocation is given by $z = 16 \times 100 + 12 \times 50 + 17 \times 40 + 19 \times 30 + 14 \times 90 + 11 \times 90 = 5700$

6.4.4 The Column Minima Method

This method is same as Row minima method except that we apply the concept of minimum cost in column instead of row.

6.4.5 Vogel's Approximation Method (VAM)

The Vogel's Approximate Method takes into account not only the least cost C_{ij} but also the costs that just exceed C_{ij}. Various steps of the method are given below:

Step I: For each row of the transportation table identify the smallest and the next to smallest cost. Determine the difference between them for each row. These are called penalties (opportunity cost). Put them along side of the transportation table by enclosing them in parenthesis against the respective rows. Similarly compute these penalties for each column.

Step II: Identify the row or column with the largest penalty among all the rows and columns. If a tie occurs, use any arbitrary tie breaking choice. Let the greatest penalty correspond to i^{th} row and let c_{ij} be the smallest cost in the i^{th} row. Allocate the largest possible amount X_{ij} = min. (a_i, b_j) in the (i, j) of the cell and cross off the i^{th} row or j^{th} column in the usual manner.

Step III: Re-compute the column and row penalty for the reduced transportation table and go to step II. Repeat the procedure until all the requirements are satisfied.

Remarks:

1. A row or column 'difference' indicates the minimum unit penalty incurred by failing to make an allocation to the smallest cost cell in that row or column.
2. It will be seen that VAM determines an initial basic feasible solution which is very close to the optimum solution that is the number of iterations required to reach optimum solution is minimum in this case.

6.4.2 Illustration of Vogel's Approximation Method (VAM):

Let us illustrate this method by considering Example 1 discussed before in this lesson .

Routes	Chilling centers				Route Capacity	Penalties
	P	Q	R	S		
A	16	18	21	12	150	(4)
B	17	19	14	13	160	(1)
C	32	11	15	10	90	(1)
Chilling Centre Capacity	140	120	90	50	400	
Penalties	(1)	(7) ↑	(1)	(2)		

For each row and column of the transportation table determine the penalties and put them along side of the transportation table by enclosing them in parenthesis against the respective rows and beneath the corresponding columns. Select the row or column with the largest penalty i.e. (7) (marked with an arrow) associated with second column and allocate the maximum possible amount to the cell (3,2) with minimum cost and allocate an amount X_{32}=Min(120,90)=90 to it. This exhausts the capacity of route C. Therefore, cross off third row. The first reduced penalty table will be:

Routes	Chilling centers				Route Capacity	Penalties
	P	Q	R	S		
A	16	18	21	12	150	(4)
B	17	19	14	13	160	(1)
Chilling Centre Capacity	140	30	90	50	310	
Penalties	(1)	(1)	(7) ↑	(1)		

In the first reduced penalty table the maximum penalty of rows and columns occurs in column 3, allocate the maximum possible amount to the cell (2,3) with minimum cost and allocate an amount X_{23}=Min(90,160)=90 to it . This exhausts the capacity of chilling center R .As such cross off third column to get second reduced penalty table as given below.

Routes	Chilling centers			Route Capacity	Penalties
	P	Q	S		
A	16	18	12	150	(4) →
B	17	19	13	70	(4)
Chilling Centre Capacity	140	30	50	220	
Penalties	(1)	(1)	(1)		

In the second reduced penalty table there is a tie in the maximum penalty between first and second row. Choose the first row and allocate the maximum possible amount to the cell (1,4) with minimum cost and allocate an amount X_{14}=Min(50,150)=50 to it . This exhausts the capacity of chilling center S so cross off fourth column to get third reduced penalty table as given below:

Routes	Chilling centers		Route Capacity	Penalties
	P	Q		
A	16	18	100	(2) →
B	17	19	70	(2)
Chilling Centre Capacity	140	30	170	
Penalties	(1)	(1)		

The largest of the penalty in the third reduced penalty table is (2) and is associated with first row and second row. We choose the first row arbitrarily whose min. cost is C_{11} = 16. The fourth allocation of X_{11}=min. (140, 100) =100 is made in cell (1, 1). Cross off the first row. In the fourth reduced penalty table i.e. second row, minimum cost occurs in cell (2,1) followed by cell (2,2) hence allocate X_{21}=40 and X_{22}=30. Hence the whole allocation is as under:

Routes	Chilling centers				Route Capacity
	P	Q	R	S	
A	100 | 16	| 18	| 21	50 | 12	**150**
B	40 | 17	30 | 19	90 | 14	| 13	**160**
C	| 32	90 | 11	| 15	| 10	**90**
Chilling Centre Capacity	140	120	90	50	400

The transportation cost according to the above allocation is given by $z = 16 \times 100 + 12 \times 50 + 17 \times 40 + 19 \times 30 + 14 \times 90 + 11 \times 90 = 5700$

NOTE: Generally, Vogel's Approximation Method is preferred over the other methods because the initial BFS obtained is either optimal or very close to the optimal solution.

Lesson 7

OPTIMAL SOLUTION

7.1 Introduction

After examining the initial basic feasible solution, the next step is to test the optimality of basic feasible solution. Though the solution obtained by Vogel's method is not optimal, yet the procedure by which it was obtained often yields close to an optimal solution. So to say, we move from one basic feasible solution to a better basic feasible solution, ultimately yielding the minimum cost of transportation. There are two methods of testing optimality of a basic feasible solution. The first of these is called the Stepping Stone method and the second method is called Modified Distribution method (MODI).By applying either of these methods, if the solution is found to be optimal, then problem is solved. If the solution is not optimal, then a new and better basic feasible solution is obtained. It is done by exchanging a non-basic variable for one basic variable i.e. rearrangement is made by transferring units from an occupied cell to an empty cell that has the largest opportunity cost and then shifting the units from other related cells so that all the rim requirements are satisfied.

7.2 Some Definitions

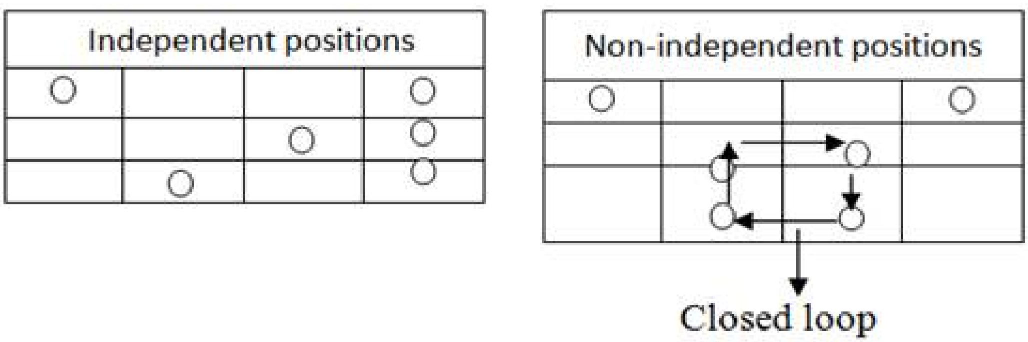
7.2.1 Non-degenerate solution

A basic feasible solution of an m x n transportation problem is said to be non-degenerate, if it has the following two properties:

(1) Starting BFS must contain exactly (m + n - 1) number of individual allocations.

(2) These allocations must be in independent positions.

Here by independent positions of a set of allocations we mean that it is always impossible to form closed loops through these allocations. The following table show the non-independent and independent positions indicated by the following diagram:



7.2.2 Degeneracy

If the feasible solution of a transportation problem with m origins and n destinations has fewer than m+n-1 positive X_{ij} (occupied cells), the problem is said to be a degenerate transportation problem. Degeneracy can occur at two stages:

a) At the initial stage of Basic Feasible Solution.

b) During the testing of the optimal solution.

To resolve degeneracy, we make use of artificial quantity.

7.2.3 Closed path or loop

This is a sequence of cells in the transportation tableau such that

a) each pair of consecutive cells lie in either the same row or the same column.

b) no three consecutive cells lie in the same row or column.

c) the first and last cells of a sequence lie in the same row or column.

d) no cell appears more than once in the sequence.

7.3 Stepping Stone Method

In this method, the net cost change can be obtained by introducing any of the non-basic variables (unoccupied cells) into the solution. For each such cell find out as to what effect on the total cost would be if one unit is assigned to this cell. The criterion for making a re-allocation is simply to know the desired effect upon various costs. The net cost change is determined by listing the unit costs associated with each cell and then summing over the path to find the net effect. Signs are alternate from positive (+) to negative (-) depending upon whether shipments are being added or subtracted at a given point. A negative sign on the net cost change indicates that a cost reduction can be made by making the change while a positive sign will indicate a cost increase. The stepping stone method for testing the optimality can be summarized in the following steps:

Step

1. Determine an initial basic feasible solution.

2. Make sure that the number of occupied cells is exactly equal to m+n-1, where m is number of rows and n is number of columns.

3. Evaluate the cost–effectiveness of transporting goods through the transportation routes not currently in solution. The testing of each unoccupied cell is conducted by following four steps given as under:

   a. Select an unoccupied cell, where transportation should be made. Beginning with this cell, trace a closed path using the most direct route through at least three occupied cells and moving with only horizontal and vertical moves. Further, since only the cells at the turning points are considered to be on the closed path, both unoccupied and occupied boxes may be skipped over. The cells at the turning points are called Stepping Stone on the path.

   b. Assign plus (+) and minus (-) sign alternatively on each corner cell of the closed path traced starting a plus sign at the unoccupied cell to be evaluated.

   c. Compute the net change in the cost along the closed path by adding together the unit cost figures found in each cell containing a plus sign and then subtracting the unit cost in each square containing the minus sign.

   d. Repeat sub step (a) through sub step (b) until net change in cost has been calculated for all unoccupied cells of the transportation table.

4. Check the sign in each of the net changes .If all net changes computed are greater than or equal to zero, an optimal solution has been reached. If not, it is possible to improve the current solution and decrease total transportation cost.

5. Select the unoccupied cell having the highest negative net cost change and determine the maximum number of units that can be assigned to a cell marked with a minus sign on the closed path, corresponding to this cell. Add this number to the unoccupied cell and to other cells on the path marked with a plus sign. Subtract the number from cells on the closed path marked with a minus sign.

6. Go to step 2 and repeat the procedure until we get an optimal solution.

To demonstrate the application of this method let us take example 1 discussed in lesson 6 for consideration.

Example 1: Find optimal solution of the problem given in example 1 of lesson 6 by stepping stone method.

Solution :

The initial basic feasible solution using Least Cost Method was found in example 1 of previous lesson with the following allocations.

Routes	Chilling centers				Route Capacity
	P	Q	R	S	
A	140 16	10 18	21	12	150
B	17	70 19	90 14	13	160
C	32	40 11	15	50 10	90
Chilling Centre Capacity	140	120	90	50	400

Closed Path at cell AR

The Transportation cost according to the above allocation is given by

$$z = 16 \times 140 + 18 \times 10 + 19 \times 70 + 11 \times 40 + 14 \times 90 + 10 \times 50 = 5950$$

Beginning at first unoccupied cell (A,Q) trace a closed path and this closed path is

$AR \rightarrow BR \rightarrow BQ \rightarrow AQ$. Assign plus (+) and minus (-) sign alternatively on each corner cell of the closed path traced starting a plus sign at the unoccupied cell .Evaluate the cost–effectiveness of transporting milk on different transportation routes of each unoccupied cell and given in the following table:

Unoccupied cell	Closed Path	Net Cost change (in Rs.)
(A,R)	$AR \rightarrow BR \rightarrow BQ \rightarrow AQ$	21-14+19-18= 8
(A,S)	$AS \rightarrow CS \rightarrow CQ \rightarrow AQ$	12-10+11-18= -5
(B,P)	$BP \rightarrow BQ \rightarrow AQ \rightarrow AP$	17-16+18-19= 0
(B,S)	$BS \rightarrow CS \rightarrow CQ \rightarrow AQ$	13-10+11-19= -5
(C,P)	$CP \rightarrow AP \rightarrow AQ \rightarrow CQ$	32-16+18-11= 23
(C,R)	$CR \rightarrow BP \rightarrow BQ \rightarrow CQ$	15-11+19-14= 9

Select the unoccupied cell having the highest negative net cost change i.e. cell (A,S) . The maximum number of units that can be transported to a cell marked with a minus sign on the closed path is 10. Add this number to the unoccupied cell and to other cells on the path marked with a plus sign. Subtract the number from cells on the closed path marked with a minus sign. New solution is given in the following table.

Routes	Chilling centers								Route Capacity
	P		Q		R		S		
A	140	16		18		21	10	12	150
B		17	70	19	90	14		13	160
C		32	50	11		15	40	10	90
Chilling Centre Capacity	140		120		90		50		400

The transportation cost according to the above allocation is given by

$$z = 16 \times 140 + 12 \times 10 + 19 \times 70 + 14 \times 90 + 11 \times 50 + 10 \times 40 = 5900$$

Beginning at the first unoccupied cell (A,Q) and repeating the steps given above. Evaluate the cost–effectiveness of transporting milk on different transportation routes of each unoccupied cell given in the following table:

$$CR \rightarrow BR \rightarrow BQ \rightarrow CQ$$

Unoccupied cell	Closed Path	Net Cost change (in Rs.)
(A,Q)	$AQ \rightarrow AS \rightarrow CS \rightarrow CQ$	18-12+10-11= 5
(A,R)	$AR \rightarrow AS \rightarrow CS \rightarrow CQ \rightarrow BQ \rightarrow BR$	21-12+10-11+19-14= 13
(B,P)	$BP \rightarrow BQ \rightarrow CQ \rightarrow CS \rightarrow AS \rightarrow AP$	17-19+11-10+12-16= -5
(B,S)	$BS \rightarrow CS \rightarrow CQ \rightarrow BQ$	13-10+11-19= -5
(C,P)	$CP \rightarrow AP \rightarrow AS \rightarrow CS$	32-16+12-10= 18
(C,R)	$CR \rightarrow BR \rightarrow BQ \rightarrow CQ$	15-11+19-14= 9

Select the unoccupied cell having the highest negative net cost change i.e. cell (B,S) . The maximum number of units that can be transported to a cell marked with a minus sign on the closed path is 40. Add this number to the unoccupied cell and to other cells on the path marked with a plus sign. Subtract the number from cells on the closed path marked with a minus sign. New solution is given in the following table.

Routes	Chilling centers				Route Capacity
	P	Q	R	S	
A	140 (16)	(18)	(21)	10 (12)	150
B	(17)	30 (19)	90 (14)	40 (13)	160
C	(32)	90 (11)	(15)	(10)	90
Chilling Centre Capacity	140	120	90	50	400

The transportation cost according to the above allocation is given by

$$z = 16 \times 140 + 12 \times 10 + 19 \times 30 + 14 \times 90 + 13 \times 40 + 11 \times 90 = 5700$$

Beginning at the first unoccupied cell (A,Q) and repeating the steps given above. Evaluate the cost–effectiveness of transporting milk on different transportation routes of each unoccupied cell and given in the following table:

Unoccupied cell	Closed Path	Net Cost change (in Rs.)
(A,Q)	$AQ \rightarrow AS \rightarrow BS \rightarrow BQ$	18-12+13-19= 0
(A,R)	$AR \rightarrow AS \rightarrow BS \rightarrow BR$	21-12+13-14= 8
(B,P)	$BP \rightarrow AP \rightarrow AS \rightarrow BR$	17-16+12-13= 0
(C,P)	$CP \rightarrow AP \rightarrow AS \rightarrow BS \rightarrow BQ \rightarrow CQ$	32-16+12-13+14-11= 18
(C,R)	$CR \rightarrow BR \rightarrow BQ \rightarrow CQ$	15-11+19-14= 9
(C,S)	$CS \rightarrow BS \rightarrow BQ \rightarrow CQ$	10-13+19-11= 5

Since all the unoccupied cells have positive values for the net cost change, hence optimal solution has reached and optimal cost is Rs. 5700.

7.4 MODI (Modified Distribution) Method

The MODI (Modified Distribution) method is an efficient method of testing the optimality of a transportation solution. In stepping stone method each of the empty cells is evaluated for the opportunity cost by drawing a closed loop. In situations where a large number of sources and destinations are involved, this would be a very time consuming exercise. This method avoids this kind of extensive scanning and reduces the number of steps required in the evaluation of the empty cells. This method allows us to compute improvement indices quickly for each unused square without drawing all of the closed paths. Because of this, it can often provide considerable time savings over other methods for solving transportation problems. It provides new means of finding the unused route with the largest negative improvement index. Once the largest index is identified, we are required to trace only one closed path. This path helps determine the maximum number of units that can be transported via the best unused route. Steps involved for finding out the optimal solution of transportation problem are as follows:

1. Determine an initial basic feasible solution using any one of the **five methods** discussed earlier. We start with a basic feasible solution consisting of (m + n – 1) allocations in independent positions.
2. Determine a set of (m +n) numbers u_r (r = 1, 2, 3, . . . , m) and v_s (s = 1, 2, 3, . . . ,n), such that, for each occupied cell (r, s) $c_{rs} = u_r + v_s$
3. Compute the opportunity cost using $d_{rs} = c_{rs} - (u_r + v_s)$.
4. Check the sign of each opportunity cost. If the opportunity costs of all the unoccupied cells are either positive or zero, the given solution is the optimum solution. On the other hand, if one or more unoccupied cell has negative opportunity cost, the given solution is not an optimum solution and further savings in transportation cost are possible.
5. Select the unoccupied cell with the smallest negative opportunity cost as the cell to be included in the next solution.
6. Draw a closed path or loop for the unoccupied cell selected in the previous step, using the most direct route through at least three occupied cells and moving with only horizontal and vertical moves.
7. Assign alternate plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.
8. Determine the maximum number of units that should be transported to this unoccupied cell. The smallest value with a negative position on the closed path indicates the number of units that can be transported to the entering cell. Now, add this quantity to all the cells on the corner points of the closed path marked with plus signs and subtract it from those cells marked with minus signs. In this way an unoccupied cell becomes an occupied cell.
9. Repeat the whole procedure until an optimum solution is obtained.

Let us illustrate this method by taking example 1 into consideration.

Example 2 : Find optimal solution of the problem given in example 1 of lesson 6 by MODI method.

Solution :

The initial basic feasible solution using Least Cost Method was found in example 1 of previous lesson with the following allocations.

Routes	Chilling centers				Route Capacity
	P	Q	R	S	
A	140 [16]	10 [18]	[21]	[12]	150
B	[17]	70 [19]	90 [14]	[13]	160
C	[32]	40 [11]	[15]	50 [10]	90
Chilling Centre Capacity	140	120	90	50	400

The transportation cost according to the above allocation is given by

$z = 16 \times 140 + 18 \times 10 + 19 \times 70 + 11 \times 40 + 14 \times 90 + 10 \times 50 = 5950$

Step 1: Since the number of occupied cells are m+n-1=3+4-1=6 the initial solution is non –degenerate. Thus optimal solution can be obtained.

Step 2: We first set up an equation for each occupied cell:

$u_1 + v_1 = 16$	$\Rightarrow u_1 = 0 \text{ and } v_1 = 16$
$u_1 + v_2 = 18$	$\Rightarrow u_1 = 0 \text{ and } v_2 = 18$
$u_2 + v_2 = 19$	$\Rightarrow u_2 = 1 \text{ and } v_2 = 18$
$u_2 + v_3 = 14$	$\Rightarrow u_2 = 1 \text{ and } v_3 = 13$
$u_3 + v_2 = 11$	$\Rightarrow u_3 = -7 \text{ and } v_2 = 18$
$u_3 + v_4 = 10$	$\Rightarrow u_3 = -7 \text{ and } v_4 = 17$

$$d_{13} = 21 - (0 + 13) = 8$$

Step 3: After the row and column numbers have been computed, the next step is to evaluate the opportunity cost for each of the unoccupied cells by using the relationship

$d_{rs} = c_{rs} - (u_r + v_s)$

Unoccupied cell	Opportunity Cost
(A,R)	$d_{13} = 21 - (0+13) = 8$
(A,S)	$d_{14} = 12 - (0+7) = -5$
(B,P)	$d_{21} = 17 - (1+16) = 0$
(B,S)	$d_{24} = 13 - (1+17) = -5$
(C,P)	$d_{31} = 32 - (-7+16) = 23$
(C,R)	$d_{33} = 15 - (-7+13) = 9$

Step 4: Select the unoccupied cell having the highest negative net cost change i.e. cell (A,S). The maximum number of units that can be transported to a cell marked with a minus sign on the closed path is 10. Add this number to the unoccupied cell and to other cells on the path marked with a plus sign. Subtract the number from cells on the closed path marked with a minus sign. New solution is given in the following table

Routes	Chilling centers				Route Capacity
	P	Q	R	S	
A	140 [16]	[18]	[21]	10 [12]	150
B	[17]	70 [19]	90 [14]	[13]	160
C	[32]	50 [11]	[15]	40 [10]	90
Chilling Centre Capacity	140	120	90	50	400

The transportation cost according to the above allocation is given by

$z = 16 \times 140 + 12 \times 10 + 19 \times 70 + 14 \times 90 + 11 \times 50 + 10 \times 40 = 5900$

To test this solution for further improvement, we recalculate the values of u_r and v_s based on the second solution for each of the occupied cells.

$u_1 + v_1 = 16$	$\Rightarrow u_1 = 0$ *and* $v_1 = 16$
$u_1 + v_4 = 12$	$\Rightarrow u_1 = 0$ *and* $v_4 = 12$
$u_2 + v_2 = 19$	$\Rightarrow u_2 = 6$ *and* $v_2 = 13$
$u_2 + v_3 = 14$	$\Rightarrow u_2 = 6$ *and* $v_3 = 8$
$u_3 + v_2 = 11$	$\Rightarrow u_3 = -2$ *and* $v_2 = 13$
$u_3 + v_4 = 10$	$\Rightarrow u_3 = -2$ *and* $v_4 = 12$

Again, we again calculate the opportunity costs for each unoccupied cell by using the relationship
$d_{rs} = c_{rs} - (u_r + v_s)$

Unoccupied cell	Opportunity Cost
(A,Q)	d_{12} = 18-(0+13) = 5
(A,R)	d_{13} = 21-(0+8) = 13
(B,P)	d_{21} = 17 – (6+16) = -5
(B,S)	d_{24} =13 – (6+12) = -5
(C,P)	d_{31} = 32-(-2+16) = 18
(C,R)	d_{33} = 15 – (-2+8) = 9

Since the opportunity cost in the cell (B,P) and (B,S) are negative , the current basic feasible solution is not optimal and can be improved . Choosing the unoccupied cell (B,S), we trace a closed path that begins and ends at this cell. The maximum number of units that can be transported to a cell marked with a minus sign on the closed path is 40. Add this number to the unoccupied cell and to other cells on the path marked with a plus sign. Subtract the number from cells on the closed path marked with a minus sign. New solution is given in the following table.

Routes	Chilling centers				Route Capacity
	P	Q	R	S	
A	140 [16]	[18]	[21]	10 [12]	150
B	[17]	30 [19]	90 [14]	40 [13]	160
C	[32]	90 [11]	[15]	[10]	90
Chilling Centre Capacity	140	120	90	50	400

To test this solution for further improvement, we recalculate the values of u_r and v_s based on the second solution for each of the occupied cells.

$u_1 + v_1 = 16$	$\Rightarrow u_1 = 0 \; and \; v_1 = 16$
$u_1 + v_4 = 12$	$\Rightarrow u_1 = 0 \; and \; v_4 = 12$
$u_2 + v_2 = 19$	$\Rightarrow u_2 = 1 \; and \; v_2 = 18$
$u_2 + v_3 = 14$	$\Rightarrow u_2 = 1 \; and \; v_3 = 13$
$u_2 + v_4 = 13$	$\Rightarrow u_2 = 1 \; and \; v_4 = 12$
$u_3 + v_2 = 11$	$\Rightarrow u_3 = -7 \; and \; v_2 = 18$

Again, we calculate the opportunity costs for each unoccupied cell by using the relationship

$$d_{rs} = c_{rs} - (u_r + v_s)$$

Unoccupied cell	Opportunity Cost
(A,Q)	$d_{12} = 18-(0+18) = 0$
(A,R)	$d_{13} = 21-(0+13) = 8$
(B,P)	$d_{21} = 17 - (1+16) = 0$
(C,P)	$d_{31} = 32-(-7+16) = 23$
(C,R)	$d_{32} = 15 - (-7+13) = 9$
(C,S)	$d_{34} = 10 - (-7+12) = 5$

In the above table the opportunity costs for each unoccupied cell is positive value, we conclude that solution is optimal one and the transportation cost is

$z = 16 \times 140 + 12 \times 10 + 19 \times 30 + 14 \times 90 + 13 \times 40 + 11 \times 90 = 5700$

7.5 Maximization in a Transportation Problem

Although the transportation problems which were dealt so far were of minimization, we may also come across of transportation problems with maximization objectives. When origins are related to destinations by a profit function instead of cost, the objective is to be maximize instead of minimize. The most convenient way to deal with the maximization case is to transform the profits to relative costs which are carried out by subtracting the profit associated with each cell from the largest profit in the matrix. Once the optimal solution for the minimization problem is obtained, the value of objective function is determined with reference to the original profit matrix.

Lesson 8

INTRODUCTION AND MATHEMATICAL FORMULATION

8.1 Introduction

In earlier module, transportation problem and the technique of solving such a problem was discussed. In this lesson, the Assignment Problem, which is a special type of transportation problem, is introduced. Here the objective is to minimize the cost or time of completion of a number of jobs by a number of persons. In other words, when the problem involves the allocation of n different facilities to n different tasks, it is often termed as Assignment Problem. The assignment problem deals in allocating the various origins (jobs) to equal number of destinations (persons) on a one to one basis in such a way that the resultant effectiveness is optimized (minimum cost or maximum profit). This is useful in solving problems such as assigning men to different operations in a milk plant, milk tankers to delivery routes, machine operators to machines, jobs to persons in a dairy plant etc.

8.2 Definition of Assignment Problem

Assignment problem is special class of the transportation problem in which the supply in each row represents the availability of a resource such as man, vehicle, product and demand in each column represents different activities to be performed, such as jobs, routes, milk plants respectively is required. The name 'Assignment Problem 'originates from the classical problem where the objective is to assign a number of origins (jobs) to equal number of destinations(persons) at a minimum cost (or Maximum profit). Suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degree of efficiency. Let C_{ij} be the cost if i^{th} person is assigned the j^{th} job, the problem is to find an assignment so that the total cost for performing all jobs is minimum. One of the important characteristics of assignment problem is that only one job (or worker) is assigned to one machine (or project). Hence, the number of sources are equal to the number of destinations and each requirement and capacity value is exactly one unit.

The assignment problem can be stated in the form n x n cost matrix [C_{ij}] of real number as given below

Sources (Milk plants)	Jobs					
	J_1	J_2	……	J_j	……	J_n
P_1	C_{11}	C_{12}	……	C_{1j}	……	C_{1n}
P_2	C_{21}	C_{22}	……	C_{2j}	……	C_{2n}
:	:	:	……	:	……	:
P_i	C_{i1}	C_{i2}	……	C_{ij}	……	C_{in}
:	:	:	……	:	……	:

P_n	C_{n1}	C_{n2}	……	C_{nj}	……	C_{nn}

Formulation of an Assignment Problem

Let us consider the case of a milk plant which has three jobs to be done on the three available machines. Each machine is capable of doing any of the three jobs. For each job the cost depends on the machine to which it is assigned. Costs incurred by doing various jobs on different machines are given below

Job	Machine		
	I	II	III
A	7	8	6
B	5	4	9
C	2	5	6

The problem of assigning jobs to machines, one to each, so as to minimize total cost of doing all the jobs, is an assignment problem. Each job machine combination which associates all jobs to machines on one –to-one basis is called an assignment. In the above example let us write all the possible assignments

Number	Assignment	Total Cost
1	Job A-Machine I, Job B –Machine II, Job C-Machine III	7+8+6=21
2	Job A-Machine I, Job B –Machine III, Job C-Machine II	7+9+5=21
3	Job A-Machine II, Job B –Machine III, Job C-Machine I	8+9+2=19
4	Job A-Machine II, Job B –Machine I, Job C-Machine III	8+5+6=19
5	Job A-Machine III, Job B –Machine I, Job C-Machine II	6+5+2=13
6	Job A-Machine III, Job B –Machine II, Job C-Machine I	6+4+2=12

As per the above assignment, the assignment number 6 having total cost 12 is minimum therefore needs to be selected. But selecting assignment in this manner is quite time consuming.

8.3 Mathematical Formulation of Assignment Problem

Using the notations described above, the assignment problem consist of finding the values of X_{ij} in order to minimize the total cost

$$Minimize\ Z = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij} \quad i = 1,2,\ldots,n\ ;\ j = 1,2,\ldots,n$$

Subject to restrictions

$$X_{ij} = \begin{cases} 1 \text{ if the } i^{th} \text{person is assigned to } j^{th} \text{ job} \\ \qquad\qquad 0 \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} X_{ij} = 1 (only\ one\ job\ is\ done\ by\ the\ i^{th}\ person)$$

$$\sum_{i=1}^{n} X_{ij} = 1 (only\ one\ person\ should\ be\ assigned\ the\ j^{th}\ job)$$

where X_{ij} denotes the j^{th} job to be assigned to the i^{th} person. An assignment problem could thus be solved by Simplex Method.

We state below, the following theorems which have potential applications in finding out of the optimal solution for assignment problems:

Theorem 1: Reduction Theorem

It states that in an assignment problem, if we add or subtract a constant to every element of any line (row or column) of the cost matrix $[C_{ij}]$, then an assignment that minimizes the total cost on one matrix also minimizes the total cost on the other matrix.

Theorem 2: In an assignment problem with cost (C_{ij}), if all $C_{ij} \geq 0$ then a feasible solution (X_{ij}) which satisfies, $\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} = 0$ is optimal for the problem

Remarks:

There are situations when a particular assignment may not be permissible. In such situations assign a very high cost (say M) for such an assignment and proceed as usual.

1. If the assignment problem involves maximization, convert the effective matrix to an opportunities loss matrix by subtracting each element from the highest element of the matrix. Minimization of the resulting matrix is the same as the maximization of the original matrix.

8.4 Similarity of Assignment Problem to Transportation Problem

The assignment problem is a particular case of transportation problem in which a number of operations are to be assigned to an equal number of operators, where each operator performs only one operation. The objective is to maximize overall profit or minimize overall cost for a given assignment schedule.

The Assignment Problem is a special case of the transportation problem in which m=n, All a_i and b_j are unity i.e., The availability and requirement at i^{th} origin and j^{th} destination are unity, and each X_{ij} is limited to one of the two values 0 and 1. Under these circumstances, exactly n of X_{ij} can be non-zero (i.e., unity), one in each row of the table and one in each column.

An assignment problem is a completely degenerate form of a transportation problem. The units available at each origin and units demanded at each destination are all equal to one. This means exactly one occupied cell in each row and each column of the transportation table. i.e., only n occupied cells in place of the required (n + n – 1) = (2n – 1).

Lesson 9

SOLUTION OF ASSIGNMENT PROBLEM

9.1 Introduction

Although assignment problem can be solved either by using the techniques of Linear Programming or by the transportation method yet the assignment method developed by D. Konig, a Hungarian mathematician known as the Hungarian method of assignment problem is much faster and efficient. In order to use this method, one needs to know only the cost of making all the possible assignments. Each assignment problem has a matrix (table) associated with it. Normally, the objects (or people) one wishes to assign are expressed in rows, whereas the columns represent the tasks (or things) assigned to them. The number in the table would then be the costs associated with each particular assignment. It may be noted that the assignment problem is a variation of transportation problem with two characteristics firstly the cost matrix is a square matrix and secondly the optimum solution for the problem would be such that there would be only one assignment in a row or column of the cost matrix.

9.2 Solution of Assignment Problem

The assignment problem can be solved by the following four methods:

a) Complete enumeration method

b) Simplex Method

c) Transportation method

d) Hungarian method

9.2.1 Complete enumeration method

In this method, a list of all possible assignments among the given resources and activities is prepared. Then an assignment involving the minimum cost, time or distance or maximum profits is selected. If two or more assignments have the same minimum cost, time or distance, the problem has multiple optimal solutions. This method can be used only if the number of assignments is less. It becomes unsuitable for manual calculations if number of assignments is large

9.2.2 Simplex Method

This can be solved as a linear programming problem as discussed in section 8.1.3 of the last lesson and as such can be solved by the simplex algorithm.

9.2.3 Transportation method

As assignment is a special case of transportation problem, it can also be solved using transportation model discussed in module 3. The solution obtained by applying this method would be degenerate. This is because the optimality test in the transportation method requires that there must be $m+n-1= (2n-1)$ basic variables. For an assignment problem of order $n \times n$ there would be only n basic variables in the solution because here n assignments are required to be made. This degeneracy problem of solution makes the transportation method computationally inefficient for solving the assignment problem.

9.2.4 Hungarian assignment method

The Hungarian method of assignment provides us with an efficient means of finding the optimal solution. The Hungarian method is based upon the following principles:

(i) If a constant is added to every element of a row and/or column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original problem or vice versa.

(ii) The solution having zero total cost is considered as optimum solution.

Hungarian method of assignment problem (minimization case) can be summarized in the following steps:

Step I: Subtract the minimum cost of each row of the cost (effectiveness) matrix from all the elements of the respective row so as to get first reduced matrix.

Step II: Similarly subtract the minimum cost of each column of the cost matrix from all the elements of the respective column of the first reduced matrix. This is first modified matrix.

Step III: Starting with row 1 of the first modified matrix, examine the rows one by one until a row containing exactly single zero elements is found. Make any assignment by making that zero in or enclose the zero inside a. Then cross (X) all other zeros in the column in which the assignment was made. This eliminates the possibility of making further assignments in that column.

Step IV: When the set of rows have been completely examined, an identical procedure is applied successively to columns that is examine columns one by one until a column containing exactly single zero element is found. Then make an experimental assignment in that position and cross other zeros in the row in which the assignment has been made.

Step V: Continue these successive operations on rows and columns until all zeros have been either assigned or crossed out and there is exactly one assignment in each row and in each column. In such case optimal assignment for the given problem is obtained.

Step VI: There may be some rows (or columns) without assignment i.e. the total number of marked zeros is less than the order of the matrix. In such case proceed to step VII

Step VII: Draw the least possible number of horizontal and vertical lines to cover all zeros of the starting table. This can be done as follows:

1. Mark (√) in the rows in which assignments has not been made.
2. Mark column with (√) which have zeros in the marked rows.
3. Mark rows with (√) which contains assignment in the marked column.
4. Repeat 2 and 3 until the chain of marking is completed.
5. Draw straight lines through marked columns.
6. Draw straight lines through unmarked rows.

By this way we draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. It should, however, be observed that in all n x n matrices less than n lines will cover the zeros only when there is no solution among them. Conversely, if the minimum number of lines is n, there is a solution.

Step VIII: In this step, we

1. Select the smallest element, say X, among all the not covered by any of the lines of the table; and
2. Subtract this value X from all of the elements in the matrix not covered by lines and add X to all those elements that lie at the intersection of the horizontal and vertical lines, thus obtaining the second modified cost matrix.

Step IX: Repeat Steps IV, V and VI until we get the number of lines equal to the order of matrix I, till an optimum solution is attained.

Step X: We now have exactly one encircled zero in each row and each column of the cost matrix. The assignment schedule corresponding to these zeros is the optimum assignment. The above technique is explained by taking the following examples

Example 1:

A plant manager has four subordinates, and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. This estimate of the times each man would take to perform each task is given in the effectiveness matrix below.

	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man hours?

Solution:

Step I : Subtracting the smallest element in each row from every element in that row, we get the first reduced matrix.

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Step II: Next, we subtract the smallest element in each column from every element in that column; we get the second reduced matrix.

Step III: Now we test whether it is possible to make an assignment using only zero distances.

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

(a) Starting with row 1 of the matrix, we examine rows one by one until a row containing exactly single zero elements are found. We make an experimental assignment (indicated by) to that cell. Then we cross all other zeros in the column in which the assignment was made.

(b) When the set of rows has been completely examined an identical procedure is applied successively to columns. Starting with Column 1, we examine columns until a column containing exactly one remaining zero is found. We make an experimental assignment in that position and cross other zeros in the row in which the assignment was made. It is found that no additional assignments are possible. Thus, we have the complete 'Zero assignment',

A – I, B – III, C – II, D – IV

The minimum total man hours are computed as

Optimal assignment	Man hours
A – I	8
B – III	4
C – II	19
D – IV	10
Total	**41 hours**

Example 2:

A dairy plant has five milk tankers I, II, III, IV & V. These milk tankers are to be used on five delivery routes A, B, C, D, and E. The distances (in kms) between dairy plant and the delivery routes are given in the following distance matrix

	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How the milk tankers should be assigned to the chilling centers so as to minimize the distance travelled?

Solution :

Step I: Subtracting minimum element in each row we get the first reduced matrix as

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60

20	0	35	45	70

Step II: Subtracting minimum element in each column we get the second reduced matrix as

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Step III: Row 1 has a single zero in column 2. We make an assignment by putting ‘ ‘around it and delete other zeros in column 2 by marking ‘X’. Now column1 has a single zero in column 4 we make an assignment by putting ‘ ‘and cross the other zero which is not yet crossed. Column 3 has a single zero in row 2; we make an assignment and delete the other zero which is uncrossed. Now we see that there are no remaining zeros; and row 3, row 5 and column 4 has no assignment. Therefore, we cannot get our desired solution at this stage.

30	[0]	35	30	15		√
15	X	[0]	10	X	L_2	
30	X	35	30	20		√
[0]	X	20	X	5	L_3	
20	X	25	15	15		√
	√ L_1					

Step IV: Draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once by using the following procedure

1. Mark (√) row 3 and row 5 as having no assignments and column 2 as having zeros in rows 3 and 5.
2. Next we mark (√) row 2 because this row contains assignment in marked column 2. No further rows or columns will be required to mark during this procedure.
3. Draw line L_1 through marked col.2.
4. Draw lines L_2 & L_3 through unmarked rows.

Step V: Select the smallest element say X among all uncovered elements which is X = 15. Subtract this value X=15 from all of the values in the matrix not covered by lines and add X to all those values that lie at the intersections of the lines L_1, L_2 & L_3.

Applying these two rules, we get a new matrix

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5

5	0	10	0	0

Step VI: Now reapply the test of Step III to obtain the desired solution.

15	~~0~~	20	15	[0]
15	15	[0]	10	~~0~~
15	[0]	20	15	5
[0]	15	20	~~0~~	5
5	~~0~~	10	[0]	~~0~~

The assignments are

A → V B → III C → II D → I E → I

Total Distance 200 + 130 + 110 + 50 + 80 = 570

Lesson 10

INTRODUCTION AND GENERAL NOTATIONS

10.1 Introduction

Inventory, in broad sense, is a stock of physical assets having some economic value, which can be either in the form of material, money or men and is defined as any idle resource of an enterprise. It is a physical stock of goods kept for the future purposes. The term is generally used to indicate raw material, work-in-progress (intermediate good), finished goods, packaging material and other stock in order to meet an expected demand or distribution in future as well as day to day functioning of any organization. Though inventory of materials is an idle resource, it is not meant for immediate use, it is always essential to maintain inventory of goods for smooth functioning of an enterprise. For example, suppose a milk plant does not have Inventory of materials at all. If this plant receives a sales order, then

- **i)** It will have to order out raw material required to complete the order.
- **ii)** It has to wait till the material arrives and then start production.
- **iii)** Customers have to wait too long for the delivery of finished goods.
- **iv)** The plant may have to purchase raw materials at very high prices which will increase the cost of production and decrease the margin of profit.

Centuries ago, inventories were viewed as measures of the wealth and power of a country or of an individual. Now-a-days, inventories are viewed as a large potential risk rather than as a measure of wealth due to the fast developments and changes in product life, therefore, it has necessitated the use of scientific techniques in the management of inventories known as inventory control. It is the technique of maintaining stock-items at desired levels. Different departments within the same organization adopt different attitudes towards inventory. This is mainly because the particular functions performed by a department influence the department's motivation. For example, the sales department might desire large stock in reserve to meet virtually every demand that comes. The production department similarly would ask for large stocks of materials so that the production system runs uninterrupted. On the other hand, the finance department would always argue for a minimum investment in stocks so that the funds could be used elsewhere for other better purpose.

10.2 Inventory (Production Management)

The study or function of directing the movements of goods through the entire manufacturing cycle from the raw material to the inventory of finished goods orderly mannered to meet the objective of maximum customers service with minimum investment and efficient plant operation. This process is known as inventory control. Inventory may be defined as stock of goods, commodities or other economic resources that are stored or reserved in order to ensure smooth and efficient running of business. Inventory, for example may include raw material which are kept in stock for in the production of goods (raw material inventory), Semi finished goods or goods in process which are stored during the production process (Work in progress inventory), finished goods awaiting shipment from plant, wholesaler etc. (finished goods inventory). Inventory also includes furniture, machinery fixtures etc. The term inventory is generally classified into two categories:

10.2.1 Direct inventory

Direct inventories play a direct role in the manufacturing and become a bigger part of finished goods. They are further classified into three groups:

i) Raw material inventories are provided :

- **a)** for economical bulk purchasing.
- **b)** to enable production rate changes.
- **c)** to provide production buffer against transportation.
- **d)** seasonal fluctuations.

ii) Work in progress inventories are provided:

- **a)** to enable economical lot production.
- **b)** to consider the variety of production for replacement of base.
- **c)** to maintain uniform production even if amount of sales may vary.

iii) Finished goods inventories are provided:

- **a)** for maintaining of self delivery.
- **b)** to allow stabilization of production level.
- **c)** for sales promotion.

10.2.2 Indirect inventory

They include those items which are necessary for manufacturing but do not become component of finished production, such as oil, grease, petrol, lubricant, office material, maintenance material etc.

10.3 Types of Inventory

10.3.1 Fluctuation inventory

These have to be carried because sales and production time can't be predicted accurately. There is fluctuation in the demand and lead times that affect the production of items such type of results stock or safety stock are called fluctuation inventory.

10.3.2 Anticipation inventory

These are built in advance for the season of large sale a promotional programme or a plant shut down period. In this inventory are store for future requirement.

10.3.3 Cycle or lot size inventory

In practical situations it seldom happens that the rate of consumption is same as rate of production so the items are purchased in large quantity than they are required. This results in cycle or lot size inventory.

10.3.4 Transportation inventory

Such inventory exists because the material is required to move from one place to another place when the transportation time is long and the items under transport can't be served to customer. Therefore these inventories exist solely because of time.

10.3.5 Decoupling inventory

Such inventory is needed for meeting of demand for decoupling the different parts of the production system.

10.4 Inventory Decisions

The following are the decisions made for every item of inventory.

i) How much amount of an item should be ordered when the inventory of that item is to be replenished.
ii) When to replenish the inventory of that item

10.5 How to Develop Inventory Control Model?

i) First take physical stock of all inventory items in an inventory organization.
ii) Second, classify all items in various categories.
iii) Each of above classification may be further divided into different groups.
iv) After classification of inventories, each item should be assigned a suitable code.
v) Since the number of items is very large, separate inventory management model should be developed for each category of items A-B-C or V-E-D classification.
vi) Estimates annual demand of each inventory item and their prevailing prices.
vii) Estimate lead time, safety stock and reorder level, if supply is not instantaneous.
viii) Develop the inventory mode.
ix) Finally, review the position and make suitable alterations, if required due to current situations or constraints.

10.6 Costs Involved in Inventory Control Models

10.6.1 Holding cost

Costs associated with carrying or holding goods in stock is known as carrying or holding cost which is denoted by C_1 or C_h per unit of goods for a unit of time, respectively. However cost is assumed to be varying directly the size of inventory as well as the time for which the item is in stock. The following components constitute the holding cost.

i) **Invested capital costs (Interest charged on capital investment):-** This is the most important component so its rate of interest should be investigated carefully**.**

ii) **Record keeping and administrative costs: -** This signifies the need of keeping funds for maintaining of records and necessary administration.

iii) **Handling cost:-** It includes all costs associated with movement of stock such as cost of labour, overhead and other machinery required for this purpose.

iv) **Storage cost: -** This involves the rent of storage space or depreciation in interest even if own space is used.

v) **Depreciation, deterioration or obsolescence cost: -** Such cost is due to the item in stock being out of fashion or the item undergoes chemical changes during storage (e.g. rusting of iron), breakage and spoilage.

vi) **Insurance cost –** The amount of money paid in the form of insurance premium is known as insurance cost. This cost is also a component of stock holding cost.

vii) **Purchase Cost or Production Cost: -** Purchase price per unit item is affected by the purchase due to discount or price break. Production cost per unit item depends upon length of production time.

viii) Selling price or Salvage Cost: - When the demand for an item is affected by its quantities in stock, the decision model of the problem depends upon profit maximum and in revenue (sales tax etc) from the sale of the items. Generally salvage cost is combined with storage cost and not considered independently.

Carrying cost can be determined as

Carrying cost= (Cost of carrying one unit of item for a given length of time) x (Average number of units carried for a given length of time)

10.6.2 Shortage cost or stock out cost

These are penalty costs that are incurred as a result of either delay in meeting the demand or inability to meet it at all due to running out of stock are known as shortage or stock out cost. It is denoted as C_2 or C_s per unit of goods for specified period. This cost arises due to shortage of goods, sales may be lost, good will may be lost either by delay in meeting the demands or being quite unable to meet the demand.

10.6.3 Setup cost

These include the fixed cost associated with obtaining goods through placing of an order or purchasing or manufacturing or setting up of a machine before start of production. So they include cost of purchase, requisition, quality control etc. These are also known as order cost or replenishment cost. It is denoted by C_3.

10.7 Objective of Inventory Control

i) It helps in smooth and efficient running of business.

ii) It provides service to the customers immediately at a short notice.

iii) Due to absence of stock the company has to pay high price because of piecewise purchasing.

iv) It also acts as a buffer stock when raw material is received late or when orders are likely to be rejected.

v) It reduces product cost.

vi) It helps to maintain the economy by absorbing some of the fluctuations and the demand for an item fluctuates or is seasonal.

Lesson 11

ECONOMIC LOT SIZE MODELS WITH KNOWN DEMAND

11.1 Introduction

In the last lesson we have seen that maintenance of proper Inventory Control System helps in keeping the investment in Inventories as low as possible and yet (i) ensures availability of materials by providing adequate protection against supply uncertainty and consumption of materials and (ii) allows full advantages of economies of bulk purchases and transportation costs. The basic Inventory Control problem therefore lies in determining firstly when should an order for materials be placed and secondly how much should be produced at the beginning of each time interval or what quantity of an item should be ordered each time. In this lesson we will learn how to develop inventory model.

11.2 Variables in Inventory Problem

The variables used in any inventory model are of two types: Controlled and Uncontrolled variables

11.2.1 Controlled variables

The following are the variables that may be considered separately or in combination:

- How much quantity acquired
- The frequency or timing of acquisition .How often or when to replenish the inventory?
- The completion stage of stocked items.

11.2.2 Uncontrolled variables

The following are the principal variables that may be controlled:

- The holding costs, shortage or penalty cost, set up costs.
- **Demand:** It is the number of units required per period and may be either known exactly or is known in terms of probabilities or is completely unknown. Further if the demand is known, it may be either fixed or variable per unit of time. The model which has fixed demand is known as deterministic model.
- **Lead Time:** This is the time of placing an order and its arrival in stock as shown in Fig. 11.1. If the lead time is known and not equal to zero and if demand is deterministic then one should order in advance by an amount of time equal to lead time. If the lead time is zero then there is no need to order in advance. If lead time is variable then it is known as probabilistically.

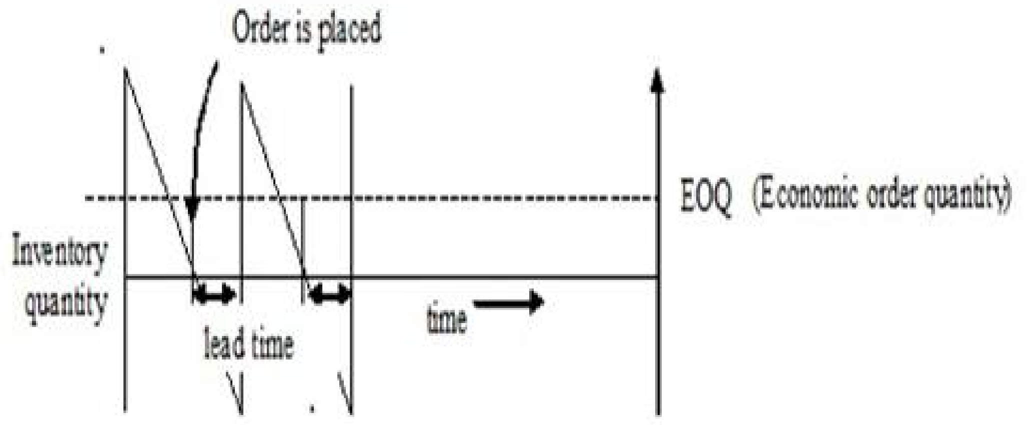


Fig 11.1 Inventory with constant demand rate and constant lead time

- **Amount delivered (supply of goods) :** The supply of goods may be instantaneous or spread over a period of time .If a quantity q is ordered for purchase, the amount delivered may very around g with known probability density function .

11.3 Symbols used in Inventory Models

$C_1 =$ Holding cost per item per unit time

$C_2 =$ Shortage cost per unit quantity per unit time for the back-log case and per unit only for no back log case.

$C_3 =$ Set up cost per production run

$R =$ Demand Rate

$K =$ Production Rate

$t =$ Scheduling time period which is not prescribed.

$t_p =$ Scheduling time period (prescribed)

$z =$ Order Level (or stock level)

$D =$ Total Demand

$Q =$ Quantity already present in the beginning

$L =$ Lead time

11.4 Classification of Inventory Models

The inventory models are broadly classified as either deterministic (variables are known with certainty) or stochastic (variables are probabilistic). The deterministic models are further divided into Static demand models and dynamic demand models. In this lesson we will discuss deterministic inventory model i.e. economic lot size system with uniform demand.

11.5 Economic Order Quantity (EOQ) Model

This concept was developed by F. Haris in 1916. The concept is as lot size (q) increases the carrying charges (C_1) will increase while the ordering cost (C_3) will decrease. On the other hand as the lot size (q) decreases, the carrying cost (C_1) will decrease but the ordering costs will increase. Economic Ordering Quantity (EOQ) is the size of order which minimizes total annual cost of carrying inventory and cost of ordering under the assumed conditions of certainly and annual demands are known.

11.5.1 The economic lot size system with uniform demand

In this model we want to derive an Economic Lot Size Formula for the optimum production quantity **q** per cycle (per production run) of a single product so as to minimize the total average variable cost per unit time. The model is based on the following basic assumptions:

- **i)** the demand for the item is uniform at the rate of R quantity units per unit time
- **ii)** the lead time is known and fixed. Thus, when the lead time is zero, the delivery of item is instantaneous.
- **iii)** production rate is infinite i.e. production is instantaneous
- **iv)** shortages are not allowed
- **v)** holding cost is Rs. C_1 in per quantity unit per unit time
- **vi)** set up cost is Rs. C_3 per set up.

This can be solved by two following methods

11.5.1.1 Algebraic method

The algebraic method is based on the following principle:

Inventory carrying costs=Annual ordering (set up) costs

(11.1)

Since the demand is uniform and known exactly and supply is instantaneous, the reorder point is that when inventory falls to zero.

Average Inventory =1/2(maximum level +minimum level) =(q+0)/2=q/2

Total inventory carrying costs are determined by using the following formula:

$$\begin{pmatrix}\text{Total inventory carrying}\\ \text{costs per unit}\end{pmatrix} = \begin{pmatrix}\text{average no. of units}\\ \text{in inventory}\end{pmatrix} \times \begin{pmatrix}\text{cost of}\\ \text{one unit}\end{pmatrix} \times \begin{pmatrix}\text{inventory carrying}\\ \text{costs percentage}\end{pmatrix}$$

$$=1/2q \times C \times I = 1/2qC_1 \qquad (11.2)$$

where C_1=CI is holding or carrying cost per unit for unit time.

Total annual ordering costs are obtained as follows:

$$\text{Total annual ordering costs} = \begin{pmatrix}\text{number of orders}\\ \text{per year}\end{pmatrix} \times \begin{pmatrix}\text{ordering cost}\\ \text{per order}\end{pmatrix}$$

$$=(R/q) \times C_3 = (R/q)C_3 \qquad (11.3)$$

Now summing up the total inventory carrying cost and total ordering cost we get total inventory cost as

$$\begin{pmatrix}\text{Total ordering}\\ \text{costs}\end{pmatrix} = \begin{pmatrix}\text{Total inventory carrying}\\ \text{costs per unit}\end{pmatrix} + \begin{pmatrix}\text{Total annual}\\ \text{ordering costs}\end{pmatrix}$$

$C(q) = 1/2qC_1 + (R/q)C_3$, this is cost equation . (11.4)

The total inventory cost C(q) is minimum when the inventory carrying costs become equal to the total ordering costs. Therefore

$1/2qC_1=(R/q)C_3$ or $q = \sqrt{\frac{2C_3R}{C_1}}$ (11.5)

Or optimal $q^{*(EOQ)} = \sqrt{\frac{2\times \text{setup cost} \times \text{demand rate}}{\text{carrying cost}}}$

To find the minimum of total inventory cost C(q) , we substitute the value of q from (11.5) in cost equation (11.4) we get

$$C_{min} = \sqrt{2C_1C_3R} \qquad (11.6)$$

Optimum inventory cost (C_{min}) = $\sqrt{2\ (\text{holding cost}) \times (\text{setup cost}) \times (\text{demand rate})}$

To obtain the optimum interval of ordering (t*) we have

$$\begin{pmatrix}\text{Economic ordering}\\ \text{quanity}\end{pmatrix} = \begin{pmatrix}\text{demand}\\ \text{rate}\end{pmatrix} \times \begin{pmatrix}\text{interval of}\\ \text{ordering}\end{pmatrix}$$

$$q = R \times t \qquad (11.7)$$

$$t = \sqrt{\frac{2C_3}{RC_1}} \text{ i.e Optimium ordering interval } (t^*) = \sqrt{\frac{2\times \text{setup cost}}{\text{demand rate} \times \text{holding cost}}}$$

(11.8)

11.5.1.2 Calculus method

Let each production cycle be made at fixed interval ‘t’ and therefore the quantity q already present in the beginning (when the business was started) should be

$$q = R.t \qquad (11.9)$$

where R is the demand rate.

Since the stock in small time dt will be $Rt.\,dt$, therefore the stock in total time t will be

$= \int_0^t Rt.\,dt = \frac{1}{2}Rt^2 = \frac{1}{2}qt$ =Area of the inventory Δ POA as shown in Fig. 11.2

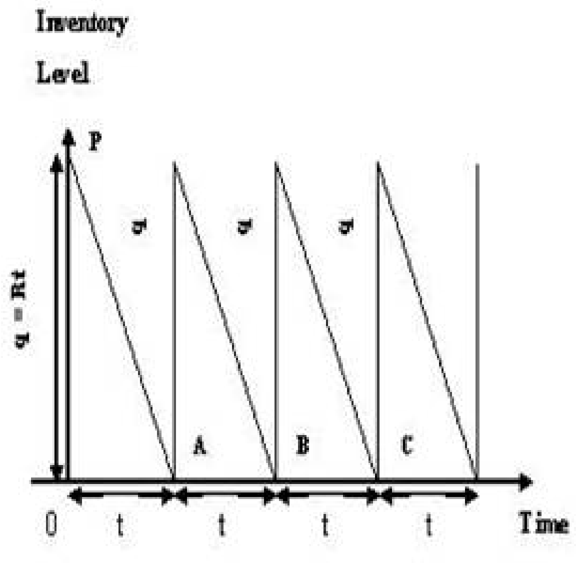


Fig. 11.2 Inventory profile of EOQ model

q=Rt

The rate of replenishment = slope of line OP= $\tan\frac{\pi}{2} \to \infty$

Thus the cost of holiday inventory is = $c_1(\text{Area of } \Delta OPA)$ per production run

$$\text{Total Cost } = C_1\left(\frac{1}{2}Rt^2\right) \tag{11.10}$$

Set up cost is C_3 per production run for interval t. (11.11)

Total cost C(t) is summing up the costs in (11.10) and (11.11) and dividing by t we get the average total cost given by ;

$$c(t) = \frac{1}{2}C_1Rt + \frac{C_3}{t} \tag{11.12}$$

The condition for minimum or maximum of C(t) is

$$\frac{d(c(t))}{dt} = 0 \ \ we\ get \quad \frac{1}{2}C_1R - \frac{C_3}{t^2} = 0$$

$$t = \sqrt{\frac{2C_3}{C_1R}} \tag{11.13}$$

$\frac{d^2c(t)}{dt^2} > 0$ for minimum which is $\frac{2C_3}{t^3}$ which is obviously positive for the value of t given by equation (11.13). Hence C(t)is minimum for

$$t = \sqrt{\frac{2C_3}{C_1R}}$$

Optimin Quantity to be produced at each interval 't'is (11.14)

$$q^* = R.t^* = R\sqrt{\frac{2C_3}{C_1.R}} = \sqrt{\frac{2C_3R}{C_1}} \tag{11.15}$$

which is known as optimal lot size formula.

Therefore

$$C_{min} = \frac{1}{2}C_1R\sqrt{\frac{2C_3}{C_1.R}} + C_3\sqrt{\frac{C_1}{2C_3}} = \sqrt{2C_1C_3R}$$

per unit time is obtained by putting in in equation (11.12) .The above procedure is illustrated through following examples

Example 1: A manufacture has to supply his customer with 600 units of a product per year, shortages are not allowed and the storage costs amounts to 0.60 per unit per year. The set up cost per run is Rs. 80.00. Find the optimum run size and the minimum average yearly cost.

Solution :

In this the demand rate (R)=600 per unit per year

$C_1 =$ Holding cost per item per unit time=Rs. 0.60 per unit per year

$C_3 =$ Set up cost per production run=Rs. 80 per production run

Optimum lot size is $q^* = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2\times80\times600}{0.60}} = 400$

$$Optimium\ ordering\ interval\ (t^*) = \sqrt{\frac{2 \times setup\ cost}{demand\ rate\ \times holding\ cost}}$$

$$= \sqrt{\frac{2\times80}{600\times0.60}}$$

=0.67 years =8 months

Thus the manufacturer should produce 400 units of his product at an interval of 8 months.

Example 2 : A company uses 3000 units of a product, its carrying cost is 30%of average inventory. Ordering cost is Rs. 100 per order. Unit cost is Rs. 20.Calculate EOQ and the total cost .

Solution :

$D =$ Total Demand=3000 units

$C_1 =$ carrying cost =30% of Rs. 20=Rs.6

$C_3 =$ Ordering cost =Rs.100

Optimum lot size is $q^* = \sqrt{\frac{2C_3D}{C_1}} = \sqrt{\frac{2\times3000\times100\times}{6}} = 316.23\ units$

The total cost is equal to

Total cost=Material cost +Total variable cost

=(3000 × 20) + $\sqrt{2\times3000\times100\times6}$ = 60000 + 1897.36 = Rs.61897.36

11.6 Limitations of EOQ Formulae

In spite of several assumptions made in the derivation of above EOQ formulae, following are the limitations while considering applications of these formulae:

i) In the EOQ model we assumed that the demand for the item under consideration is constant, while in practical situations demand is neither known with certainty nor it is uniform.

ii) It is difficult to measure the ordering cost and also it is not linearly related to number of orders.

iii) In EOQ model it is assumed that the annual demand can be estimated in advance which is just a guess in practice.

iv) In EOQ model it is assumed that the entire inventory which is ordered arrives simultaneously. In many situations it may not be true.

v) In EOQ, it is assumed that the demand is uniform; which may not hold in practice.

vi) In EOQ model, the replenishment time is assumed to be zero which is not possible in real life always.

Lesson 12

INTRODUCTION AND ELEMENTARY CONCEPTS

12.1 Introduction

The replacement problems are concerned with the situations that arise when some items such as machines, men, electric appliance etc. need replacement due to their decreased efficiency, failure or breakdown. The deteriorating efficiency or complete breakdown may be either gradual or all of a sudden. A replacement is called for whenever new equipment offers more efficient or economical service than the existing one. The problem in such situation is to determine the best policy to be adopted with respect to replacement of the equipment. In case of items whose efficiency go on decreasing according to their age, we have to spend more and more money on account of increased operating cost, increased repair cost, increased scrap, etc. In such cases the replacement of an old item with a new one is the only alternative to prevent such increased expenses. Thus, it becomes necessary to determine the age at which replacement is more economical rather than continuing with the same.

12.2 Types of Replacement Situations

The replacement situations may be classified into four categories:

a) Replacement of items that become worse with time e.g. milk plant machinery, tools, vehicles, equipment etc.

b) Replacement of items which do not deteriorate with time but break down completely after certain usage e.g. electric tubes, machinery parts etc.

c) Replacement of items that becomes obsolete due to new developments.

d) The existing working staff in an organization gradually reduces due to death, retirement and other reasons.

The problem is to decide the best policy to adopt with regard to replacement. The need for replacement arises in a number of different situations so that different types of decisions may have to be taken. For example:

a) It may be necessary to decide whether to wait for certain items to fail, which might cause some loss, or to replace the same in advance, even at a higher cost.

b) An item can be considered individually to decide whether or not to replace immediately.

c) It is necessary to decide whether to replace by the same item or by an improved type of item.

12.3 Types of Failure

There are two types of failure: i) Gradual failure ii) Sudden failure

12.3.1 Gradual failure

It means slow or progressive failure as the life of the item increases, its efficiency decreases resulting in decreased productivity, increased operating cost and decrease in the value of the item, e.g. machines/equipment etc.

12.3.2 Sudden failure

In this type of failure the items do not deteriorate markedly with service but which ultimately fail after some period of usage, thus precipitating cost of failure. Sometimes sudden failure of an item may cause loss of production or may also account for damaged or faulty products. The period between installation and failure is not constant for any particular type of equipment but will follow some probability distribution which may be progressive, retrogressive or random in nature.

12.3.2.1 Progressive failure

Under this mechanism, the probability of failure increases with the increase in the life of an item.

12.3.2.2 Retrogressive failure

Certain items have more probability of failure in the beginning of their life and as time passes, the chances of failure become less. In other words, the ability of the unit to survive the initial period of life increases its expected life.

12.3.2.3 Random failure

Under this mechanism, constant probability of failure is associated with items that fail from random causes such as physical shocks, not related to age.

12.4 Assumptions

Following assumptions are essentially required for replacement decisions:

i) The quality of the output remains constant.
ii) Replacement and maintenance costs remain constant.
iii) The operational efficiency of the equipment remains constant.
iv) There is no change in technology of the asset under consideration.

12.5 OR Methodology of Solving Replacement Problem

OR provides a methodology for tackling replacement problem which is discussed below:

i) Identify the items to be replaced and also their failure mechanism.
ii) Collect the data relating to the depreciation cost and the maintenance cost for the items which follow gradual failure mechanism. In case of sudden failure of items, collect the data for replacement cost of the failed items.
iii) Select a suitable replacement model as discussed in Lesson 13.

Lesson 13

REPLACEMENT OF ITEMS DETERIORATING WITH TIME

13.1 Introduction

In any establishment, sooner or later equipment needs to be replaced, particularly when new equipment gives more efficient or economical service than the old one. In some cases, the old equipment might fail and work no more or is worn out. In such situations it needs more expenditure on its maintenance than before. The problem in such situation is to determine the best policy to be adopted with respect to replacement of the equipment. The replacement theory provides answer to this question in terms of optimal replacement period. Replacement theory deals with the analysis of materials and machines which deteriorate with time and fix the optimal time of their replacement so that total cost is the minimum.

13.2 Replacement Decisions

The problem is to decide the best policy to adopt with regard to replacement. The need for replacement arises in a number of different following situations so that different types of decisions may have to be taken.

- It may be necessary to decide whether to wait for a certain item to fail which might cause some loss or to replace earlier at the expense of higher cost of the item.
- The item can be considered individually to decide whether to replace now or if not when to reconsider the item in question.
- It is necessary to decide whether to replace by the same item or by a different type of item.

13.3 Types of Replacement Problems

i) Replacement policy for items, efficiency of which declines gradually with time without change in money value.

ii) Replacement policy for items, efficiency of which declines gradually with time but with change in money value.

iii) Replacement policy of items breaking down suddenly

a) Individual replacement policy

b) Group replacement policy

iv) Staff replacement

In this lesson we confine ourselves to first two situations only

13.4 Replacement of Items that Deteriorate with Time

There are certain items which deteriorate gradually with usage and such items decline in efficiency over a period of time. Generally, the maintenance cost of certain items always increase gradually with time and a stage comes when the maintenance cost becomes so large that it is better and economical to replace the item with a new one. There may be number of alternatives and we may have a comparison between various alternatives by considering the costs due to waste, scrap, loss of output, damage to equipment and safety risks etc.

13.4.1 Replacement of items whose maintenance cost increases with time and the value of money remains same during the period

The following costs are considered in such decisions:

C: Capital cost of a certain item say a machine

s(t): The selling or scrap value of the item after t years

f(t): Operating (or maintenance) cost of the item at time t

n : Optimal replacement period of the item

The operating cost function f(t) is assumed to be strictly positive. It may be continuous or discrete.

13.4.2 When t is a continuous variable

Now the annual cost of the machine at time **t** is given by $C - S(t) + f(t)$ and since the total maintenance cost incurred on the machine during **n** years is $\int_0^n f(t)dt$

Total cost T incurred on machine during **n** years is given by

$$T = C - S(t) + \int_0^n f(t)dt$$

Thus the average annual cost incurred on the machine per year during **n** years is given by

$$T_A = \frac{T}{n} = \frac{1}{n}\left[C - S(t) + \int_0^n f(t)dt\right]$$

To determine the value of optimal period (n), the principle of minima will be employed.

$$\frac{d(T_A)}{dn} = 0 \text{ and } \frac{d^2}{dn^2}(T_A) \geq 0$$

$$\frac{d(T_A)}{d_n} = -\frac{1}{n^2}[C - S(t)] - \frac{1}{n^2}\left[\int_0^n f(t)dt\right] + \frac{1}{n}f(n) \text{and equating } \frac{d(T_A)}{d_n} = 0 \text{ we get}$$

$$f(n) = \frac{1}{n}[C - S(t)] + \frac{1}{n}\left[\int_0^n f(t)dt\right], n \neq 0$$

Clearly

$$\frac{d^2}{dn^2}(T_A) \geq 0 \; at \; f(n) = T_A$$

Therefore, it can be seen that $A(n)$ or $T_A = f(n)$ is a minimum for **T** provided that $f(t)$ is non–decreasing and $f(0) = 0$. Hence, if time is measured continuously, then the averageannual cost will be minimized by replacing the item when the average cost becomes equal to the current maintenance cost.

13.4.3 When time is a discrete variable

Here the period of time is considered as fixed and **n** takes values $1, 2, 3, \ldots.$ then

$$A(n) = \frac{T}{n} = \frac{1}{n}[C - S(t) + \Sigma_1^n f(t)]$$

By using finite differences, A(n) will be a minimum for that value of **n** for which

$$A(n+1) \geq A(n) \text{ and } A(n-1) \geq A(n)$$

or $$A(n+1) - A(n) \geq 0 \text{ and } A(n) - A(n-1) \leq 0$$

For this, we write

$$A(n+1) = \frac{1}{n+1}[C - S(t) + \Sigma_1^{n+1} f(t)]$$

$$A(n+1) - A(n) = \frac{1}{n+1}[C - S(t) + \Sigma_1^{n+1} f(t)] - \frac{1}{n}[C - S(t) + \Sigma_1^n f(t)]$$

$$= \frac{f(n+1)}{n+1} - \Sigma_1^n \frac{f(t)}{n(n+1)} - \frac{[C - S(t)]}{n(n+1)}$$

$$A(n+1) - A(n) = \frac{1}{n+1}[f(n+1) - A(n)]$$

$$\text{Thus } A(n+1) - A(n) \geq 0 \Longrightarrow f(n+1) \geq A(n)$$

Similarly,it can be shown

$$A(n) - A(n-1) \leq 0 \Longrightarrow f(n) \leq A(n-1)$$

This suggests the replacement policy that if time is measured in discrete units, then the average annual cost will be minimized by replacing item when the next period's maintenance cost become greater than the current average cost. Hence, replace the equipment at the end of n years, if the maintenance cost in the $(n+1)^{th}$ year is more than the average total cost in the $(n)^{th}$ year and the $(n)^{th}$ year's maintenance cost is less than the previous year's average total cost. The following examples will illustrate this methodology

Example 1: A milk plant is considering replacement of a machine whose cost price is Rs. 12,200 and the scrap value Rs. 200. The running (maintenance and operating) costs in Rs. are found from experience to be as follows:

Year: 1 2 3 4 5 6 7 8

Running Cost: 200 500 800 1200 1800 2500 3200 4000

When should the machine be replaced?

Solution: The computations can be summarized in the following tabular form:

Table 13.1 Calculations for average cost of machine

					(In Rupees)
Year (t) (1)	**Running Cost** $f(t)$ (2)	**Cumulative Running Cost** $\sum f(t)$ (3)	**Depreciation Cost** $(C - S)$ (4)	**Total Cost TC** (5) = (3) + (4)	**Average Cost** $A(n)$ (6) = (5)/(1)
1	200	200	12000	12200	12200
2	500	700	12000	12700	6350
3	800	1500	12000	13500	4500
4	1200	2700	12000	14700	3675
5	1800	4500	12000	16500	3300
6	2500	7000	12000	19000	3167
7	3200	10200	12000	22200	3171
8	4000	14200	12000	26200	3275

From the table it is noted that the average total cost per year, A(n) is minimum in the 6th year (Rs. 3167). Also the average cost in 7th year (Rs.3171) is more than the cost in 6th year. Hence the machine should be replaced after every 6 years.

Example 2:

A Machine owner finds from his past records that the maintenance costs per year of a machine whose purchase price is Rs. 8000 are as given below:

Year: 1 2 3 4 5 6 7 8

Maintenance Cost: 1000 1300 1700 2200 2900 3800 4800 6000

Resale Price: 4000 2000 1200 600 500 400 400 400

Determine at which time it is profitable to replace the machine.

Solution : $C = \text{Rs.}\,8000$ .Table 13.2 shows the average cost per year during the life of machine. Here The computations can be summarized in the following tabular form:

Table 13.2 Calculations for average cost of machine

Year (t)	$f(t)$	**Cumulative maintenance cost** $\sum f(t)$	Scrap value $s(t)$	Total cost $TC = c - s(t) + \sum f(t)$	T_A
1	1000	1000	4000	5000	5000
2	1300	2300	2000	8300	4150
3	1700	4000	1200	10800	3600
4	2200	6200	600	13600	3400
5	2900	9100	500	16600	3200
6	3800	12900	400	20500	3417
7	4800	17700	400	25300	3614
8	6000	23700	400	31300	3913

The above table shows that the value of T_A during fifth year is minimum. Hence the machine should be replaced after every fifth year.

Example 3:

The cost of a machine is Rs. 6100 and its scrap value is only Rs.100. The maintenance costs are found to be

Year:	1	2	3	4	5	6	7	8
Maintenance Cost (in Rs.):	100	250	400	600	900	1250	1600	2000

When should the Machine be replaced?

Solution :

$C = 6100 \quad s(t) = 100$ The computations can be summarized in the following tabular form:

Table 13.3 Calculations for average cost of machine

Replace at the end of year	f(t)	**Cumulative maintenance cost** $\sum f(t)$	Total cost $TC = C - s(t) + \sum f(t)$	T_A
1	100	100	6100	6100
2	250	350	6350	3175
3	400	750	6750	2250
4	600	1350	7350	1737.50
5	900	2250	8250	1650
6	1250	3500	9500	1583.33
7	1600	5100	11100	1585.7
8	2000	7100	13100	1637.50

It is now observed that the machine should be replaced at the end of sixth year otherwise the average cost per year will start to increase.

13.4.4 Replacement of items whose maintenance cost increases with time and the money value changes at a constant rate

To understand this let us define the following terms:

Money Value: Since money has a value over time, therefore the explanation of the statement: 'Money is worth 10% per year' can be given in two ways:

(a) In one way, spending Rs.100 today would be equivalent to spend Rs.110 in year's time. In other words if we plan to spend Rs.110 after a year from now, we could spend Rs.100 today and an investment which would be worth Rs.110 next year.

(b) Alternatively if we borrow Rs.100 at the interest of 10% per year and spend Rs.100 today, we have to pay Rs.100 after one year (next year).

Thus, we conclude that Rs.100 is equal to Rs.110 a year from now. Consequently Rs. 1 from a year now is equal to $(1+0.1)^{-1}$ rupee today.

Present Worth Factor: As we have seen, a rupee a year from now will be equivalent to $(1+0.1)^{-1}$ rupee today at the discount rate of 10% per year. So, one rupee in **n** years from now will be equal to $(1+0.1)^{-n}$. Therefore, the quantity $(1+0.1)^{-n}$ is called the Present Worth Factor (PWF) or Present Value (PV) of one rupee spent in **n** years from now. In general, if **r** is the rate of interest, then $(1+r)^{-n}$ is called PWF or PV of one rupee spent in **n** years from now onwards. The expression $(1+r)^{-n}$ is known as compound amount factor of one rupee spent in **n** years.

Discount Rate: Let **r** be the rate of interest. Therefore present worth factor of unit amount to be spent after one year is $v = \frac{1}{1+r}$. Then **v** is known as the discount rate. The optimum replacement policy for replacement of item where maintenance costs increase with time and money value changes with constant rate can be determined by following method:

Suppose that the item (which may be machine or equipment) is available for use over a series of time periods of equal intervals (say one year). Let

C = Purchase price of the item to be replaced

R_t= Running (or maintenance) cost in the t^{th} year

r=Rate of interest

$v = \frac{1}{1+r}$ is the present worth of a rupee to be spent in a year hence.

Let the item be replaced at the end of every n^{th} year. The year wise present worth of expenditure on the item in the successive cycles of **n** years can be calculated as follows:

Year	1	2----	n	n+1	n+2	----	2n	2n+1
Present worth	$C+R_1$	R_2v	R_nv^{n-1}	$(C+R_1)v^n$	R_2v^{n+1}		R_nv^{2n-1}	$(C+R_1)v^{2n}$

Assuming that machines has no resale value at the time of replacement, the present worth of the machine in **n** years will be given by

$$p(n) = \lfloor(C+R_1)+R_2v+R_3v^2+---+R_nv^{n-1}\rfloor+\lfloor(C+R_1)v^n+R_2v^{n+1}+R_3v^{n+2}+---+R_nv^{2n-1}\rfloor+\lfloor(C+R_1)v^{2n}+R_2v^{2n+1}+R_3v^{2n+2}+---+R_nv^{3n-1}\rfloor+-- \text{and so on}$$

Summing up the right-hand side, column-wise

$$P(n) = (C+R_1)\lfloor 1+v^n+v^{2n}+--\rfloor+R_2v\lfloor 1+v^n+v^{2n}+--\rfloor+-- +R_nv^{n-1}\lfloor 1+v^n+v^{2n}+--\rfloor$$

$$= (C+R_1+R_2v+---+R_nv^{n-1})\lfloor 1+v^n+v^{2n}+--\rfloor$$

$$= (C+R_1+R_2v+---+R_nv^{n-1})\left[\frac{1}{1-v^n}\right]$$, using sum of an infinite G.P.

$$P(n) = \left[\frac{f(n)}{1-v^n}\right] \text{ where } f(n) = (C+R_1+R_2v+---+R_nv^{n-1}),$$

$$(n+1) = \left[\frac{f(n+1)}{1-v^{n+1}}\right]$$

f(n) and f(n+1) given above at n = 0,1,2…, are called the weighted average cost of previous **n** years with weights 1,v, v^2, ----v^{n-1}respectively. P(n) is the amount of money required now to pay all the future costs of acquiring and operating the equipment when it is renewed every **n** years. However, if P (n) is less than P (n+1) then replacing the equipment each **n** year is preferable to replacing each **n** years is preferable to replacing each (n+1) years. Further, if the best policy is replacing every **n** years, then the two inequalities P (n+1) – P (n) > 0 and P (n-1) - P (n) < 0 must hold, without giving the proof we shall state the following two inequalities which holds good at **n**, the optimal replacement interval.

$$P(n) < \frac{(C+R_1)+R_2v+R_3v^2+\cdots\ldots\ldots+R_{n-1}v^{n-2}}{1+v+v^2+\cdots+v^{n-2}}$$

$$\text{and } P(n+1) > \frac{(C+R_1)+R_2v+\cdots\ldots\ldots+R_nv^{n-1}}{1+v+v^2+\cdots+v^{n-1}}$$

As a result of these two inequalities, rules for minimizing costs may be stated as follows:

1. Do not replace if the operating cost of next period is less than the weighted average of previous costs.
2. Replace if the operating cost of the next period is greater than the weighted average of the previous costs.

Working Procedure

The step-by-step procedure for solving the problem is stated as under:

1. Write in a column the running/maintenance costs of machine or equipment for different years, R_n.

2. In the next column write the discount factor indicating the present value of a rupee received after (i-1) years, $v^{n-1} = (1/1+r)^{n-1}$

3. The two column values are multiplied to get present value of the maintenance costs, i.e., $R_n v^{n-1}$.

4. These discounted maintenance costs are then cumulated to the i[th] year to get $\sum R_n v^{n-1}$.

5. The cost of machine or equipment is added to the values obtained in Step 4 above to

Obtain C+ $\sum R_n v^{n-1}$.

6. The discount factors are then cumulated to get $\sum v^{n-1}$.

7. The total costs obtained in (Step 5) are divided by the corresponding value of the accumulated discount factor for each of the years.

8. Now compare the column of maintenance costs which is constantly increasing with the last column. Replace the machine in the latest year that the last column exceeds the column of maintenance costs.

Example 4

A milk plant is offered an equipment A which is priced at Rs.60,000 and the costs of operation and maintenance are estimated to be Rs.10,000 for each of the first 5 years, increasing every year by Rs. 3000 per year in the sixth and subsequent years. If money carries the rate of interest 10% per annum what would the optimal replacement period?

Solution :

Table 13.4 Determination of optimal replacement period

At the end of year (n)	Operating & maintenance cost R_n	Discounted factor v^{n-1}	Discounted operation & maintenance cost $R_n v^{n-1}$	Cumulative Discounted operation & maintenance cost	Discounted total cost $C + \sum R^{n-1}$	Cumulative discounted factor $\sum v^{n-1}$	Weighted average annual cost $\frac{C + R_n v^{n-1}}{\sum v^{n-1}}$
(1)	(2)	(3)	(4)=(2)x(3)	(5)	(6)=(5)+60000	(7)	(8)=(6)+(7)
1	10000	1.0000	10000.00	10000.00	70000.00	1.00	70000.00
2	10000	0.9091	9091.00	19091.00	79091.00	1.91	41428.42
3	10000	0.8264	8264.00	27355.00	87355.00	2.74	31933.83
4	10000	0.7513	7513.00	34868.00	94868.00	3.49	27207.75
5	10000	0.6830	6830.00	41698.00	101698.00	4.17	24389.18
6	13000	0.6209	8071.70	49769.70	109769.70	4.79	22913.08
7	16000	0.5645	9032.00	58801.70	118801.70	5.36	22184.36
8	19000	0.5132	9750.80	68552.50	128552.50	5.87	21905.89
9	22000	0.4665	10263.00	78815.50	138815.50	6.33	21912.82
10	25000	0.4241	10602.50	89418.00	149418.00	6.76	22106.52

From Table 13.4 we find the weighted cost is minimum at the end of 8[th] year, hence the equipment should be replaced at the end of 8[th] year.

Example 5 :

A Manufacturer is offered two machines A and B. Machine A is priced at Rs. 5000 and running cost is estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth and subsequent years. Machine B, with the same capacity as A, costs Rs. 2500, but has running cost of Rs. 1200 per year for six years, thereafter increasing by Rs. 200 per year. If money is worth 10% per year, which machine should be purchased? (Assume that the machines will eventually be sold for scrap at a negligible price).

Solution

Since money is worth 10% per year, therefore discount rate is $v = \frac{1}{(1+0.10)} = 0.9091$

Table 13.5 Computation of weighted average cost for machine A

At the end of year (n)	Operating & maintenance cost R_n	Discounted factor v^{n-1}	Discounted operation & maintenance cost $R_n v^{n-1}$	Cumulative Discounted operation & maintenance cost	Discounted total cost $C+\sum R^{n-1}$	Cumulative discounted factor $\sum v^{n-1}$	Weighted average annual cost $\frac{C+R_n v^{n-1}}{\sum v^{n-1}}$
(1)	(2)	(3)	(4)=(2)x(3)	(5)	(6)=(5)+ 5000	(7)	(8)=(6)+(7)
1	800	1.0000	800	800	5800	1	5800
2	800	0.9091	727	1527	6527	1.9091	3419.035
3	800	0.8264	661	2188	7188	2.7355	2627.819
4	800	0.7513	601	2789	7789	3.4868	2233.98
5	800	0.6830	546	3336	8336	4.1698	1999.098
6	1000	0.6209	621	3957	8957	4.7907	1869.61
7	1200	0.5645	677	4634	9634	5.3552	1799.025
8	1400	0.5132	718	5353	10353	5.8684	1764.13
9	1600	0.4665	746	6099	11099	6.3349	1752.043
10	1800	0.4241	763	6862	11862	6.759	1755.053

From table 13.5 we conclude that for machine A 1600<1752.043<1800. Since the running cost of 9th year is 1600and that of 10th year is 1800 and 1800>1752.043, it would be economical to replace machine A at the end of nine years.

Table 13.6 Computation of weighted average cost for machine B

At the end of year (n)	Operating & maintenance cost R_n	Discounted factor v^{n-1}	Discounted operation & maintenance cost $R_n v^{n-1}$	Cumulative Discounted operation & maintenance cost	Discounted total cost $C+\sum R^{n-1}$	Cumulative discounted factor $\sum v^{n-1}$	Weighted average annual cost $\frac{C+R_n v^{n-1}}{\sum v^{n-1}}$
(1)	(2)	(3)	(4)=(2)x(3)	(5)	(6)=(5)+ 2500	(7)	(8)=(6)+(7)
1	1200	1.0000	1200.00	1200.00	3700.00	1.00	3700.00
2	1200	0.9091	1090.92	2290.92	4790.92	1.91	2509.52
3	1200	0.8264	991.68	3282.60	5782.60	2.74	2113.91
4	1200	0.7513	901.56	4184.16	6684.16	3.49	1916.99
5	1200	0.6830	819.60	5003.76	7503.76	4.17	1799.55
6	1200	0.6209	745.08	5748.84	8248.84	4.79	1721.84
7	1400	0.5645	790.30	6539.14	9039.14	5.36	1687.92
8	1600	0.5132	821.12	7360.26	9860.26	5.87	1680.23
9	1800	0.4665	839.70	8199.96	10699.96	6.33	1689.05
10	2000	0.4241	848.20	9048.16	11548.16	6.76	1708.56

In table13.6 we find that 1800<1689<2300 so it is better to replace the machine B after 8th year. The equivalent yearly average discounted value of future costs is Rs. 1748.60 for machine A and it is 1680.23for machine B. Hence, it is more economical to buy machine B rather than machine A.

Lesson 14

INTRODUCTION AND GENERAL NOTATIONS

14.1 Introduction

Sequencing problems are concerned with an appropriate order (sequence) for a series of jobs to be done on a finite number of service facilities (like machines) in some well-defined technological order so as to optimize some efficiency measure such as total elapsed time or overall cost etc. In such cases, the effectiveness is a function of the order or sequence in which the tasks are performed. The effectiveness may be measured in terms of cost, time or mileage etc. A sequencing problem could involve jobs in a manufacturing plant, aircraft waiting for landing and clearance, maintenance scheduling in a factory, programmers to be run on a computer centre, customers in a bank, and so forth.

14.2 Sequencing

In Sequencing we are concerned with a situation where the effectiveness measure is a function of the order or sequence in which a series of tasks or jobs are performed. Suppose we have **n** jobs (1,2,3,---,n), each of which has to be processed or performed one at the time on each of m machines A,B,C,---. The order (sequence) of processing each job through the machines as well as the actual or expected time required by the jobs on each of the machine is also given. The effectiveness in terms of cost, times or mileage etc. can be measured for any given sequence of jobs at each machine and our aim is to select the most suitable sequence (which optimizes the effectiveness measure) among all theoretical possible sequences whose number will be $(n!)^m$. Although theoretically it is always possible to select the best sequence by testing each one but it is practically impossible due to large number of computations. Hence we have to compute effectiveness for each of $(n!)^m$ sequences before selecting the most suitable one. But this is practically impossible to do. Let us explain as to what the sequencing problem is.

14.2.1 Problem of sequencing

Definition: Suppose there are **n** jobs (1, 2, ---, n) each of which has to be processed one at a time at each of **m** machines A, B, C, ………… The order of processing each job through the machines is given (for example, Job 1 is processed through machines A, C, B, - in that order). The time required for each job at each machine is also given. The problem is to find among $(n!)^m$ number of all possible sequences (or combinations) that sequence (or order) for processing the jobs so that the total elapsed time for all the jobs will be a minimum. Mathematically, let us define:

A_i = Time for i^{th} job on machine A $\forall\, i$

B_i = Time for i^{th} job on machine B $\forall\, i$

T = Time from start of the first job to completion of the last job

We wish to determine for each machine a sequence $(i_1, i_2, \ldots \ldots \ldots \ldots i_n)$ where $(i_1, i_2, \ldots \ldots \ldots \ldots i_n)$ is a permutation of the integers (1,2, .. … … n) which will minimize total elapsed time T.

14.3 Basic Terminology and Notations

The following terminology and notations will be used in this unit:

14.3.1 Number of machines

It means service facilities through which a job must pass before it is completed. In a milk plant, milk packed in pouches has to be processed through boiling, cooling, packaging etc. In this milk packing in pouches constitutes the job and various processes constitute the number of machines.

14.3.2 Processing time

It means the time required by each job on each machine. It is denoted by T_{ij} which means processing time required by the i^{th} machine (i=1, 2, 3---, n; j=1, 2, 3---, m).

14.3.3 Processing order

It refers to the order in which various machines are required for completing the job.

14.3.4 Idle time on a machine

This is the time a machine remains idle during the total elapsed time. Let X_{ij} denote the idle time of machine j between the end of $(i-1)^{th}$ job and the start of i^{th} job.

14.3.5 Total elapsed time

This is the time between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines. This will be denoted by T.

14.3.6 No passing rule

It means that passing is not allowed i.e. same order of jobs is maintained over each machine. If each of the n jobs is to be processed through two machines A and B in the order AB then this means that each job will go to machine A and then to B.

14.4 Basic Assumptions of Sequencing Problem

- No machine can process more than one operation at a time.
- Each job once started on a machine is to be performed up to its completion on that machine.
- A job is an entity i.e. even though the job represents a lot of individual parts, no lot may be processed by more than one machine at a time.
- The time intervals for processing are independent of the order in which the operations are performed.
- There is only one of each type of machine.
- A job is processed as soon as possible subject only to ordering requirements.
- All jobs are known and are ready to start processing before the period under consideration begins.
- The processing times are on different machines are independent of the order of the job in which they are to be processed.
- The time taken by the jobs in moving from one machine to another is very negligible and is taken as equal to zero.

Lesson 15

SOLUTION OF A SEQUENCING PROBLEM

15.1 Introduction

When a number of jobs are given to be done and they require processing on two or more machines, the main concern of a manager is to find the order or sequence to perform these jobs. We shall consider the sequencing problems in respect of the jobs to be performed in a factory and study the method of their solution. Such sequencing problems can be broadly divided in two groups. In the first one, there are n jobs to be done, each of which requires processing on some or all of the k different machines. We can determine the effectiveness of each of the sequences that the technologically feasible (that is to say, those satisfying the restrictions on the order in which each job must be processed through the machines) and choose a sequence which optimizes the effectiveness. To illustrate, the timings of processing of each of the n jobs on each of the **k** machines, in a certain given order, may be given and the time for performing the jobs may be the measure of effectiveness. We shall select the sequences for which the total time taken in processing all the jobs on the machines would be the minimum.

In this unit we will look into solution of a sequencing problem. In this lesson the solutions of following cases will be discussed:

a) n jobs and two machines A and B, all jobs processed in the order AB.
b) n jobs and three machines A, B and C all jobs processed in the order ABC
c) Problems with n jobs and m machines.

15.1.1 Processing of n jobs through two machines

The simplest possible sequencing problem is that of n job two machine sequencing problem in which we want to determine the sequence in which **n**-job should be processed through two machines so as to minimize the total elapsed time T. The problem can be described as:

a) Only two machines A and B are involved;
b) Each job is processed in the order AB.
c) The exact or expected processing times $A_1, A_2, A_3, ---, A_n; B_1, B_2, B_3 ---, B_n$ are known and are provided in the following table

Machine	Job(s)								
	1	2	3	--	-	i	- -	-	n
A	A_1	A_2	A_3	--	-	A_i	--	-	A_n
B	B_1	B_2	B_3	--	-	B_i	--	-	B_n

The problem is to find the sequence (or order) of jobs so as to minimize the total elapsed time T. The solution of the above problem is also known as Johnson's procedure which involves the following steps:

Step 1. Select the smallest processing time occurring in the list $A_1, A_2, A_3, - - -, A_n; B_1, B_2, B_3 - - -, B_n$ if there is a tie, either of the smallest processing times can be selected.

Step 2. If the least processing time is A_r, select the r th job first. If it is B_s, do the sth job last as the given order is AB

Step 3. There are now (n-1) jobs left to be ordered. Repeat steps I and II for the remaining set of processing times obtained by deleting the processing time for both the machines corresponding to the job already assigned.

Step 4. Continue in the same manner till the entire jobs have been ordered. The resulting ordering will minimize the total elapsed time T and is called the optimal sequence.

Step 5. After finding the optimal sequence as stated above find the total elapsed time and idle times on machines A and B as under:

Total elapsed time = The time between starting the first job in the optimal sequence on machine A and completing the last job in the optimal machine B.

Idle time on machine A = (Time when the last job in the optimal sequence on sequences is completed on machine B)-(Time when the last job in the optimal sequences is completed on machine A)

Idle time on machine B = (Time when the first job in the optimal sequences is completed on machine A)+

$\sum_{k=2}^{n}[(time\ when\ k^{th} job\ starts\ on\ machine\ B) - (time\ (k-1)^{st}\ job\ finshed\ on\ machine\ B)]$

The Johnson's procedure can be illustrated by following examples:

Example 1;

There are nine jobs, each of which must go through two machines P and Q in the order PQ, the processing times (in hours) are given below:

Machine	Job(s)								
	A	B	C	D	E	F	G	H	I
P	2	5	4	9	6	8	7	5	4
Q	6	8	7	4	3	9	3	8	11

Find the sequence that minimizes the total elapsed time T. Also calculate the total idle time for the machines in this period.

Solution :

The minimum processing time on two machines is 2 which correspond to task A on machine P. This shows that task A will be preceding first. After assigning task A, we are left with 8 tasks on two machines

Machine	B	C	D	E	F	G	H	I

P	5	4	9	6	8	7	5	4
Q	8	7	4	3	9	3	8	11

Minimum processing time in this reduced problem is 3 which correspond to jobs E and G (both on machine Q). Now since the corresponding processing time of task E on machine P is less than the corresponding processing time of task G on machine Q therefore task E will be processed in the last and task G next to last. The situation will be dealt as

A							G	E

The problem now reduces to following 6 tasks on two machines with processing time as follows:

Machine	B	C	D	F	H	I
P	5	4	9	8	5	4
Q	8	7	4	9	8	11

Here since the minimum processing time is 4 which occurs for tasks C and I on machine P and task D on machine Q. Therefore, the task C which has less processing time on P will be processed first and then task I and task D will be placed at the last i.e., 7^{th} sequence cell.

The sequence will appear as follows:

A	C	I				D	E	G

The problem now reduces to the following 3 tasks on two machines

Machine	B	F	H
P	5	8	5
Q	8	9	8

In this reduced table the minimum processing time is 5 which occurs for tasks B and H both on machine P. Now since the corresponding time of tasks B and H on machine Q are same i.e. 8. Tasks B or H may be placed arbitrarily in the $4^{th} and 5^{th}$ sequence cells. The remaining task F can then be placed in the 6^{th} sequence cell. Thus the optimal sequences are represented as

A	I	C	B	H	F	D	E	G

or

A	1	C	H	B	F	D	E	G

Further, it is also possible to calculate the minimum elapsed time corresponding to the optimal sequencing $\mathbf{A \rightarrow I \rightarrow C \rightarrow B \rightarrow H \rightarrow F \rightarrow D \rightarrow E \rightarrow G}$

Job Sequence	Machine A		Machine B	
	Time In	Time Out	Time In	Time Out
A	0	2	2	8
I	2	6	8	19
C	6	10	19	26
B	10	15	26	34
H	15	20	34	42
F	20	28	42	51
D	28	37	51	55
E	37	43	55	58
G	43	50	58	61

Hence the total elapsed time for this proposed sequence staring from job A to completion of job G is 61 hours .During this time machine P remains idle for 11 hours (from 50 hours to 61 hours)and the machine Q remains idle for 2 hours only (from 0 hour to 2 hour) .

15.2 Processing of n Jobs through Three Machines

The type of sequencing problem can be described as follows:

a) Only three machines A, B and C are involved;

b) Each job is processed in the prescribed order ABC

c) No passing of jobs is permitted i.e. the same order over each machine is maintained.

d) The exact or expected processing times $A_1, A_2, A_3, ---, A_n; B_1, B_2, B_3 ---, B_n$ and $C_1, C_2, C_3, ---, C_n$ are known and are denoted by the following table

Machine	Job(s)								
	1	2	3	--	-	i	- -	-	n
A	A_1	A_2	A_3	--	-	A_i	--	-	A_n
B	B_1	B_2	B_3	--	-	B_i	--	-	B_n
C	C_1	C_2	C_3			C_i			C_n

Our objective will be to find the optimal sequence of jobs which minimizes the total elapsed time. No general procedure is available so far for obtaining an optimal sequence in such case. However, the Johnson's procedure can be extended to cover the special cases where either one or both of the following conditions hold:

a) The minimum processing time on machine A $\geq$ the maximum processing time on machine B.

b) The minimum processing time on machine C $\geq$ the maximum processing time on machine B.

The method is to replace the problem by an equivalent problem involving n jobs and two machines. These two fictitious machines are denoted by G and H and the corresponding time G_i and H_i are defined by

$$G_i = A_i + B_i \text{ and } H_i = B_i + C_i$$

Now this problem with prescribed ordering GH is solved by the method with n jobs through two machines, the resulting sequence will also be optimal for the original problem. The above methodology is illustrated by following example:

Example 2:

There are five jobs (namely 1,2,3,4 and 5), each of which must go through machines A, B and C in the order ABC. Processing Time (in hours) are given below:

Jobs	1	2	3	4	5
Machine A	5	7	6	9	5
Machine B	2	1	4	5	3
Machine C	3	7	5	6	7

Find the sequence that minimum the total elapsed time required to complete the jobs.

Solution

Here $\mathbf{Min A_i = 5; B_i = 5 \text{ and } C_i = 3}$ since the condition of $\text{Min.} A_i \geq \text{Max.} B_i$ is satisfied the given problem can be converted into five jobs and two machines problem.

Jobs	$G_i = A_i + B_i$	$H_i = B_i + C_i$
1	7	5
2	8	8
3	10	9
4	14	11
5	8	10

The Optimal Sequence will be

2	5	4	3	1

Total elapsed Time will be

Jobs	Machine A		Machine B		Machine C	
	In	Out	In	Out	In	Out
2	0	7	7	8	8	15
5	7	12	12	15	15	22
4	12	21	21	26	26	32
3	21	27	27	31	32	37
1	27	32	32	34	37	40

Min. total elapsed time is 40 hours.

Idle time for Machine A is 8 hrs. (32-40)
Idle time for Machine B is 25 hours (0-7, 8-12, 15-21, 26-27, 31-32 and 34-40)
Idle time for Machine C is 12 hours (0-8, 22-26.)

15.3 Problems with n Jobs and m Machines

Let there be n jobs, each of which is to be processed through m machines, say $M_1, M_2, ---, M_m$ in the order $M_1 M_2 M_3 --- M_m$. Let T_{ij} be the time taken by the i^{th} machine to complete the j^{th} job.
The iterative procedure of obtaining an optimal sequence is as follows:

Step I: Find i) $\min_j(T_{1j})$ ii) and $\min_j(T_{mj})$

iii) $\max_j\left(T_{2j}, T_{3j}, T_{4j} --- T_{(m-1)_j}\right)$ for j=1,2,---, n
Step II: Check whether

a. $\min_j(T_{1j}) \geq \max_j(T_{ij})$ for i=2,3,----,m-1
Or

b. $\min_j(T_{mj}) \geq \max_j(T_{ij})$ for i=2,3,---,m-1
Step III: If the inequalities in Step II are not satisfied, method fails, otherwise, go to next step.
Step IV: Convert the m machine problem into two machine problem by introducing two fictitious machines G and H, such that

$$T_{Gj} = T_{1j} + T_{2j} + --- + T_{(m-1)_j} \text{ and } T_{Hj} = T_{2j} + T_{3j} + --- + T_{mj}$$

Determine the optimal sequence of n jobs through 2 machines by using optimal sequence algorithm.

Step V: In addition to condition given in Step IV, if $T_{ij} = T_{2j} + T_{3j} + --- + T_{mj} = C$ is a fixed positive constant for all $i = 1, 2, ----, n$, then determine the optimal sequence of n jobs and two machines M_1 and M_m in the order $M_1 M_m$ by using the optimal sequence algorithm.
Example 3:

Find an optimal sequence for the following sequencing problem of four jobs and five machines when passing is not allowed, of which processing time (in hours) is given below:

Job	Machine				
	A	B	C	D	E
1	7	5	2	3	9
2	6	6	4	5	10
3	5	4	5	6	8
4	8	3	3	2	6

Also find the total elapsed time.

Solution

Here $\text{Min.}A_i = 5$, $\text{Min.}E_i = 6$

$\text{Max.}(B_i, C_i, D_i) = 6, 5, 6$ respectively

Since $\text{Min.}E_i = \text{Max.}(B_i, D_i)$ and $\text{Min.}A_i = \text{Max.}C_i$ satisfied therefore the problem can be converted into 4 jobs and 2 fictitious machines G and H as follows:

Job	Fictitious Machine	
	$G_i = A_i + B_i + C_i + D_i$	$H_i = B_i + C_i + D_i + E_i$
1	17	19
2	21	25
3	20	23
4	16	14

The above sequence will be:

1	3	2	4

Total Elapsed Time Corresponding to Optimal Sequence can be obtained as follows:

	Machine A		Machine B		Machine C		Machine D		Machine E	
Job	In	Out	In	Out	In	Out	In	Out	In	Out
1	0	7	7	12	12	14	14	17	17	26
3	7	12	12	16	16	21	21	27	27	35
2	12	18	18	24	24	28	28	33	35	45
4	18	26	26	29	29	32	33	35	45	51

Thus the minimum elapsed time is 51 hours.

Idle time for machine A= 25 hours(26-51)

Idle time for machine B= 33 hours(0-7,16-18,24-26,29-51)

Idle time for machine C= 37 hours(0-12,14-16,21-24,28-29,32-51)

Idle time for machine D= 35 hours (0-14,17-21,27-28,35-51)

Idle time for machine E= 18 hours (0-17,26-27)

Lesson 16

INTRODUCTION AND CLASSIFICATION OF QUEUES

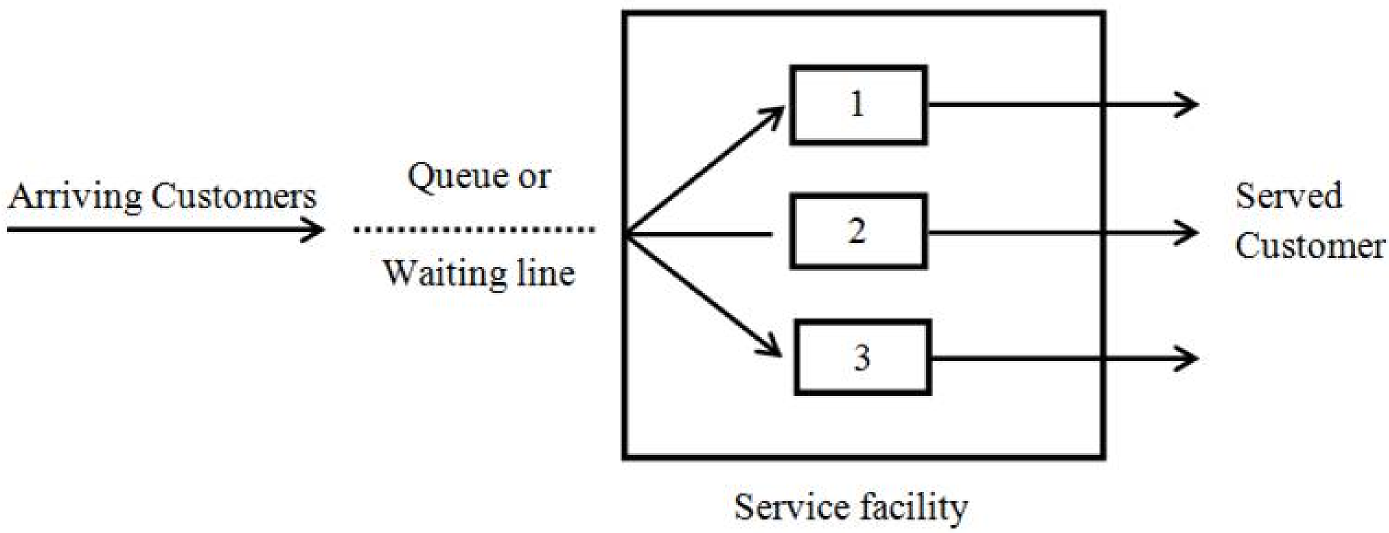
16.1 Introduction

The study of waiting lines, called queuing theory is one of the oldest and most widely used Operations Research techniques. Waiting lines are the most frequently encountered problem in our daily life. The queuing theory, also called the waiting line theory, owes its development to A. K. Erlang's efforts to analyze telephone traffic congestion with a view to satisfying the randomly arising demand for the services of the Corpenhagen automatic telephone system, in the year 1909. The theory is applicable to situation where the 'customers' arrive at some 'service stations' for some service; wait (occasionally not); and then leave the system after getting the service.

A flow of customers from finite/ infinite population towards the services facility form a queue (waiting line) on account of lack of capability to serve them all at a time. The queues may consist of customers for buying milk and other milk products at a milk parlour, machines waiting to be repaired, trucks or vehicles waiting at the milk plant, patients in a hospital who need treatment and so on. In the absence of a perfect balance between the service facilities and the customers, waiting is required either of the services facilities or for the customer's arrival. In general a queue is formed when either units requiring services-commonly referred as customers, wait for service or the service facilities, stand idle and wait for customers. The queuing models are basically relevant to service oriented organizations and suggest ways and means to improve the efficiency of the service. Queuing methodology indicates the optimal usage of existing manpower and other resources to improve the service. Waiting lines can't be eliminated completely but suitable techniques can be used to reduce the waiting line of an object in the system.

16.2 Queuing System

The mechanism of a queuing process is very simple. Customers arrive at services counter are attended by one or more of the servers. As soon as a customer is served, he departs from the system. Thus a queuing system can be described as composed of customers arriving for service, waiting for service if it is not immediate, and if having wanted for service, leaving the system after being served.



16.3 Component of a Queuing System

A queuing system can be described by the following components:

16.3.1 Input process (or Arrival pattern)

This is considered with the pattern in which the customers arrive and join the system. An input source is characterized by

a) Size of the calling population.
b) Pattern of arrivals at the system.
c) Behaviour of the arrivals.

Customers requiring service are generated at different times by an input source, commonly known as population. The rate at which customers arrive at the service facility is determined by the arrival process.

16.3.1.1 Size of the calling population

The size represents the total number of potential customers who will require service. The source of customers can be either finite or infinite. It is considered infinite if the number of people being very large e.g. all people of a city or state (and others) could be the potential customers at a milk parlour. Whereas there are many situations in industrial conditions where we cannot consider the population to be infinite—it is finite. The customers may arrive for service individually or in groups. Single arrivals are illustrated by a customer visiting a milk parlour, students arriving at a library counter etc. On the other hand, families visiting restaurants, ships discharging cargo at a dock are examples of bulk or batch arrivals.

16.3.1.2 Pattern of arrivals at the system

Customers arrive in the system at a service facility according to some known schedule (for example one patient every 15 minutes or a candidate for interview every half hour) or else they arrive randomly. Arrivals are considered at random when they are independent of one another and their occurrence cannot be predicted exactly. The queuing models wherein customers' arrival times are known with certainty are categorized as deterministic models and are easier to handle. On the other hand, a substantial majority of the queuing models are based on the premise that the customers enter the system stochastically, at random points in time. The arrival process (or pattern) of customers to the service system is classified into two categories: **static** and **dynamic**. These two are further classified based on the nature of arrival rate and the control that can be exercised on the arrival process.

In static arrival process, the control depends on the nature of arrival rate (random or constant). Random arrivals are either at a constant rate or varying with time. Thus to analyze the queuing system, it is necessary to describe the probability distribution of arrivals. From such distributions average time between successive arrivals, is obtained also called inter-arrival time (time between two consecutive arrivals), and the average arrival rate (i.e. number of customers arriving per unit of time at the service system).

The dynamic arrival process is controlled by both service facility and customers. The service facility adjusts its capacity to match changes in the demand intensity, by either varying the staffing levels at different timings of service, varying service charges (such as telephone call charges at different hours of the day or week) at different timings, or allowing entry with appointments. Frequently in queuing problems, the number of arrivals per unit of time can be estimated by a probability distribution known as the Poisson distribution, as it adequately supports many real world situations

16.3.2 Service Mechanism (or Service Pattern)

The service is provided by a service facility (or facilities). This may be a person (a bank teller, a barber, a machine (elevator, gasoline pump), or a space (airport runway, parking lot, hospital bed), to mention just a few. A service facility may include one person or several people operating as a team. There are two aspects of a service system

a) the configuration of the service system
b) the speed of the service.

16.3.2.1 Configuration of the service system

The customers' entry into the service system depends upon the queue conditions. If at the time of customers' arrival, the server is idle, then the customer is served immediately. Otherwise the customer is asked to join the queue, which can have several configurations. By configuration of the service system we mean how the service facilities exist. Service systems are usually classified in terms of their number of channels, or numbers of servers.

i) **Single Server – Single Queue** -- The models that involve one queue – one service station facility are called single server models where customer waits till the service point is ready to take him for servicing. Students arriving at a library counter are an example of a single server facility.

ii) **Single Server – Several Queues** – In this type of facility there are several queues and the customer may join any one of these but there is only one service channel.

iii) **Several (Parallel) Servers – Single Queue** – In this type of model there is more than one server and each server provides the same type of facility. The customers wait in a single queue until one of the service channels is ready to take them in for servicing.

iv) **Several Servers – Several Queues** – This type of model consists of several servers where each of the servers has a different queue. Different cash counters in an electricity office where the customers can make payment in respect of their electricity bills provide an example of this type of model. Different ticket issue encounters in a trade fair and different boarding pass encounters at an airport are also other possible examples of this type of model.

v) **Service facilities in a series** – In this, a customer enters the first station and gets a portion of service and then moves on to the next station, gets some service and then again moves on to the next station. …. and so on, and finally leaves the system, having received the complete service. For example in a milk plant packaging of milk pouches consist of boiling, pasteurization, cooling and packaging operations, each of which is performed by a single server in a series.

16.3.2.2 Speed of service

In a queuing system, the speed with which service is provided can be expressed in either of two ways—as service rate and as service time. The service rate describes the number of customers serviced during a particular time period and the service time indicates the amount of time needed to service a customer. Service rates and times are reciprocal of each other and either of them is sufficient to indicate the capacity of the facility. Thus if a cashier can attend, on an average 5 customers in an hour, the service rate would be expressed as 5 customers/hour and service time would be equal to 12 minutes/customer. Generally, we consider the service time only. If these service times are known exactly, the problem can be handled easily. But, as generally happens, if these are different and not known with certainty, we have to consider the distribution of the service times in order to analyze the queuing system. Generally, the queuing models are based on the assumption that service times are exponentially distributed about some average service time.

16.3.3 Queue discipline

In the queue structure, the important thing to know is the queue discipline. The queue discipline is the rule determining the formation of queue, manner in which customers form the queue are selected for service. There are a number of ways in which customers in the queue are served. Some of these are:

16.3.3.1 Static queue disciplines

These are based on the individual customer's status in the queue. The most common queue disciplines are:

i) **First-Come-First-Served (FCFS):** If the customers are served in the order of their arrival, then this is known as the FCFS service discipline. For example, this type of queue discipline is observed at a milk parlour, railway station etc. FCFS is also known as First In First Out (FIFO).

ii) **Last-Come-First-Served (LCFS):** Sometimes, the customers are serviced in the reverse order of their entry so that the ones who join the last are served first and the system is referred to as LCFS. For example, in a big godown the items which come last are taken out first. Similarly, the people who join an elevator last are the first ones to leave it.

16.3.3.2 Dynamic queue disciplines

These are based on the individual customer attributes in the queue. Few of such disciplines are:

i) **Service in Random Order (SIRO):** Under this rule customers are selected for service at random, irrespective of their arrivals in the service system. In this, every customer in the queue is equally likely to be selected. The time of arrival of the customers is, therefore, of no relevance in such a case.

ii) **Priority Service:** Under this rule customers are grouped in priority classes on the basis of some attributes such as service time or urgency or according to some identifiable characteristic, and FCFS rule is used within each class to provide service. Treatment of VIPs in preference to other patients in a hospital is an example of priority service.

16.3.4 Customer's behaviour

Another thing to consider in the queuing structure is the behaviour or attitude of the customers entering the queuing system. On this basis, the customers may be classified as being patient, or impatient. If a customer, on arriving at the service system stays in the system until served, no matter how much he has to wait for service is called a patient customer whereas the customer, who waits for a certain time in the queue and leaves the service system without getting service due to certain reasons such as a long queue in front of him is called an impatient customer. The customers generally behave in four ways

i) **Balking:** A customer may leave the queue because the queue is too long or the estimated waiting time is too long or waiting space is inadequate, for desired service and may decide to return for service at a later time. In queuing theory this is known as **balking**.

ii) **Reneging:** A customer, after joining the queue, waits for some time and leaves the service system due to intolerable delay or due to impatience.

iii) **Jockeying:** A customer who switches from one queue to another, hoping to receive service more quickly, is said to be jockeying.

iv) **Priorities:** In certain applications some customers are served before others regardless of their order of arrival. These customers have priority over others.

Lesson 17

SOLUTION OF QUEUING MODELS

17.1 Introduction

The ultimate objective of the analysis of queuing systems is to understand the behavior of their underlying processes so that informed and intelligent decisions can be made in their management. In a specified queuing system the problem is to determine the probability distribution of queue length, waiting time of customers and the busy period. Queuing theory uses queuing models to represent the various types of queuing systems that arise in practice. Formulae for each model indicate how the corresponding queuing system should perform, including the average amount of waiting time under a variety of circumstances. Therefore, these queuing models are helpful in determining how to operate a queuing system in the most effective way. Providing too much service capacity to operate the system involves excessive costs. But not providing enough service capacity results in excessive waiting and all its unfortunate consequences. The models enable finding an appropriate balance between the cost of service and the amount of waiting.

17.2 Characteristics of A Queuing System

Queuing models enable the analyst to study the effect of manipulating decision variables on the operating characteristics of a service system. The most commonly used characteristics are stated as under:

17.2.1 Queue length

The average number of customers in the queue waiting to get service is known as 'queue length'. 'Short queues' could mean either good customer service or large waiting space while 'long queues' could indicate low service efficiency or a little waiting space.

17.2.2 System length

It is the average number of customers in the system waiting to be served and those being served. Long queues imply congestion, potential customer dissatisfaction and need for more capacity.

17.2.3 Waiting time in queue

Waiting time is the average time that a customer has to wait in the queue to get service. Long waiting times may indicate a need to adjust the service rate of the system or change the arrival rate of customers.

17.2.4 Total time in system

The average time that customer spends in the system from entry in the queue to completion of service. If this time is more then there may be a need to change the priority discipline, increase productivity or adjust the capacity.

17.2.5 Server idle time

The relative frequency with which the service system is idle which is directly related to cost. Queuing theory analysis involves the study of systems' behaviour over time.

17.2.6 Transient and Steady States

When a service system is started, it progresses through a number of changes. However, it attains stability after some time. Before the service operations start, it is very much influenced by the initial conditions (number of customers in the system) and the elapsed time. This period of transition is termed as **transient state**. A system is said to be in transient-state when its operating characteristics are dependent on time.

However, after sufficient time has passed, the system becomes independent of the initial conditions and of the elapsed time (except under very special conditions) and enters a **steady state** condition. A steady state condition is said to prevail when the behaviour of the system becomes independent of time. Let $P_n(t)$ denote the probability that are **n** units in the system at time **t**. We know that the change of $P_n(t)$ with respect to **t** is described by the derivative $\frac{d}{dt}P_n(t)$. Then the queuing system is said to be stable eventually, in the sense that the probability $P_n(t)$ is independent of time, i.e. remains the same as time passes $(t \to \infty)$. Mathematically, in a steady state,

$$\lim_{t\to\infty} P_n(t) = P_n \text{ (independent of t)}$$

This implies that

$$\mathrm{Lim}_{t\to\infty} \frac{d}{dt} P_n(t) = \frac{d}{dt} \mathrm{Lim}_{t\to\infty} P_n = \frac{d}{dt} P_n = 0$$

From the practical point of view period of the steady state behaviour of the system, queuing system under the existence of steady state condition are being considered.

17.3 Notations and Symbols

The notations used in the analysis of a queuing system are as follows:

n =	Number of customers in the system (waiting and in service)
$P_n(t)$ =	Transient state probability that n calling units are in the queuing system at time t
E_n =	The state in which there are n calling units in the system
P_n =	Steady state probability of having n units in the system
λ =	Average (expected) customer arrival rate or average number of arrivals per unit of time in the queuing system
μ =	Average (expected) service rate or average number of customers served per unit time at the place of service
ρ =	

Traffic intensity or server utilization factor

s = Number of service channels (service facilities or servers)

N = Maximum number of customers allowed in the system.

L_s = Average (expected) number of customers in the system (waiting and in service)

L_q = Average (expected) number of customers in the queue (queue length)

L_b = Average (expected) length of non-empty queue

W_s = Average (expected) waiting time in the system (waiting and in service)

W_q = Average (expected) waiting time in the queue

P_w = Probability that an arriving customer has to wait

17.3.1 Kendall's notation for representing queuing models

Generally queuing model can be specified by the symbolic representation $(a|b|c):(d|e)$

where,

a : Probability distribution of the arrival (or inter-arrival) time

b : Probability distribution of the service time.

c : Number of channels (or service stations)

d : Capacity of the system

e : Queue discipline

The first three characteristics (a|b|c) in the above notation were introduced by D. Kendall in 1953. Later in 1966, A. Lee added the fourth (d) and fifth (e) characteristics to the notation. Traditionally, the exponential distribution in queuing problems is denoted by M. Thus, (M|M|1): (∞|FIFO) indicates a queuing system when the inter-arrival times and service times are exponentially distributed having one server in the system with first in first out discipline and the number of customers allowed in the system can be infinite.

17.4 Traffic Intensity (or Utilization factor)

An important measure of simple queue is its traffic intensity, where

$$Traffic\ intensity(\rho) = \frac{mean\ arrival\ rate}{mean\ service\ rate} = \frac{\lambda}{\mu}$$

$$i.e.\ \rho = \frac{1/\mu}{1/\lambda} = \frac{mean\ service\ time}{mean\ inter-arrival\ time}$$

The unit of traffic intensity is Erlang.

A necessary condition for a system to have settled down to steady state is that

$$\rho < 1\ or\ \lambda/\mu < 1\ or\ \lambda < \mu$$

i.e. arrival rate < service rate. If $\rho > 1$, the arrival rate is greater than the service rate and consequently , the number of units in the queue tends to increase indefinitely as the time passes on, provided the rate of service is not affected by the length of queue.

17.5 Queuing Models

The queuing models are categorized as 'deterministic' or 'probabilistic'. If each customer arrives at known intervals and the service time is known with certainty, the queuing model will be deterministic in nature. When both arrival and service rate are unknown and assumed to be random variable then this type of queuing model is known as probabilistic.

17.6 Probability Distributions in Queuing systems

The arrival of customers at a queuing system varies between one system and another, but in practice one pattern of completely random arrivals is observed.

17.6.1 Distribution of arrivals 'the Poisson process' (pure birth process)

The models in which only arrivals are counted and no departure take place are called pure birth models. In terms of queuing, birth-death process that is increased by birth or arrival in the system and decreased by death or departure of serviced customer from the system. If the arrivals are completely random, then the probability distribution of number of arrivals in a fixed time–interval, follows Poisson probability distribution with parameter (mean) λt. Thus,

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \text{ for } n \geq o$$

17.6.2 Distribution of Inter–Arrival times (Exponential Process)

Inter–arrival times are defined as the time intervals between two successive arrivals. It will be proved that if the arrival process follows Poisson distribution, an associated random variable defined as the time between successive arrivals (inter-arrival time) follows an exponential distribution $f(t) = \lambda e^{-\lambda t}$ and vice-versa.

The expected (or mean) time of inter arrival is given by $E(t) = \int_0^\infty t.\, f(t)\, dt = \int_0^\infty \lambda t e^{-\lambda t}.\, dt = \frac{1}{\lambda}$ where λ is the mean arrival rate.

Thus, t has the exponential distribution with mean $1/\lambda$, we would intuitively expect that if the mean arrival rate is λ, then the mean time between arrival is $1/\lambda$. Conversely, it can also be independent and have the exponential distribution then the arrival rate follows the Poisson distribution.

17.7 Classification of Queuing Models (LISTING OF FOUR MODELS)

The queuing models are classified as follows:

Model I	:	(M|M|1): (∞|FCFS)
Model II	:	(M|M|s): (∞|FCFS)
Model III	:	(M|M|1): (N|FCFS)
Model IV	:	(M|M|s): (∞|FCFS)

In this lesson Model I and II have been discussed as Model III and IV are beyond the scope of this course.

17.7.1 Model I (single channel queuing model with Poisson arrivals and exponential service times)

This model is symbolically represented as by (M|M|1): (∞|FCFS). This denotes Poisson arrival (exponential inter-arrival), Poisson departure (exponential service time), single server, Infinite capacity and 'First come, First served' service discipline. This model is also called 'birth and death model'. This model is one of the most widely used and simplest models. It assumes the following conditions:

(i) Arrivals are served on a 'first come-first served' (FCFS) basis.

(ii) Every arrival waits to be served regardless of the length of the line.

(iii) Arrivals are independent of preceding arrivals, but the average number of arrivals does not change over time.

(iv) Arrivals are described by a Poisson probability distribution and come from an infinite population.

(v) Service times also vary from one customer to the next and are independent of one another, but their average rate is known.

(vi) Service times occur according to the negative exponential probability distribution.

(vii) The average service rate is greater than the average arrival rate.

17.7.1.1 To obtain the system of steady- state equations

The probability that there will be n units (n>0) in the system at time (t+Δt) may be expressed as the sum of three independent compound properties by using the fundamental properties of probability, Poisson arrivals, and of exponential service times.

(i) The product of three probabilities (Fig. 17.1)

a) that there are n units in the system at time $t = P_n(t)$

b) that there is no arrival in time $\Delta t = P_0(\Delta t) = 1 - \lambda\Delta t$

c) that there is no service in time $\Delta t = \varphi_{\Delta t}(0) = 1 - \mu\Delta t$ is given by

$$P_n(t).(1-\lambda\Delta t).(1-\mu\Delta t) \cong P_n(t)[1-(\lambda+\mu)\Delta t] + 0_1(\Delta t)$$

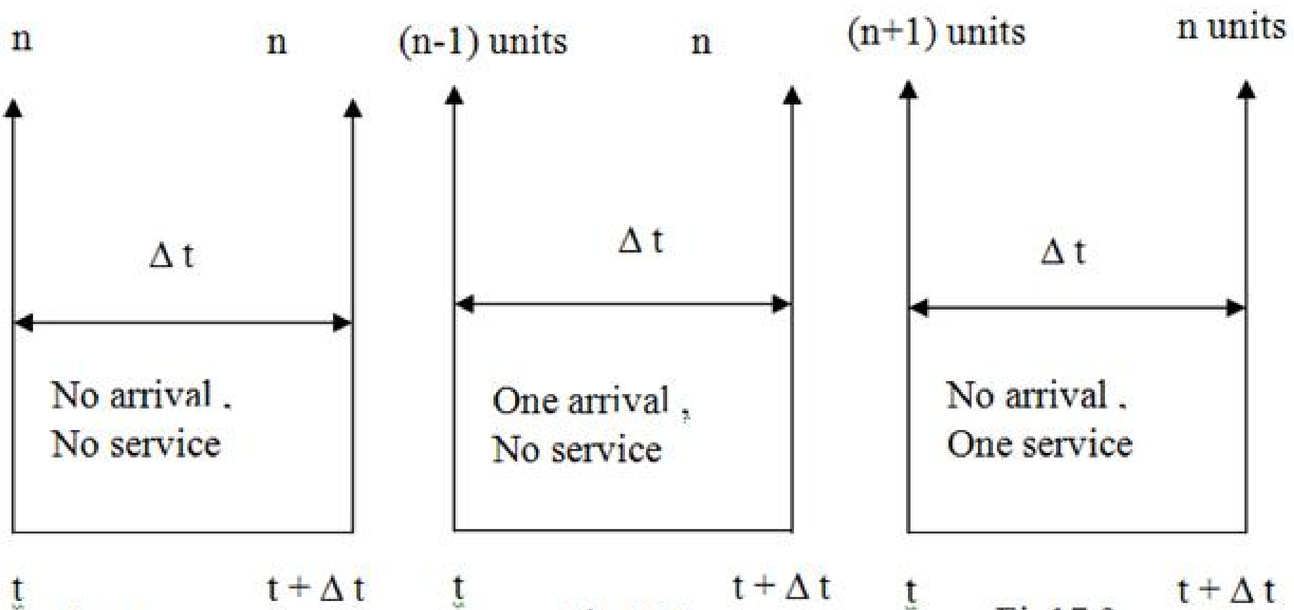


Fig 17.1 Fig 17.2 Fig17.3

(ii) The product of three probabilities (Fig. 17.2)

a) that there are (n-1) units in the system at time t = $P_{n-1}(t)$

b) that there is one arrival in time $\Delta_t = P_1(\Delta t) = \lambda\Delta t$

c) that there is no service in time $\Delta_t = \varphi_{\Delta r}(0) = 1 - \mu\Delta t$ is given by

$$P_{n-1}(t).(\lambda\Delta t).(1 - \mu\Delta t) \cong \lambda P_{n-1}(t)\Delta t + 0_2(\Delta t).$$

(iii) The product of probabilities (see Fig. 17.3)

a) that there are (n+1) units in the system at time t = $P_{n+1}(t)$

b) that there is no arrival in time $\Delta t = P_0(\Delta t) = 1 - \lambda\Delta t$

c) that there is one service in time $\Delta t = \varphi_{\Delta t}(1) = \mu\Delta t$ is given by

$$P_{n+1}(t).(1 - \lambda\Delta t).\mu\Delta t \cong P_{n+1}(t)\mu\Delta t + 0_3(\Delta t)$$

Now, by adding above three independent compound probabilities, we obtain the probability of n units in the system at time (t + Δt), i.e.,

$$P_n(t + \Delta t) = P_n(t)[1 - (\lambda + \mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta_t + 0(\Delta) \qquad (17.1)$$

where $0(\Delta t) = 0_1(\Delta t) + 0_2(\Delta t) + 0_3(\Delta t)$

The equation may be written as

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda + \mu)P_n t + \lambda P_{n+1}(t) + \frac{0(\Delta t)}{\Delta t}$$

Now, taking limit as $\Delta t \to 0$ on both sides,

$$\lim_{\Delta t \to 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \lim_{\Delta t \to 0} \left[-(\lambda + \mu)P_n t + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{0(\Delta t)}{\Delta t} \right]$$

or

$$\frac{dP_n t}{dt} = -(\lambda+\mu)P_n t + \lambda P_{n-1}(t) + \mu P_{n+1}(t); n > 1 \left(since \lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0\right) \quad (17.2)$$

In a similar fashion, the probability that there will be no unit (i.e. n = 0) in the system at time (t + Δt) will be the sum of the following two independent probabilities:

(i) P[that there is no unit in the system at time t and no arrival in time

$$\Delta t] = P_0(t)(1-\lambda\Delta t)$$

(ii) P[that there is one unit in the system at time t, one unit serviced in Δt and no arrival in time Δt]= $P_1(t)\mu\Delta t(1-\lambda\Delta t) \equiv P_1(t)\mu\Delta t + O(\Delta t)$

Now, adding these two probabilities we get

$$P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta t] + P_1(t)\mu\Delta t + O(\Delta) \quad (17.3)$$

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0 t + \mu P_1(t) + \frac{O(\Delta t)}{\Delta t}$$

Now, taking limit as Δt→0 on both sides,

$$\frac{dP_0 t}{dt} = -\lambda P_0 t + \mu P_1(t); n = 0 \quad (17.4)$$

Consequently the equations (17.2) and (17.4) can be written as

$$0 = -(\lambda+\mu)P_n t + \lambda P_{n-1}(t) + \mu P_{n+1}(t); n > 0 \quad (17.5)$$

$$0 = -\lambda P_0 t + \mu P_1(t); n = 0 \quad (17.6)$$

Equations (17.5) and (17.6) constitute the system of steady state difference equations for the model.

17.7.1.2 To solve the system of difference equations

We solve the difference equations given in (17.5) and (17.6) by the method of successive substitution

Since $P_0 = P_0$ and putting n=0 in equation (17.5) we get $P_1 = \frac{\lambda}{\mu}P_0$

$$P_2 = \frac{\lambda}{\mu}P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0,\ P_3 = \frac{\lambda}{\mu}P_2 = \left(\frac{\lambda}{\mu}\right)^3 P_0, \text{---},\ P_n = \frac{\lambda}{\mu}P_{n-1} = \left(\frac{\lambda}{\mu}\right)^n P_0, \quad (17.7)$$

Now using the fact that $\sum_{n=0}^{\infty} P_n = 1$

$$P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + - - - + \left(\frac{\lambda}{\mu}\right)^n P_0 = 1 \quad \text{or } P_0\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + - - - + \left(\frac{\lambda}{\mu}\right)^n\right) = 1$$

$$\text{or } P_0\left[\frac{1}{1-(\lambda/\mu)}\right] = 1 \quad \text{or} \quad P_0 = 1 - (\lambda/\mu) \qquad (17.8)$$

Now, substituting the value of P_0 from (17.8) in (17.7), we get

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = \rho^n(1-\rho) \qquad (17.9)$$

The equations (17.8) and (17.9) give the probability distribution of queue length

17.7.1.3 Measure of model 1

Expected (average) number of units in system (L_s) is given by

$$L_s = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right) \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left[1 + 2\left(\frac{\lambda}{\mu}\right) + 3\left(\frac{\lambda}{\mu}\right)^2 + - - -\right] = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho} \qquad (17.10)$$

Expected (average) queue length (L_q) is given by:

Since there are (n-1) units in the queue excluding one being serviced

$$L_q = \sum_{n=1}^{\infty}(n-1) P_n = \sum_{n=1}^{\infty} n P_n - \sum_{n=1}^{\infty} P_n = \sum_{n=1}^{\infty} nP_n - \left[\sum_{n=o}^{\infty} P_n - P_0\right] = L_s - (1 - P_0)$$

Substituting the value of P_0 from equation 17.8

$$L_q = L_s - 1 + \left(1 - \frac{\lambda}{\mu}\right) = L_s - \left(\frac{\lambda}{\mu}\right) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho} \qquad (17.11)$$

Expected (average) waiting time in the queue (excluding service time) (W_q) is given by:

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} \qquad (17.12)$$

Expected (average) waiting time in the system (including service time) (W_s) is given by:

$$W_s = \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{\mu-\lambda} \qquad (17.13)$$

Expected (average) length of non-empty queue, (L/L > 0) is given by:

$$(L|L > 0) = \frac{\left(\lambda/\mu\right) / \left(1-\lambda/\mu\right)}{\lambda/\mu} = \frac{\mu}{\mu-\lambda} = \frac{1}{1-\rho}$$

Expected variance of queue length is given by:

$$\mathrm{Var.}\{n\} = \frac{\rho}{(1-\rho)^2}$$

Example 1:

Customers arrive at a milk parlour being manned by a single Individual at rate of 25 per hour. The time required to serve a customer has exponential distribution with a mean of 30 per hour. Discuss the various characteristics of the queuing system, assuming that there is only one server.

Solution

Arrival rate (λ) = 25 per hour, Service rate (μ) = 30 per hour

$$Traffic\ intensity\ (Utilization\ factor)(\rho) = \frac{mean\ arrival\ rate}{mean\ service\ rate} = \frac{\lambda}{\mu} = \frac{25}{30} = \frac{5}{6}$$

Expected number of units in system (L_S) = $\frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\rho}{1-\rho} = \frac{5/6}{1-5/6} = 5\ cutomers$

Expected queue length (L_q) = $\frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho} = \frac{(5/6)^2}{1-5/6} = \frac{25}{6}$ customers

Expected waiting time in the queue $\left(W_q\right) = \frac{\lambda}{\mu(\mu\ \ \lambda)} = \frac{25}{30(30\ \ 25)} = \frac{1}{6}\text{hour}$

Example 2 :

In a service department manned by one server, on an average 8 customers arrive every 5 minutes while the server can serve 10 customers in the same time assuming Poisson distribution for arrival and exponential distribution for service rate. Determine:

a) Average number of customers in the system.

b) Average number of customers in the queue.

c) Average time a customer spends in the system.

d) Average time a customer waits before being served.

Solution

Arrival rate (λ) $=\frac{8}{5}$ =1.6 customers per minute.

Service rate (μ)$=\frac{10}{5}=$ 2 customers per minute.

$$Traffic\ intensity\ (\rho) = \frac{mean\ arrival\ rate}{mean\ service\ rate} = \frac{\lambda}{\mu} = \frac{1.6}{2} = 0.8$$

a) Average number of customer in the system.

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1-\rho} = \frac{0.8}{1-0.8} = 4\ customers$$

b) Average number of customer in the queue.

$$(L_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho} = \frac{0.8^2}{1-0.8} = 3.2 \text{ customers}$$

c) Average time a customer spends in the system.

d) $(W_s) = \frac{1}{\mu \quad \lambda} = \frac{1}{2-1.6} = \frac{1}{0.4} hr = 2.5\ hrs.$

e) Average time a customer waits before being served

$$W_q = \frac{\lambda}{\mu(\mu \quad \lambda)} = \frac{1.6}{2(2 \quad 1.6)} = 2\text{hrs.}$$

17.7.2 Model II (A) general erlang queuing model (Birth-Death Process)

This model is also represented by (M|M|1): (∞|FCFS), but this is a general model in which the rate of arrival and the service depend on the length n of the line.

17.7.2.1 To obtain the system of steady state equation

Let arrival rate $\lambda = \lambda_n$ service rate $\mu = \mu_n$; [depending upon n]

Then, by the same argument as for equations 17.1 and 17.3

$$P_n(t+\Delta t) = P_n(t)[1-(\lambda_n+\mu_n)\Delta t] + P_{n-1}(t)\lambda_{n-1}\Delta t + P_{n+1}(t)\mu_{n+1}\Delta t + O(\Delta t),$$

$$n > 0 \qquad (17.14)$$

$$P_0(t+\Delta t) = P_o(t)[1-\lambda_0\Delta t] + P_1(t)\mu_1\Delta t + O(\Delta t), n = 0 \qquad (17.15)$$

Now dividing equations (17.14) and (17.15) by Δt, taking limits as $\Delta t \to 0$ and following the same procedure as in Model I, obtain

$$\frac{dP_n(t)}{dt} = (\lambda_n+\mu_n)P_n t + \lambda_{n\ 1}P_{n\ 2}(t) + \mu_{n+1}P_{n+1}(t) \qquad (17.16)$$

$$\frac{dP_0(t)}{dt} = \lambda P_0 t + \mu_1 P_1(t) respectively \qquad (17.17)$$

The above written two equations are differential equations which could be solved if a set of initial values P_0 (0), P_1 (0),…, is given. Such a system of equations can be solved if the time dependent solution is required. But, for many problems it suffices to look at the steady state solution.

In the case of steady state, the boundary conditions are

$$P'_n(t) = 0 \text{ and } P'_0(t) = 0$$

So the equations (17.16) and (17.17) become,

$$0 = -(\lambda_n+\mu_n)P_n + \lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1}, n > 0 \qquad (17.18)$$

$$0 = -\lambda_n P_0 + \mu_1 P_1, n = 0 \qquad (17.19)$$

The equations (17.18) and (17.19) constitute the system of steady state difference equations for this model.

To solve the system of difference equations

Since $\quad P_0 = P_0$

$$P_1 = \frac{\lambda_0}{\mu_1}P_0$$

$$P_2 = \frac{\lambda_1}{\mu_2}P_1 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}P_0$$

$$P_3 = \frac{\lambda_2}{\mu_3}P_2 = \frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1}P_0$$

… … …

$$P_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} \qquad (17.20)$$

Now, in order to find P_0, use the fact that

$\sum_{n=0}^{\infty} P_n = 1$ or $P_0 + P_1 + P_2 + \cdots \infty = 1$ or $P_0\, 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_0 \mu_1} + \cdots \infty = 1$ or $P_0 = {}^{1}/_{S}$,

where $$S = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_0}{\mu_1 \mu_2} + \cdots \infty. \qquad (17.21)$$

The result obtained above is a general one and by suitably defining μ_n and λ_n many interesting cases could be studied. Now two particular cases may arise:

Case 1

$\lambda_n = \lambda, \mu_n = \mu$

In this case, the series S becomes

$$S = 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots \infty = \frac{1}{1 - {}^{\lambda}/_{\mu}}$$

Therefore, from equation 17.21 and 17.20

$$P_0 = \frac{1}{S} = 1 - \frac{\lambda}{\mu} \text{ and } P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Here, it is observed that this is exactly the case of Model 1.

Case 2

$$\left(\lambda_n = \frac{\lambda}{n+1}, \mu = \mu\right)$$

The case, in which the arrival rate λ_n depends upon n inversely and the rate of service μ_n is independent of n, is called the case of 'Queue with Discouragement'.

In this case, the series S becomes

$$S = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{2.3\mu^3} + \cdots \infty = 1 + \rho + \frac{\rho^2}{2!} + \frac{\rho^3}{3!} + \cdots \infty \left(\rho = {}^{\lambda}/_{\mu}\right) = e^{\rho}$$

The equation 17.21 gives $\quad P_0 = {}^{1}/_{S} = e^{-\rho}$

$$P_1 = \frac{\lambda}{\mu} P_0 = \rho e^{-\rho},$$

$$P_2 = \frac{\lambda^2}{2\mu^2} P_0 = \frac{\rho^2 e^{-\rho}}{2!},$$

$$\cdots \quad \cdots \quad \cdots$$

$$P_n = \frac{\lambda^2}{n!\mu^n} P_0 = \frac{\rho^2 e^{-\rho}}{n!} \text{ for all } n = 0,1, \ldots \infty.$$

It is observed in this case that P_n follows the Poisson distribution, where ${}^{\lambda}/_{\mu} = \rho$ is constant, however $\rho > 1$ or $\rho < 1$ but must be finite. Since the series S is convergent and hence sum able in both the cases.

17.7.3 Model II (B) : (M|M|1):(∞ |SIRO)

This model is actually the same as Model I, except that the service discipline follows the Service in Random Order (SIOR) rule in place of FCFS rule. Since the derivation of P_n in Model I is independent of any specific queue discipline, so for the SIOR rule also

$$P_n = (1 - \rho)\rho^n$$

Consequently, the average number of customers in the customers in the system will be the same whether queue discipline follows SIRO rule or FCFS rule.

Lesson 18

INTRODUCTION AND BASIC DEFINITIONS IN NETWORK ANALYSIS

18.1 Introduction

A Project such as setting up of a new milk plant, research and development in an organization, development of a new milk product, marketing of a product etc. is a combination of interrelated activities (tasks) which must be executed in a certain order before the entire task can be completed. The activities are interrelated in a logical sequence in such a way that same activities can not start until some others are completed. An activity in a project usually viewed as job requiring resources for its completion. The objectives of project management can be described in terms of a successful project which has been finished on time, within the budgeted cost and to technical specifications and to the satisfaction level of end users. Normally for any project, one may be interested in answering questions such as

i) What will be the expected time of project completion?
ii) What is the effect of delay of any activity on the overall completion of project?
iii) How to reduce the time to perform certain activities in case of availability of additional funds?
iv) What is the probability of completion of project in time?

The OR techniques used for planning, scheduling and controlling large and complex projects are often referred to as network analysis. A network is a graphical representation consisting of certain configuration of arrows and nodes for showing the logical sequence of various tasks to be performed to achieve the project objectives. Around five decades ago the planning tool was *Gantt bar chart* which specifies start and finish time for each activity on a horizontal time scale. The disadvantage is that there is no interdependency among the many activities which control the progress of the project. Now-a-days we use a technical tool for planning, scheduling and controlling stages of the projects known as Critical Path Method (CPM) and Project Evaluation & Review Technique (PERT). The techniques of PERT and CPM prove extremely valuable in assisting the mangers in handling such projects and thus discharging their project management responsibilities both at planning and controlling stages of the projects. Commonly used project management techniques are:

a) Critical Path Method (CPM) and
b) Project Evaluation and Review Technique (PERT)

18.2 Historical Development

CPM/PERT or Network Analysis as the technique is sometimes called, developed along two parallel streams, one industrial and the other military. CPM was developed in 1957 by J. E. Kelly of Remington Rand and M. R. Walker of E. I. Du Pont de Nemours & Co. PERT was devised in 1958 for the POLARIS missile program by the Program Evaluation Branch of the Special Projects office of the U. S. Navy, helped by the Lockheed Missile Systems division and the Consultant firm of Booz-Allen & Hamilton.

Both are basically time oriented methods laid to determination of a time schedule for project. The major difference between these two techniques is that **PERT** is a **Probabilistic** approach for the determination of time estimates of different activities not exactly known to us**.** In the case of **CPM,** different estimates are known as they are **deterministic** in nature. But now a days both these techniques are used for one purpose. Initially the PERT technique was applied to research and development projects while the CPM was used towards construction projects.

18.3 Methodology in CPM/PERT Technique

The methodology involved in network scheduling by CPM/PERT for any project consists of the following four stages:

18.3.1 Planning

It is started by splitting the total project into small projects. The smaller projects are further divided into different activities and are analyzed by a department or section. The relationship of each activity with respect to other activities are defined and established.

18.3.2 Scheduling

The objective of scheduling is to give the earliest and the latest allowable start and finish time of each activity as well as its relationship with other activities in the project. The schedule must pinpoint the critical path i.e. time activities which require special attention if the project is to be completed in time.

18.3.3 Allocation of resources

Allocation of resources is performed to achieve the desired objective. Resource is a physical variable such as labour, finance, space, equipment etc. which will impose a limitation for completion of a project.

18.3.4 Controlling

The final phase in the project management is controlling. After making the network plan and identification of the Critical path, the project is controlled by checking progress against the schedule, assigning and scheduling manpower and equipment and analyzing the effects of delays. This is done by progress report from time to time and updating the network continuously. Arrow diagram and time charts are used for making periodic progress reports.

18.4 Basic Terminology used in Network Analysis

Network analysis is the general name given to certain specific techniques which can be used for the planning, management and control of projects. A fundamental method in both PERT and CPM is the use of network systems as a means of graphically depicting the current problems or proposed projects in network diagram. A network diagram is the first thing to sketch an arrow diagram which shows inter-dependencies and the precedence relationship among activities of the project. Before illustrating the network representation of a project, let us define some basic definitions:

18.4.1 Activity

Any individual operation, which utilizes resources and has a beginning and an end is called an activity. An arrow is used to depict an activity with its head indicating the direction of progress in the project. It is of four types:

a) **Predecessor activity:** activity that must be completed immediately prior to the start of another activity.

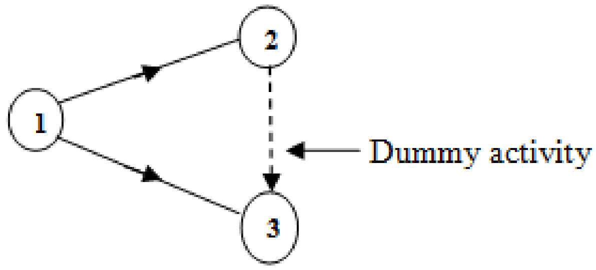
b) **Successor activity:** activity which cannot be started until one or more of other activities are completed but immediately succeed them are called successor activity.

c) **Concurrent:** Activity which can be accomplished concurrently is known as concurrent activity. An activity can be predecessor or successor to an event or it may be concurrent with the one or more of the other activities.

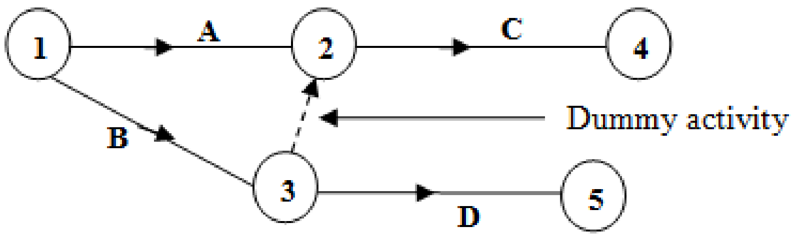
d) **Dummy activity:** An activity which does not consume any kind of resources but merely depicts the technological dependence is called a dummy activity. Dummy activity is inserted in a network to classify the activity pattern in the following situations:

i) To make activities with common starting and finishing points distinguishable.

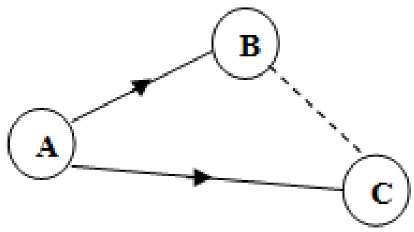
ii) To identify and maintain the proper precedence relationship between activities those are not connected by events.



Let's consider a situation where A and B are concurrent activities and activity D is dependent on B and C is dependent on both A and B. Such a situation can be handled by use of dummy activity.



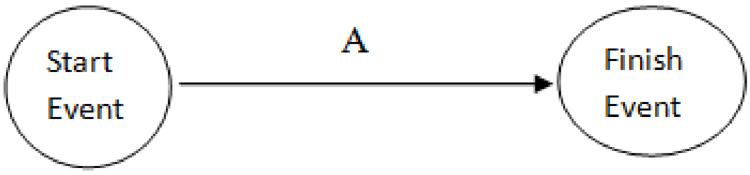
When two or more activities are exactly parallel such that they would start at the same node (event) and finish at the same node. A dummy would be inserted between the end of one of the activities and the common finishing node.



This is to ensure that each activity has a unique description when refer to by its start and finish node number. Dummy are often used to improve the layout of network. When they may not strictly necessary to represents the logic involved. This often happens at the start or finish of a network where a number of activities either start from a certain point or converge to particular point.

18.4.2 Event

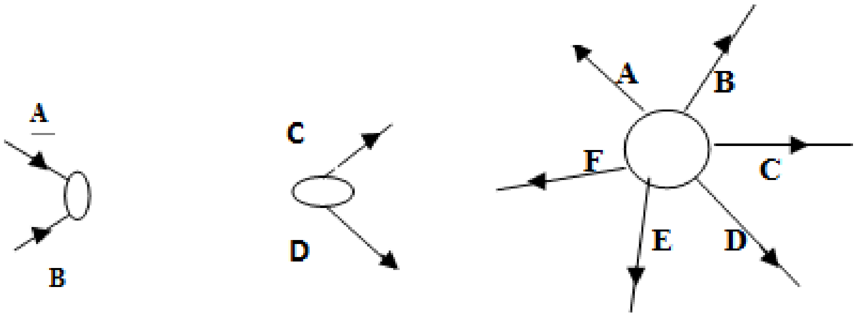
The beginning and end points of an activity are called events or nodes or connector. This is usually represented by circle in a network.



Here, A is known as the activity.

The events can be further classified into three categories:

a) **Merge Event**: When two or more activities come from an event it is known as merge event.
b) **Burst Event:** When more than one activity leaves an event is known as burst event.
c) **Merge & Burst Event:** An activity may be merged and burst at the same time.



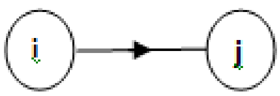
Merge event Burst event Merge and Burst event

18.4.3 Difference between event and activity

An event is that particular instant of time at which some specific part of project is to be achieved while an activity is the actual performance of a task. An activity requires time and resources for its completion. Events are generally described by such words as complete, start, issue, approves, taste etc. while the word like design, process, test, develop, prepare etc. shows that a work is being accomplished and thus represent activity. While drawing networks, it is assumed that

a) The movement is from left to right and
b) Head event has a number higher than the tail event.

Thus the activity (i-j) always means that job which begins at event (i) is completed at event (j).



Network representation is based on the following two axioms.

a) An event is not said to be complete until all the activities flowing into it are completed.
b) No subsequent activities can begin until its tail event is reached or completed.

Lesson 19

RULES FOR DRAWING NETWORK ANALYSIS

19.1 Introduction

A fundamental ingredient in both PERT and CPM is the use of network systems as a means of graphically depicting a project. When a network is being constructed, certain conventions are followed to represent a project graphically, for it is essential that the relationship between activities and events are correctly depicted. Drawing a network diagram is a relatively easy task, and can be accomplished by listing each task on a piece of paper and representing the sequence in which the tasks take place. Lines and arrows are drawn between the pieces of paper to show which tasks follow on from others. The diagram aims to portray how the tasks relate to one another i.e. which tasks have to be completed before others begin and which tasks can be performed simultaneously. Creating the network is an iterative process and may involve a number of revisions before an optimum solution is found.

19.2 Sequencing

The initial step in project scheduling process is the determination of all specific activities that comprise the project and their interdependence relationships. In order to make a network following points should be taken into consideration.

- What job or jobs precede it?
- What job or jobs run concurrently?
- What job or jobs follow it?
- What controls the start and finish of a job?

Since all further calculations are based on the network, it is necessary that a network be drawn with full care. There are many ways to draw a network, in this lesson we will describe the method which follows the precedence table. The following example of preparation of Paneer (Cottage cheese) shows the basic steps required in drawing a network.

Example 1:

For preparation of Paneer (Cottage Cheese) the following list represents major activities

i) Receive whole cow/buffalo milk
ii) Standardize milk to obtain desired level of fat percentage
iii) Take citric acid and prepare 1% solution
iv) Heat the citric acid to 70 ^{o} C
v) Bring the standardized milk to boil on medium heat
vi) Cool the milk to 70 ^{o} C and add slowly the solution of citric acid till yellowish whey separates
vii) Strain the mixture through a clean muslin cloth.
viii) Hold it under running water for a minute and then press out the excess water.
ix) Hang the muslin for 15-20 minutes so that all the whey is drained out.
x) Prepare mould to form Paneer block
xi) Fill the mass into the block and tie the muslin
xii) Place it under something heavy for up to two hours
xiii) Cut the paneer into chunks and used as required.

Based on above list of different activities a precedence table may be formed which is given in Table 19.1

Table 19.1 Precedence table

Activity	Description	Preceding Activity
A	Receive whole cow/buffalo milk	-
B	Standardize milk to obtain desired level of fat percentage	A
C	Take citric acid and prepare 1% solution	-

D	Heat the citric acid to 70 ^{o} C	C
E	Bring the standardized milk to boil on medium heat	B
F	Cool the milk to 70 ^{o} C and add slowly the solution of citric acid till yellowish whey separates.	D,E
G	Strain the mixture through a clean muslin cloth.	F
H	Hold it under running water for a minute and press out the excess water.	G
I	Hang the muslin for 15-20 minutes and drain out all the whey.	H
J	Prepare mould to form Paneer block	H
K	Fill the mass into the block and tie the muslin	J
L	Place it under something heavy for up to two hours.	K
M	Cut the paneer into chunks and use as required.	L

In the above table due consideration has been given to precedings of an activity. While drawing the network, other factors will be considered. The activity A has no preceding activity and it is represented by an arrow line (Fig. 19.1). Likewise activity C has no preceding activity and both activities A and C can be done simultaneously so they are shown as concurrent activities. Activities B and D are preceded by the activities A and C respectively. The complete network is shown in Fig 19.1

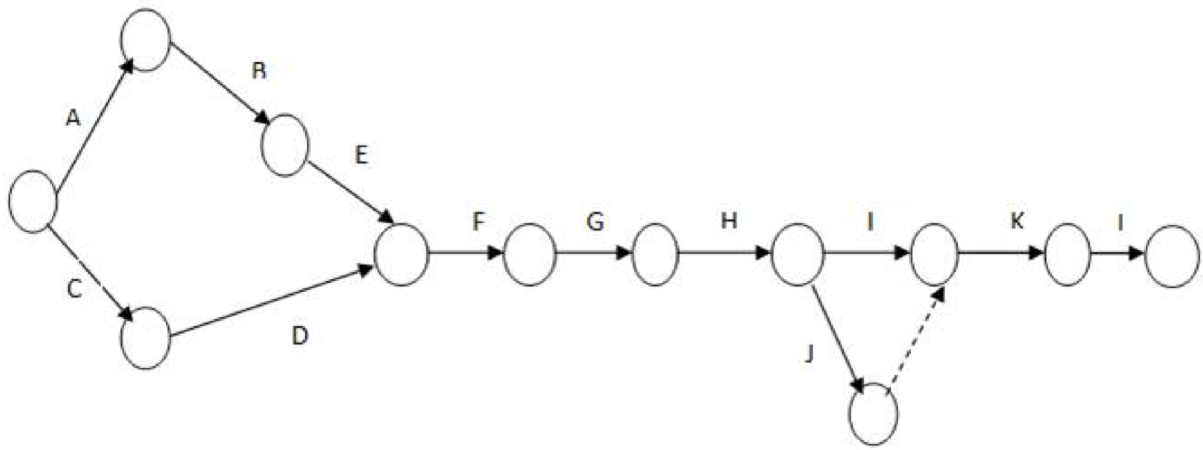


Fig. 19.1 Network diagram

19.3 Guidelines for Drawing Network Diagram

There are number of rules in connection with the handling of events and activities of a project network which are given below:

a) Each activity is represented by one and only one arrow in the network. This implies that no single activity can be represented twice in the network. This is to be distinguished from the case where one activity is broken into segments. In such a case each segment may be represented by a separate arrow.

b) No two activities can be identified by the same beginning and end event. In such cases, a dummy activity is introduced to resolve the problem as shown in Fig. 19.2

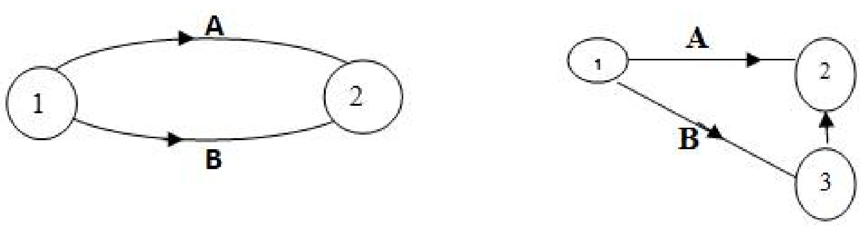


Fig. 19.2 Using a dummy activity

c) In order to ensure the correct precedence relationship in arrow diagram following question must be checked whenever any activity is added to a network.

What activity must be completed immediately before this activity can start?

What activities must follow this activity?

What activities must occur simultaneously with this activity?

d) Thus a network should be developed on the basis of logical or technical dependence.

e) The arrows depicting various activities are indicative of logical precedence only; hence length and bearing of the arrows are of no significance.

f) The flow of the diagram should be from left to right.

g) Two events are numbered in such a way that the event of higher number can happen only after the event of lower number is completed.

h) Arrows should be kept straight and not curved. Avoid arrow which cross each other.

i) Avoid mixing two directions vertical and standing arrows may be used if necessary.

j) Use dummy activity freely in rough graph but final network should have only reluctant dummy.

k) The network has only one entry point called the start event and one point of emergence called end event.

l) Angle between the arrows should be as large as possible.

19.4 Error in Drawing Network

There are three types of errors which are common in network diagrams

19.4.1 Dangling error

To disconnect an activity before the completion of all activities in a network diagram is known as dangling.

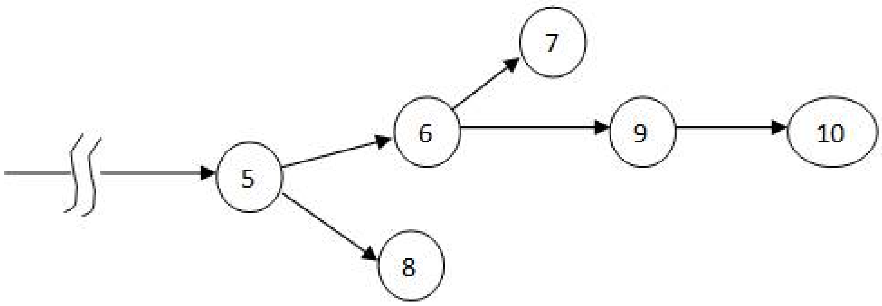


Fig. 19.3 Dangling error

In Fig. 19.3 the activity 5 to 8, 6 to 7 are known as dangling error. These are not last activities in the network.

19.4.2 Looping error

Looping error is also known as cyclic error in the network. Drawing an endless loop in a network diagram is known as error of looping as shown in Fig. 19.4

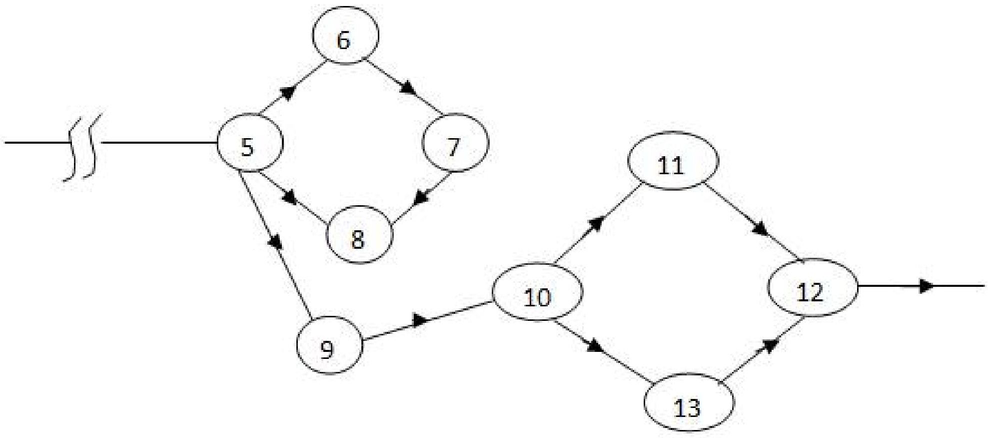


Fig. 19.4 Appearance of a loop in the network

19.4.3 Reductancy Error:

Unnecessarily inserting the dummy activity in a network diagram is known as error of reductancy as shown in Fig 19.5 in which putting an dummy activity from 10 to 12 is a reductancy error.

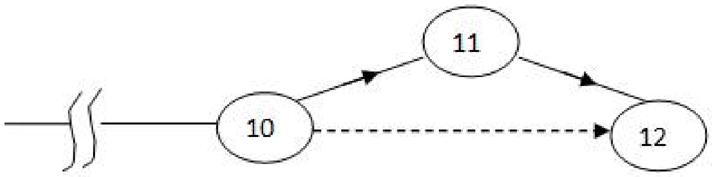


Fig. 19.5 Reductancy error

19.5 Labeling of a Network Diagram

For network representation it is necessary that various nodes be properly labeled. For convenience, labeling is done on a network diagram. A standard procedure called i-j rule developed by D.R.F Fulkerson is most commonly used for this purpose.

Fulkerson's i-j Rule:

Step 1: First, a start event is one which has arrows emerging from it but not entering it. Find the start event and label it as number1.

Step 2: Delete all arrows emerging from all numbered events. This will create at least one new start event out of the preceding events.

Step 3: Number all new start events '2', '3' and so on. No definite rule is necessary but numbering from top to bottom may facilitate other users using the network when there are more than one new start event.

Step 4: Go on repeating step no. 2 & 3 until the end reached.

These rules are illustrated by taking into consideration the Example 19.1 and network diagram as shown in Fig.19.6

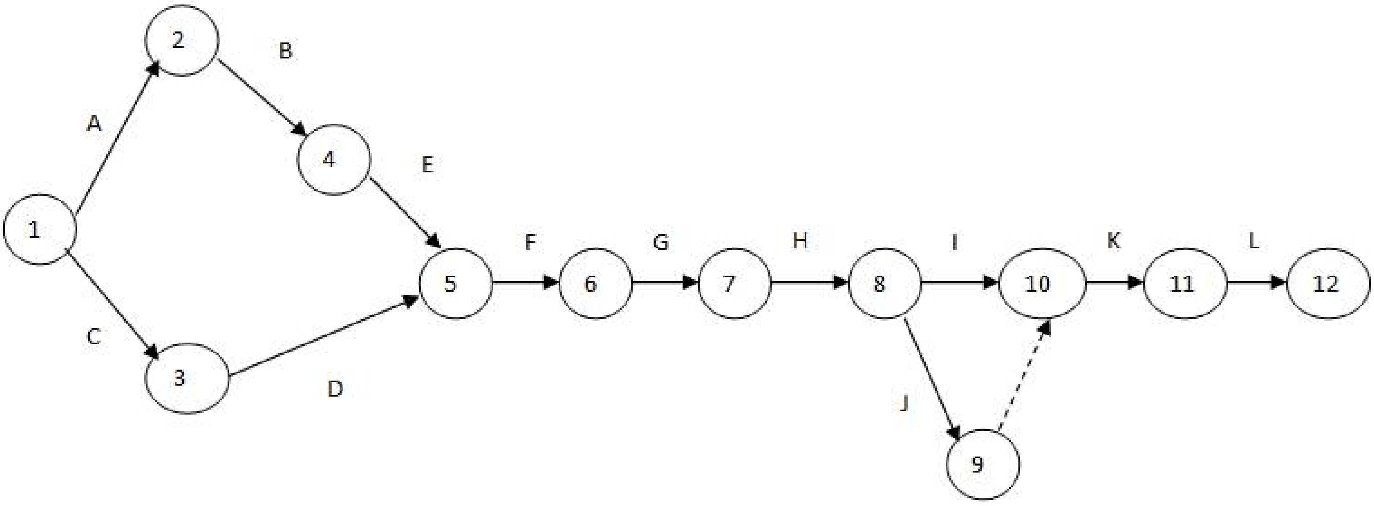


Fig. 19.6 Network diagram of preparation of paneer

Lesson 20

CRITICAL PATH METHOD (CPM)

20.1 Introduction

After the project network plan is constructed and activity times are known, the time analysis of the network becomes essential for planning various activities of the project as well as obtaining answers to questions like when the various activities are scheduled to be performed, how long it will take the project work to be completed and what are the crucial activities. In this lesson we will learn about computation of time estimates and determination of critical path.

20.2 Time Estimate Analysis

Time estimates analysis is a critical path in network analysis once the network of a project is constructed. Time analysis of network becomes essential for planning various activities of the project. An activity time is a forecast of the time and activity expected today from its starting point to it completion point (under normal conditions). The main objective of time analysis is to prepare a planning schedule of a project. Planning schedule should include the following factors.

i) Total completion time of the project.
ii) Earliest time and each activity start.
iii) Latest time each activity can be started without delaying the total project
iv) Float for each activity i.e. the amount of time by which the completion of an activity can be delaying the total project completion
v) Identification of critical activities and critical path.

20.3 Basic Notations

The following notations for basic scheduling computations will be used

(i–j)	=	Activity (i, j) with tail event i and head event j.
T_E or E_i	=	Earliest occurrence time of event (j).
T_L or L_f	=	Latest allowable occurrence time of event (j).
D_{ij}	=	Estimated completion time of activity (i, j).
$(E_S)_{ij}$	=	Earliest starting time of activity (i, j)
$(E_f)_{ij}$	=	Earliest finishing time of activity (i, j).
$(L_s)_{ij}$	=	Latest start time for activity (i, j).
$(L_f)_{ij}$	=	Latest finish time for activity (i, j).

20.4 Forward Pass Computation (For Earliest Event Time)

Before starting computations, the occurrence time of the initial network event is fixed. The forward pass computation computes the earliest start time (E_s) and earliest finish time (E_f) for each activity. The earliest time indicates the earliest time that a given activity can be scheduled and earliest finish time indicates the time by which the activity can be completed at the earliest. This is done in following three steps:

Step 1: The computations begin from the start node and move towards the end node. For the convenience in the forward pass computation start with an assumed earliest occurrence time equal to zero for the initial project events i.e. $E_1=0$

Step 2: i) Earliest starting time of activity (i, j) is the earliest event time of the tail event i.e. $(E_s)_{ij} = E_i$

ii) Earliest finish time of activity (i, j) is the addition of earliest starting time and the activity time i.e.

$$(E_f)_{ij} = (E_s)_{ij} + D_{ij} \text{ or } (E_f)_{ij} = E_i + D_{ij}$$

Step 3: Earliest event time for activity j is the maximum of the earliest finish time of all activities ending into that event. That is

$$E_f = \max_{i} [(E_f)_{ij} \; for\ all\ immediate\ preceding\ activities\ (i,j)]$$

$$= \max_{i} [E_i + D_{ij}]$$

The computed values of E_j are put over the respective circles representing each event.

20.5 Backward Pass Computations (For latest allowable time)

The idea of the backward pass is to compute the latest allowable times of starting and finishing of each of the activities of a project without delaying the completion of the project. These can be computed by reversing the method of calculation used for earliest event time. This is done by using following steps

Step 1: For ending event it is presumed that E = L where all E's are computed by previous method.

Step 2: Latest finish time for activity (i, j) is equal to the latest event time of event j. i.e., $(L_f)_{ij}=L$.

Step 3: Latest starting time for activity (i, j) is equal to latest completion time of (i, j) minus activity time $(Ls)_{ij} = (L_f)_{ij} - D_{ij}$

Step 4: Latest event time for event i is a minimum of the latest start time of all activities originating from that event.

$$L_j = \min_{j} [(L_S)_{ij} \; for\ all\ immediate\ successors\ of\ (i,j)]$$

$$L_j = \min_{j} \left[(L_f)_{ij} - D_{ij} \right] = \min_{j} [L_j - D_{ij}]$$

All the computed L values are put over respective circles representing each event.

20.6 Determination of Floats and Slack Times

When the network diagram is completely drawn, properly labeled and earliest (E) and latest (L) event times are computed, the next objective is to determine the floats and slack times defined as follows. There are three kinds of floats as given below :

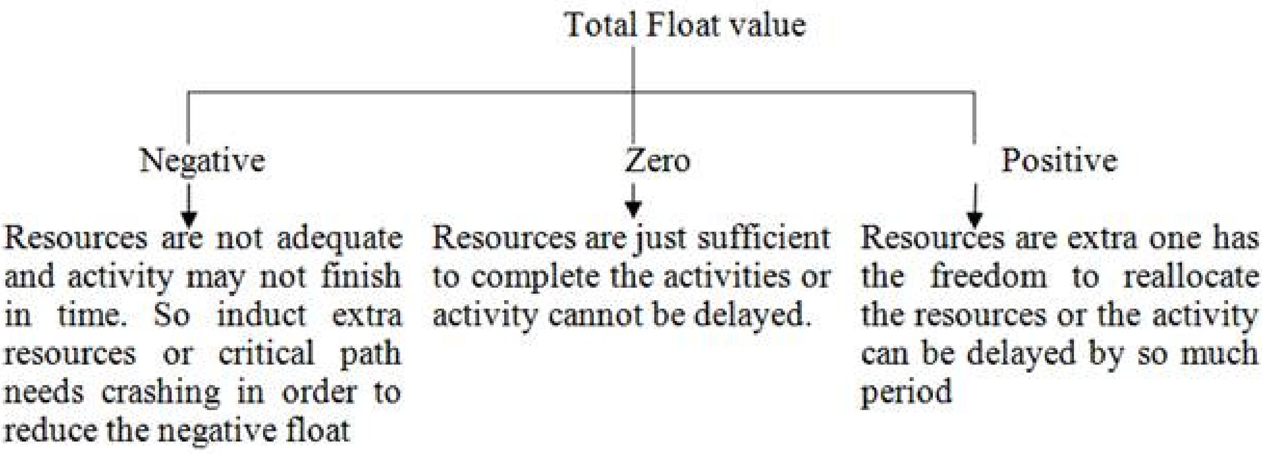
20.6.1 Total float

The amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time. In other words, total float of an activity (i-j) is the difference between latest start time and earliest start time of that activity. Hence total float for activity (i-j) , denoted by $(T_f)_{ij}$ is given as

$$(T_f)_{ij} = \text{(Latest start-Earliest start) time for activity (i-j)}$$

$$(T_f)_{ij} = (L_s)_{ij} - (E_s)_{ij} = (L_f - D_{ij}) - E_i$$

It refers to the amount of free time associated with an activity which can be used before, during or after the performance of this activity. This is the most important type of float because this is concerned with the overall project duration. Total float on critical activities is always taken as zero. The value of total floats for any activity is useful for drawing the following conclusions.



20.6.2 Free float

The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent (succeeding) activity. This is that value of the float which is consumable when the succeeding activities are started at their earliest starting times. Mathematically, free float for activity (i-j) denoted by $(F_f)_{ij}$ is calculated as $(F_f)_{ij} = (E_j - E_i) - D_{ij}$

Free float for (i-j)=(Earliest time for event j–Earliest time for event i)– Activity time for (i-j)

Thus free float is concerned with the commencement of subsequent activity.

$(T_f)_{ij} = (L_j - E_i) - D_{ij}$,but $L_j \geq E_i$ as latest event time is always greater than equal to earliest event time. Therefore for all activities free float can take values from zero up to total float but will not exceed total float. Free float is always useful for rescheduling the activities with minimum disruption of earliest plan.

20.6.3 Independent float

The amount of time by which the start of an activity can be delayed without affecting the earliest start time of any immediately following activities, assuming that the preceding activity has finished at its latest finish time. Mathematically, independent float of an activity (i ,j) denoted by $(I_f)_{ij}$ can be calculated by the formula $(I_f)_{ij} = (E_j - E_i) - D_{ij}$. The negative independent float is always taken as zero. This float is concerned with prior & subsequent activity. The independent float thus provides a measure of variation in starting time of a job without affecting preceding and succeeding activities.

Note:

- It can be observed that Independent float ≤ Free float ≤ Total float.
- The concept of float is useful for the management in representing underutilized resources and flexibility of the schedule and the extent to which the resources will be utilized on different activities.
- Float can be used for redeployment of resources to label the same or to reduce the project duration. Whenever a float in a particular activity is utilized the float of not only that activity but that of other

activities would also change.

20.7 Slack of an Event

The basic difference between slack and float times is that slack is used for events only whereas float is applied for activities. For any given event, the event slack is defined as the difference between the latest event and earliest event times. Mathematically, for a given activity (i-j)

$$\text{Head event slack} = L_j - E_j \text{ and Tail event slack} = (L_i - E_i)$$

All the floats defined earlier can be defined in terms of head or tail events slack as under:

$$\text{Total float} = L_j - E_i - D_i$$

Free float = $(E_j - E_i - D_{ij}) = (L_j - E_i - D_{ij}) - (L_{ij} - E_{ij})$ = Total float – Head event slack

Independent float $= E_j - L_i - D_{ij} = (E_j - E_i - D_{ij}) - (L_i - E_i)$

= Free float – Tail event slack

20.8 Determination of Critical Path

After determining the earliest and latest scheduled times for various activities, the next step is to find the minimum time required for the completion of whole project. Before defining this let us first discuss about the meaning of critical event and critical activity.

20.8.1 Critical event

The slack of an event is the difference between latest and earliest event time i.e. Slack (i) = $L_i - E_i$. The event with zero slack time is called critical event. In other words, the event (i) is said to be critical when $L_i = E_i$

20.8.2 Critical activity

Since the difference between the latest Start time & earliest start time of an activity is usually called as total float. Activity with zero total float are known as critical activities. In other words, an activity is said to be critical if its delay in its start will cause a further delay in the completion date of entire project.

20.8.3 Non-critical activity

A non-critical activity is such that the time between its earliest start and its latest completion date is longer than its actual duration.

20.8.4 Critical path

The sequence of critical activity in a network is called a critical path. This path is the longest path in the network from the starting event to the end of event and defines the minimum time required to complete the project. The term path is defined as a sequence of activities such that it begins at the starting event and end at the final event.

The length of the path is the sum of the individual time of the activities lying on the path. If the activities on critical path are delayed by a day, the project would also be delayed by a day unless the time of the future critical activity is reduced by a day by different means. The critical path is denoted by double or darker lines in order to distinguish from the other non critical path.

20.8.4.1 Main features of critical path

- If the project has to be shortened then some of the activities on that path must also be shortened. The application of additional resources on other activities will not give the desired results unless that critical path is shortened first.

- The variation in actual performance from the expected duration time will be completely reflected in one to one fashion in anticipation completion of the whole project.

- It plays an important role in scheduling and controlling large projects.

- It identifies all the critical activities of the project.

The computation procedure used for the time analysis of the project is described in the examples 1and 2

Example 1

Draw a network diagram of the following schedule of activities and find its critical path. Also calculate slack time for each event

Activity	1-2	1-3	1-4	2-6	3-7	3-5	4-5	5-9	6-8	7-8	8-9
Duration (in days)	2	2	1	4	5	8	3	5	1	4	3

Solution :

First construct the network diagram which is depicted in fig. 20.1

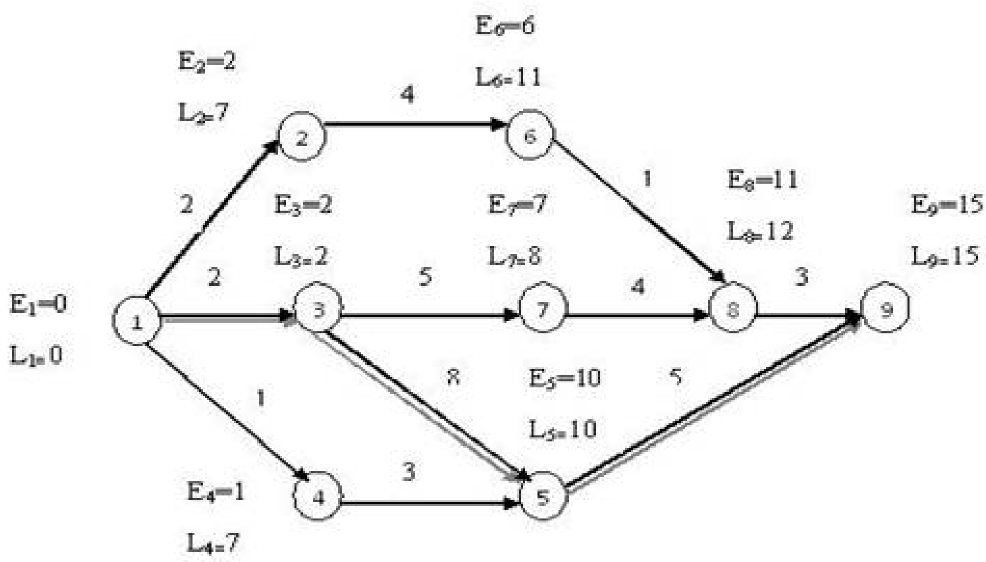


Fig. 20.1 Network diagram

To determine the critical path, compute the earliest start E_i and latest finish L_j for each activity (i, j)

Forward Pass Computation (For earliest time event): As shown in fig. 20.1 assuming $E_1=0$ and $E_2=E_1+D_{12}=0+2=2$, $E_3=E_1+D_{13}=0+2=2$ and $E_4=E_1+D_{14}=0+1=1$

$$E_6=E_2+D_{26}=2+4=6, E_7=E_3+D_{37}=2+5=7$$

$$E_8 = max \begin{bmatrix} E_7 + D_{78} & = & 7+4=11 \\ E_6 + D_{68} & = & 6+1=7 \end{bmatrix} = 11$$

$$E_9 = max \begin{bmatrix} E_8 + D_{89} = 11+3=14 \\ E_5 + D_{59} = 10+5=15 \end{bmatrix} = 15$$

From this computation it can be inferred that this project will take 15 days to complete.

Backward Pass computations (For latest allowable time) In backward computation method assign the latest allowable time determined in forward pass computation method i.e. put $L_9=15$

$L_8=L_9-D_{98}=15-3=12$, $L_6=L_8-D_{86}=12-1=11$, $L_7=L_8-D_{87}=12-4=8$

$L_5=L_9-D_{95}=31-6=25$,

$$L_4 = min \begin{bmatrix} L_9 - D_{94} = 31-8=23 \\ L_7 - D_{94} = 23-6=17 \end{bmatrix} = 17$$

$L_4=L_5-D_{54}=10-3=7$ $L_2=L_6-D_{62}=11-4=7$

$$L_3 = min \begin{bmatrix} L_5 - D_{52} = 31-6=25 \\ L_6 - D_{31} = 2-2=0 \\ L_4 - D_{41} = 7-1=6 \end{bmatrix} = 0$$

The path 1-3-5-9 is the critical path which is shown in Fig. 20.1 with double lines joining all those events where $E_i=L_j$ The total duration of project is equal to 2+8+5=15 days

Computation of Float: For each non –critical activity, the total float, free float and independent float calculations are given Table 20.1

Table 20.1 Calculations of time estimates and floats

Activity (i-j) (1)	Duration D_{ij} (2)	Start		Finish		Float		
		Earliest (3)	Latest (4)= (6)– (2)	Earliest (5) = (3) + (2)	Latest (6)	Total (7) =(4) - (3)	Free (8) =(5) - (3) -(2)	Independent (9)= (8) - [(3)-(2)]
1-2	2	0	5	2	7	5	0	0
1-3	2	0	0	2	2	0	0	0

1-4	1	0	6	1	7	6	0	0
2-6	4	2	7	6	11	5	0	0
3-7	5	2	3	7	8	1	0	0
3-5	8	2	2	10	10	0	0	0
4-5	3	1	7	4	10	6	6	0
5-9	5	10	10	15	15	0	0	0
6-8	1	6	11	7	12	5	4	0
7-8	4	7	8	11	12	1	0	0
8-9	3	11	12	14	15	1	1	0

Example 2:

Draw the network diagram for the following project and find the critical path and maximum time for completion of the project.

Activity		A	B	C	D	E	F	G	H	I	J	K	L
Preceded by		-	A	A	B	B	C	C	F	D	G, H	E	I
Duration (weeks)		10	9	7	6	12	6	8	8	4	11	5	7

Solution :

Network diagram of the above problem is shown in Fig. 20.2.

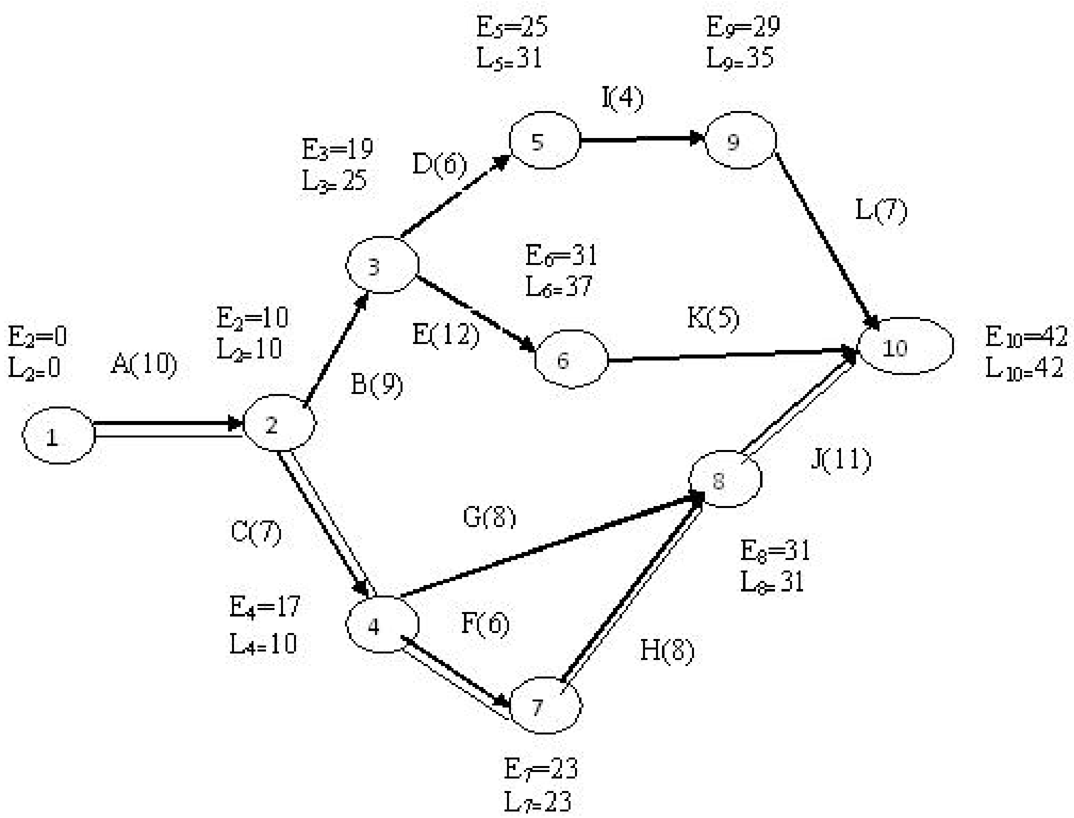


Fig. 20.2 Network diagram

To determine the critical path, compute the earliest start E_i and latest finish time L_j for each activity (i,j).The calculations are given below

Forward Pass Computation (For earliest time event): As shown in fig 20.2 assuming E_1=0 and $E_2=E_1+D_{12}=0+10=10$, $E_3=E_2+D_{23}=10+9=19$, $E_4=E_2+D_{24}=10+7=17$, $E_5=E_3+D_{35}=19+6=25$, $E_6=E_3+D_{36}=19+12=31$, $E_7=E_4+D_{47}=17+6=23$,

$$E_8 = max\begin{bmatrix} E_4 + D_{48} & = & 17+8=25 \\ E_7+D_{78} & = & 23+8=31 \end{bmatrix} = 31$$

$$E_9=E_5+D_{59}=25+4=29,$$

$$E_{10} = max\begin{bmatrix} E_9 + D_{910} = 29+7=36 \\ E_6+D_{610} = 31+5=36 \\ E_8+D_{810} = 31+11=42 \end{bmatrix} = 42$$

From this computation it can be inferred that this project will be completed in 42 weeks.

Backward Pass computations (For latest allowable time) in backward computation method assign the latest allowable time determined in forward pass computation method i.e. put L_{10}=42

$$L_9=L_{10}-D_{109}=42-7=35,\ L_6=L_{10}-D_{106}=42-5=37,\ L_8=L_{10}-D_{108}=42-11=31,$$

$$L_5=L_9-D_{95}=35-4=31,\ L_7=L_8-D_{87}=31-8=23$$

$$L_4 = min\begin{bmatrix} L_9 - D_{94} = 31-8=23 \\ L_7-D_{74} = 23-6=17 \end{bmatrix} =17$$

$$L_3 = min\begin{bmatrix} L_5 - D_{53} = 31-6=25 \\ L_6-D_{63} = 37-12=25 \end{bmatrix} = 25$$

$$L_2 = min\begin{bmatrix} L_3 - D_{32} = 25-10=15 \\ L_4-D_{42} = 17-7=10 \end{bmatrix} = 10$$

$$L_1=L_2-D_{21}=10-10=0$$

The path 1-2-4-7-8-10 is the critical path which shown in Fig. 20.2 with double lines joining all those events where $E_i=L_j$ The total duration of project is equal to 10+7+6+8+11=42 weeks

Activity (i-j) (1)	Duration D_{ij} (2)	Start		Finish		Float		
		Earliest (3)	Latest (4)	Earliest (5)	Latest (6)	Total (7) =(4) - (3)	Free (8)= (5) -(3) -	Independent (9)= (8) - [(3)-(2)]

			= (6)− (2)	=(3)+(2)			(2)	
A(1-2)	10	0	0	10	10	0	0	0
B(2-3)	9	10	16	19	25	6	0	0
C(2-4)	7	10	10	17	17	0	0	0
D(3-5)	6	19	25	25	31	6	0	0
E(3-6)	12	19	25	31	37	6	0	0
F(4-7)	6	17	17	23	23	0	0	0
G(4-8)	8	17	23	25	31	6	6	0
H(7-8)	8	23	23	31	31	0	0	0
I(5-9)	4	25	31	29	35	6	2	0
J(8-10)	11	31	31	42	42	0	0	0
K(6-10)	5	31	37	36	42	6	6	0
L(9-10)	7	29	35	36	42	6	6	0

20.9 Critical Path Method (CPM)

Critical Path Method (CPM) , was developed by M.R.Walker and J. E. Kelly. They came up with arrow diagram as the most logical representation of the interrelationships between the jobs in a project to be executed in a well defined sequence. The arrow diagram designed by them, as well as the method of calculating the critical path are the same as in PERT network, except that they used the single time estimate and did not enter the problem of uncertainty of the duration of time for the individual jobs.

CPM emphasizes the relationship between applying more men or other resources to shorten the duration of given jobs in a project and the increased cost of these additional resources. With CPM the amount of time needed to complete various parts of the project is assumed to be known with certainty. Moreover, the relation between the amount of resources employed and the time needed to complete the project is also assumed to be known. The interactive procedure of determining the critical path involves the following steps:

i) Break down the project into various activities systematically. Label all activities. Arrange all the activities in logical sequence. Construct the arrow diagram.

ii) Number all the nodes (events) and activities. Find the time for each activity considering it to be deterministic. Indicate the activity times on the arrow diagram.

iii) Calculate earliest start time, earliest finish time, latest start time and latest finish time. Tabulate activity normal times, earliest time and latest time.

iv) Determine the total float for each activity by taking difference between the earliest time and the latest time for each node.

v) Identify the critical activities (the activities with zero float) and connect them with the beginning node and the ending node in the network diagram by double line arrow. This gives the critical path.

vi) Calculate the total project duration.

Lesson 21

PROJECT EVALUATION AND REVIEW TECHNIQUE (PERT)

21.1 Introduction

The network method discussed so far may be termed as deterministic, since estimated activity times are assumed to be known with certainty. While this assumption holds for the CPM analysis. In most of the projects, these activity times are random variables. A new technique known as Project Evaluation and Review Technique (PERT) was devised in 1958 for the POLARIS missile program by the Program Evaluation Branch of the Special Projects office of the U. S. Navy, helped by the Lockheed Missile Systems division and the Consultant firm of Booz-Allen & Hamilton.

21.2 Project Evaluation & Review Technique (PERT)

In research project of designing a new machine or development of a new dairy product, various activities to be performed are based on judgment. A reliable time estimate is difficult to get because the technology is changing rapidly. Time values are subjected to variation. The main objective of the analysis through PERT is to find out the completion for a particular event within specified date. What are the chances of completing the job? This approach takes into account uncertainties. In this approach three time values are estimated with each activity: Optimistic time, most likely time and Pessimistic time. The three time values provide a measure of uncertainty associated with that activity.

21.2.1 Optimistic time

It is the shortest possible time in which the activity can be finished and assumes that everything goes very well. In other words, it is the estimate of the minimum possible time, which an activity takes to complete under ideal conditions i.e. no provision are made for breakdown, delays etc. They are generally denoted by (t_0) or (a).

21.2.2 Most likely time

This is estimate of the normal time the activity would take. This assumes normal delays. It is denoted by (t_m) or (m). If a graph is plotted between the time of completion and frequency of completion in that period, the highest frequency of occurrence is denoted by most likely time as shown in Fig. 21.1.

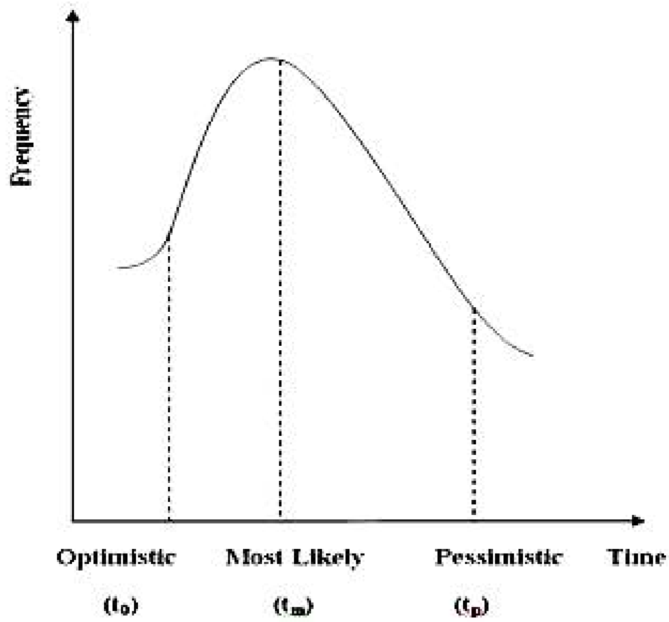


Fig. 21.1 Time distribution curve

21.2.3 Pessimistic time

The longest time, the activity could take if everything goes wrong. In other words, it is the longest time the activity can conceivably take. This is generally denoted by (t_p) or (b).

The three time values are shown in Fig. 21.1.

The PERT technique makes the following assumptions:

- Activity times are statistically independent and usually associated with 'beta' distribution.
- There are enough activities involved in the network and totals of activity times based on their means and variances will be normally distributed.
- The three estimates of the activity duration can be obtained for each activity.

In PERT calculation, all values are used to obtain the expected value.

21.2.4 Estimated time

This is the average time an activity will take if it is to be repeated large number of times and is based on the assumption that the activity time follows Beta distribution.

$$t_e = \frac{t_0 + 4t_m + t_p}{6} \quad or \quad \frac{a + 4m + b}{6}$$

21.2.5 Variance

Variance of each activity is given by formula:

$$\sigma^2 = \left(\frac{t_p - t_e}{6}\right)^2 = \left(\frac{b - a}{6}\right)^2$$

where t_o (a) , t_p (b) and t_m(m) are optimistic , pessimistic and mostly likely times respectively.

Once the expected times of the activities are obtained, the critical path of the project network is determined using three time estimates. Having found the critical path, the PERT methodology assumes that the aggregation of the mean times and the summation of the variances of critical jobs would yield the expected project duration and its variance. PERT uses the variance of critical path activity to help in determining the variance of the overall project. Project variance is computed by summing variance of just critical activities.

21.3 PERT Algorithm

The various steps involved in the PERT network for analyzing any project are summarized below:

i) Develop a list of activities involved in the project including the immediate predecessors.

ii) Draw the network diagram using the rules and conventions as discussed before.

iii) Number the events in ascending order from left to right

iv) From the three time estimates compute the expected time (t_e) for each activity using the formula

$$t_e = \frac{t_0 + 4t_m + t_p}{6}$$

Using the expected activity time estimates, determine the earliest start time and earliest finish time for each activity .

v) Compute the latest start time and latest finish time and the float associated with each activity. Find the activities with zero total float which are known as critical activities. From these critical activities find the critical path.

vi) Using the value for t_p (b) and t_o(a) , calculate the variance (σ^2) by using the formula

$$\sigma^2 = \left(\frac{t_p - t_e}{6}\right)^2 = \left(\frac{b - a}{6}\right)^2$$

vii) Use the variability in the activity times to estimate the variability of the project completion date; using this estimate compute the probability of meeting a specified date by using the standard normal equation

$$Z = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$$

where Z is a standard normal variate

21.4 Illustrative Examples on PERT

The computation procedure used for PERT is described in the examples 1and 2

Example 1:

A small project is composed of nine activities whose time estimates are listed in the following table:

Activity	t_0	t_p	t_m
1-2	5	10	8
1-3	18	22	20
1-4	26	40	33
2-5	16	20	18
2-6	15	25	20
3-6	6	12	9
4-7	7	12	10
5-7	7	9	8
6-7	3	5	4

a) Find the expected task time and their variance.

b) Earliest and latest expected time of each node.

c) Critical path

d) Probability that project will complete in 41.5 weeks and 44weeks.

Solution:

The expected task time and variances of different activities are computed by the following formulae:

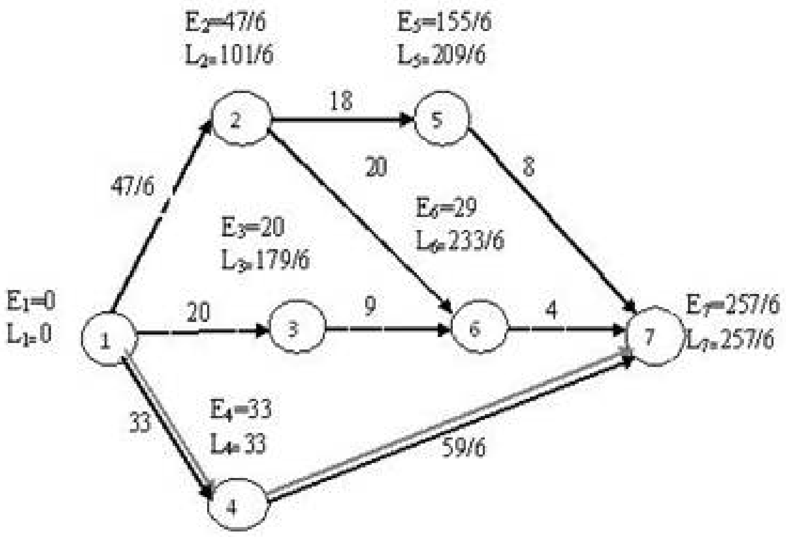
$$t_e = \frac{t_0 + 4t_m + t_p}{6}$$

and

$$\sigma^2 = \left(\frac{t_p - t_e}{6}\right)^2$$ and these values are given in following table

Activity	t_0	t_p	t_m	$t_e = \frac{t_0 + 4t_m + t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_e}{6}\right)^2$
1-2	5	10	8	47/6	25/36
1-3	18	22	20	20	16/36
1-4	26	40	33	33	196/36
2-5	16	20	18	18	16/36
2-6	15	25	20	20	100/36
3-6	6	12	9	9	36/36
4-7	7	12	10	59/6	25/36
5-7	7	9	8	8	4/36
6-7	3	5	4	4	4/36

Construct the network diagram showing earliest and latest expected time of each node.



Rule to find out variance of Events:-

We take initial value of variance $V_1 = 0$

$$V_j = V_i + \sigma^2{}_{ij}$$

$V_2 = V_1 + \sigma_{12}^2 = 0 + \frac{25}{36} = \frac{25}{36}$, $V_3 = V_1 + \sigma_{13}^2 = 0 + \frac{16}{36} = \frac{16}{36}$,

$V_4 = V_1 + \sigma_{14}^2 = 0 + \frac{196}{36} = \frac{196}{36}$,

$V_5 = V_2 + \sigma_{25}^2 = \frac{25}{36} + \frac{16}{36} = \frac{41}{36}$,

$$V_6 = \begin{cases} V_2 + \sigma^2{}_{26} = \frac{25}{36} + \frac{100}{36} = \frac{125}{36} \\ V_3 + \sigma^2{}_{36} = \frac{16}{36} + \frac{36}{36} = \frac{52}{36} \end{cases} = \frac{125}{36}$$

$$V_7 = \begin{cases} V_4 + \sigma^2{}_{47} = \frac{196}{36} + \frac{25}{36} = \frac{221}{36} \\ V_5 + \sigma^2{}_{57} = \frac{41}{36} + \frac{4}{36} = \frac{45}{36} \\ V_6 + \sigma^2{}_{67} = \frac{25}{36} + \frac{100}{36} = \frac{125}{36} \end{cases} = \frac{221}{36}$$

NOTE: At merge point the variance is computed along the longest path in case of two path having the same length the larger of the two variance of that event.

Critical path is 1-4-7 and total duration is 257/36 = 42.833 weeks with variance = 221/36

Now probability that project will complete in 41.5 weeks is

$$\text{Prob}\left(D \le 41.5\right) = P\left[Z\frac{41.5 - 42.833}{\sqrt{221/_{36}}}\right]\Bigg) = P(Z \le -0.538)$$

$$= 0.5 - P(0.5 - (0 \le Z \le -0.53) = 0.2981$$

Hence, there are 29.81% chances that project will be completed before 41.5 weeks.

Now probability that project will complete in 44 weeks is

Hence, there are 99.81% chances that project will be completed before 44 weeks.

Example 2

The following table gives the estimates of optimistic time (t_0), most likely time (t_m) and pessimistic time (t_p) of different activities of a project.

Activity	t_0	t_m	t_p
1-2	4	8	12
2-3	1	4	7
3-4	8	12	16
3-5	3	5	7
4-5	0	0	0
4-6	3	6	9
5-7	3	6	9
5-8	4	8	6
6-10	4	6	8
7-9	4	8	12
8-9	2	5	8
9-10	4	10	16

i) Construct the network diagram when it is given that scheduled completion is 40 days.

ii) Calculate the probability of finishing the project

   a) within the scheduled time

   b) less than 45days

   c) less than 38 days.

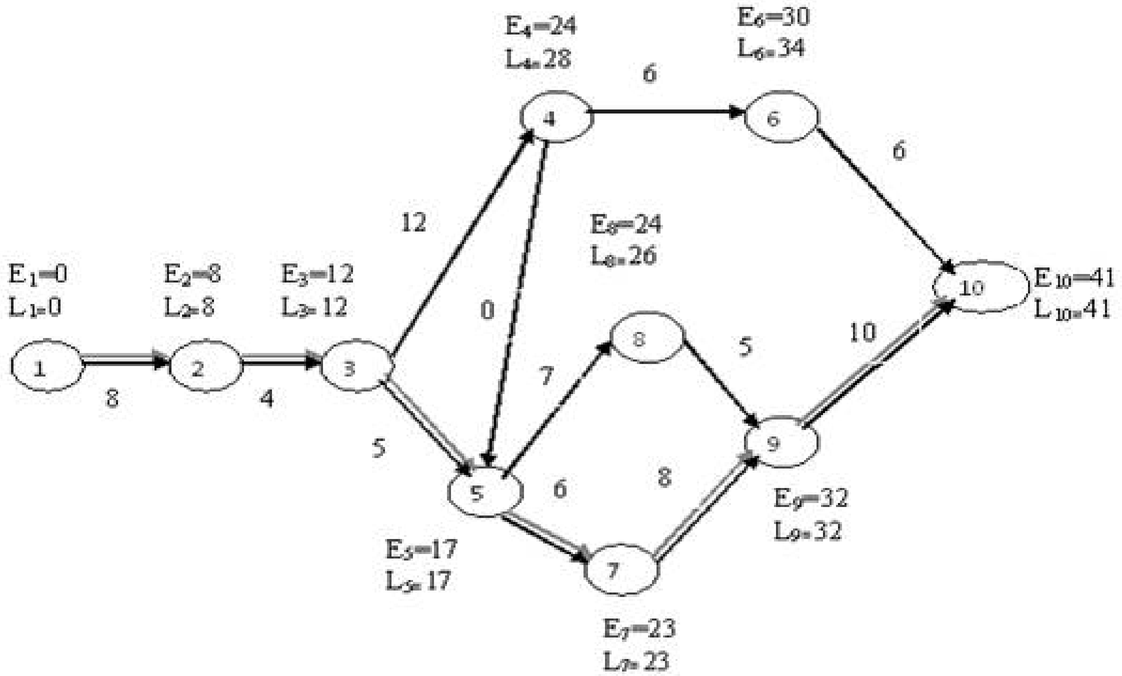
Solution :

The expected task time and their variances are given in following table

Activity	t_0	t_m	t_p	t_e	σ^2
1-2	4	8	12	8	64/36
2-3	1	4	7	4	36/36
3-4	8	12	16	12	64/36
3-5	3	5	7	5	16/36
4-5	0	0	0	0	0
4-6	3	6	9	6	36/36
5-7	3	6	9	6	36/36
5-8	4	8	6	7	4/36
6-10	4	6	8	6	16/36
7-9	4	8	12	8	64/36
8-9	2	5	8	5	36/36
9-10	4	10	16	10	144/36

From the table it is clear that the activity 4-5 is dummy activity.

The network diagram with earliest time and latest times are given below



Rule to find out variance of Events:

We take initial value of variance $V_1 = 0$

$$V_j = V_i + \sigma_{ij}^2$$

$$V_2 = V_1 + \sigma_{12}^2 = 0 + \frac{64}{36} = \frac{64}{36}, \quad V_3 = V_2 + \sigma_{23}^2 = \frac{64}{36} + \frac{36}{36} = \frac{100}{36},$$

$$V_4 = V_3 + \sigma^2{}_{34} = \frac{100}{36} + \frac{64}{36} = \frac{164}{36},$$

$$V = V + \sigma^2 = \frac{100}{36} + \frac{16}{36} = \frac{116}{36}$$

$$V_5 = V_3 + \sigma^2{}_{35} = \frac{\quad}{36} + \frac{\quad}{36} = \frac{\quad}{36},$$

$$V_6 = V_4 + \sigma^2{}_{46} = \frac{164}{36} + \frac{36}{36} = \frac{200}{36},$$

$$V_7 = V_5 + \sigma^2{}_{57} = \frac{116}{36} + \frac{36}{36} = \frac{152}{36},$$

$$V_8 = V_5 + \sigma^2{}_{58} = \frac{116}{36} + \frac{4}{36} = \frac{120}{36},$$

$$V_9 = \begin{cases} V_7 + \sigma^2{}_{79} = \frac{152}{36} + \frac{64}{36} = \frac{216}{36} \\ V_8 + \sigma^2{}_{89} = \frac{120}{36} + \frac{36}{36} = \frac{156}{36} \end{cases} = \frac{216}{36},$$

$$V_{10} = \begin{cases} V_6 + \sigma^2{}_{610} = \frac{200}{36} + \frac{216}{36} = \frac{216}{36} \\ V_9 + \sigma^2{}_{910} = \frac{216}{36} + \frac{144}{36} = \frac{360}{36} \end{cases} = \frac{360}{10} = 10$$

The critical path is 1-2-3-5-7-9-10 with expected time for completion of the project is 41 days with variance =10.

a) Probability that project will complete in given time i.e. 40 days is

$$\text{Prob}\left(D \le 40\right) = P\left[Z = \frac{40 - 41}{\sqrt{10}}\right]\Big) = P(Z \le -0.316)$$
$$= 0.5 - P(0.5 - (0 \le Z \le 0.32) = 0.5 - 0.1217 = 0.3783$$

Hence, there are 37.83% chances that project will be completed in the stipulated time of 40 days.

b) Probability that project will complete in 45 days is

$$\text{Prob}\left(D \le 45\right) = P\left[Z = \frac{45 - 41}{\sqrt{10}}\right]\Big) = P(Z \le -1.26) = 0.5 + P(0 \le Z \le 1.26)$$
$$= 0.5 + 0.3962 = 0.8962$$

Hence , there are 89.62% chances that project will be completed before 45 days.

c) Probability that project will complete in 38 days is

$$\text{Prob}\left(D \le 45\right) = P\left[Z = \frac{38 - 41}{\sqrt{10}}\right]\Big) = P(Z \le -0.948)$$
$$= 0.5 - P(0 \le Z \le 0.948) = 0.5 - 0.3264 = 0.1736$$

Hence ,there are only 17.36% chances that project will be completed before 38 days.

21.5 Comparison between PERT and CPM

As stated earlier both PERT and CPM techniques were developed independently with different set of objectives. However, the basic differences between the two are given below:

	PERT	CPM
1.	It is probabilistic model with uncertainty in activity duration. The duration of each activity is normally computed from multiple time estimates.	A deterministic model with well known activity (single) time based upon the past experience. It does not deal with uncertainty with time.
2.	PERT is said to be an event oriented as the result of analysis are expressed in terms of events.	It is an activity oriented as its results are calculated on the basis of activities.
3.	It uses dummy activities to represent project sequencing of the activities.	It does not make use of dummy activities to represent the project sequencing.
4.	PERT is usually used for those projects where time required to complete various activities is not known a priori.	This is commonly used for those projects which are repetitive in nature and here one has prior experience of handling similar projects.
5.	PERT is generally applied for planning and scheduling research program and developing projects.	CPM is generally used for construction and business problems.
6.	PERT analysis usually does not consider cost.	CPM deals with the cost of project schedules and their minimization.
7.	PERT is an important control device as it assists the management in controlling a project by constant review of such delays in the activities.	It is difficult to use CPM as controlling device because it requires repetition of the entire evaluation of project each time the changes are introduced in the network.
8.	PERT helps the manager to schedule and coordinate various activities so that project can be completed on scheduled time.	CPM plans dual emphasis on time cost and evaluates the tradeoff between project cost and time.
9.	It makes use of the statistical devices in the determination of time estimates.	It does not make use of the statistical devices in the determination of time estimates