STRENGTH OF MATERIALS

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Strength of Materials

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MODULE 1. Analysis of Statically Determinate Beams

LESSON 1. Analysis of Statically Determinate Beams

1.1 Introduction

There are two important aspects of structural design. The first one is safety which means a structure should be able to withstand all external loads without any considerable damage or collapse. The second aspect is 'serviceability' which refers to the conditions (other than the strength) under which a structure is still considered useful. The primary objective of this course is to study how to determine parameters which constitute the basis of structural design both for safety and serviceability. We begin this course with a brief review of statics. In this review we discuss concept of degrees of freedom, constraints, characteristics of forces and conditions for static equilibrium.

1.2 Degrees of Freedom (DOF)

The degree of freedo*m* of a mechanical system is the number of independent coordinates required to completely specify the configuration of the system. Motion of an object in a three-dimensional space can be completely described by three displacements (along three coordinate axes) and three rotations (about three coordinate axes) as shown in Figure 1.1a. Therefore in three-dimensional space an object has six degrees of freedom. Similarly in two dimensional plane an object has three degrees of freedom viz. two displacements along two coordinate axes and one rotation about an axis normal to the plane as shown in Figure 1.1b.

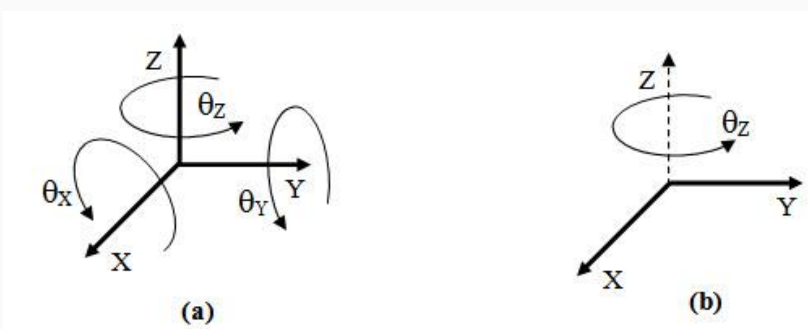


Fig. 1.1.

1.3 Force

To solve typical structural problems, we use equations involving forces or their components. Forces may consist of either a linear force that tends to produce translation or a couple that tends to produce rotation of the body on which it acts.

A system of planner forces acting on a rigid structure can always be reduced to two resultant forces (see Figure 1.2):

(a) A linear force *R* passing through the centre of gravity of the structure where R equals the vector sum of the linear forces.

(b) A moment M about the centre of gravity. The moment M is evaluated by summing the moments of all forces and couples acting on the structure with respect to an axis through the centre of gravity and perpendicular to the plane of the structure.

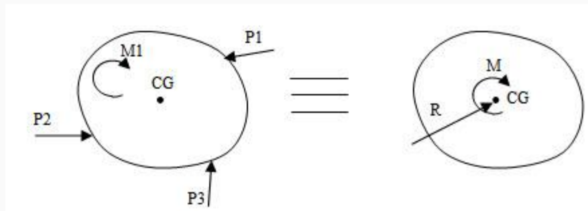
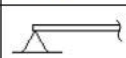
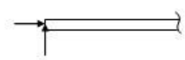
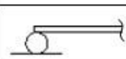
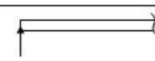
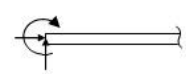
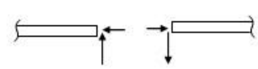


Fig. 1.2.

1.4 Supports and Support Reaction

To ensure that a structure or a structural element remains in its required position under all loading conditions, it is attached to a foundation or connected to other structural members by supports. Support constraints the motion of a structure by exerting resistive force called Support reaction. Depending on the number and type of constraints there are different kinds of supports. Characteristics of different kinds of supports in two-dimensions are summarized in Table 1.1.

Table 1.1 Support characteristics

Type	Sketch	Constraints	Reactions
Pinned		Horizontal and vertical translation.	
Roller		Vertical translation	
Fixed		Horizontal and vertical translation. Rotation	
Internal Hinge		Relative displacements of member ends	

1.5 Free Body Diagram

As a first step in the analysis of a structure, we draw a simplified sketch of the structure or a portion of the structure under consideration. A free body diagram is a pictorial device used in order to analyze the forces and moments acting on a body. This sketch, which shows the required dimensions together with all the external and internal forces acting on the structure is called free-body diagram (FBD). For instance Figure 1.3b shows the free-body diagram of the entire structure of a simply supported beam subjected to concentrated load at its midspan (Figure 1.3a). Let the beam is cut by a section *mn*. Figure 1.3c shows the free body diagram of the portion of the beam to the left part of the section *mn*.

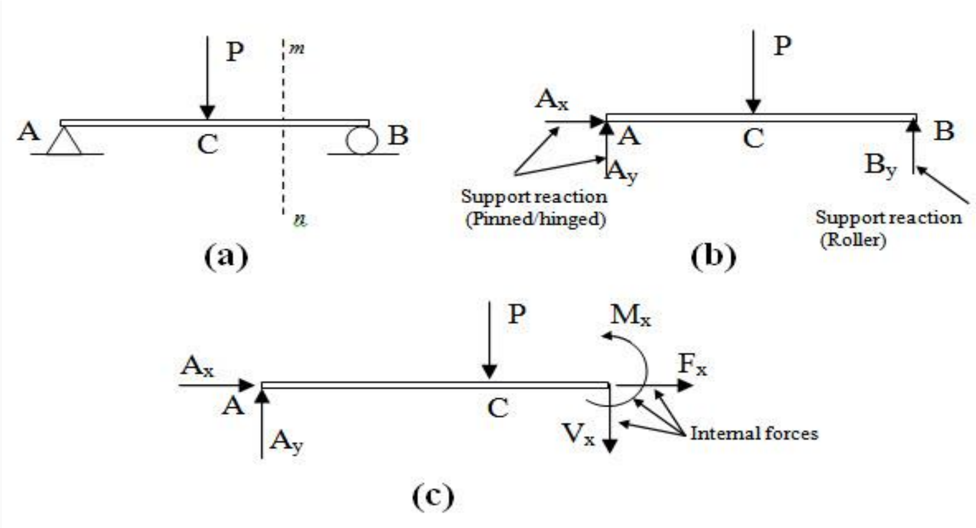


Fig. 1.3.

1.6 Equations of Static Equilibrium

An object at rest is said to be in static equilibrium if the net forces acting on the body is zero. In three dimensions the equations of static equilibrium are as follows,

$\sum F_x=0$; $\sum F_y=0$; $\sum F_z=0$ [Net force in any direction is zero]

$\sum M_x=0$; $\sum M_y=0$; $\sum M_z=0$ [Net moment about any axis is zero]

Similarly in two dimensions the above equations become,

$\sum F_x=0$; $\sum F_y=0$ [Net force in any direction is zero]

$\sum M_z=0$ [Net moment about any axis is zero]

1.7 Determinate and Indeterminate Structure

A structure is said to be determinate if the equations of static equilibrium alone are sufficient to permit a complete analysis of the structure. If the structure cannot be analyzed by the equation of statics, the structure is termed as indeterminate. To analyze an indeterminate structure, additional equations considering the geometry of the deflected shape are required. This will be discussed in details in the next module. In this module we will learn several methods to analyze statically determinate beams.

LESSON 2. Axial Force, Shear Force and Bending Moment in Beam

2.1 Introduction

To start with, consider a simply supported beam subjected to some arbitrary external load as shown in Figure 2.1a. Let the beam is cut by a section *mn*. Figure 2.1b shows the free body diagram of the portion of the beam to the left part of the section *mn*. F_{in} represents the internal force on section *mn*. Now using the concept of equivalent force-couple system, F_{in} may be represented by a force F_c applied at the centroid of the cross-section and a couple M_x as shown in Figure 2.1c. The force F_c may further be decomposed into two orthogonal components, F_x, normal to the plane of the cross-section and V_x, tangential to the plane of the cross-section (see Figure 2.1d).

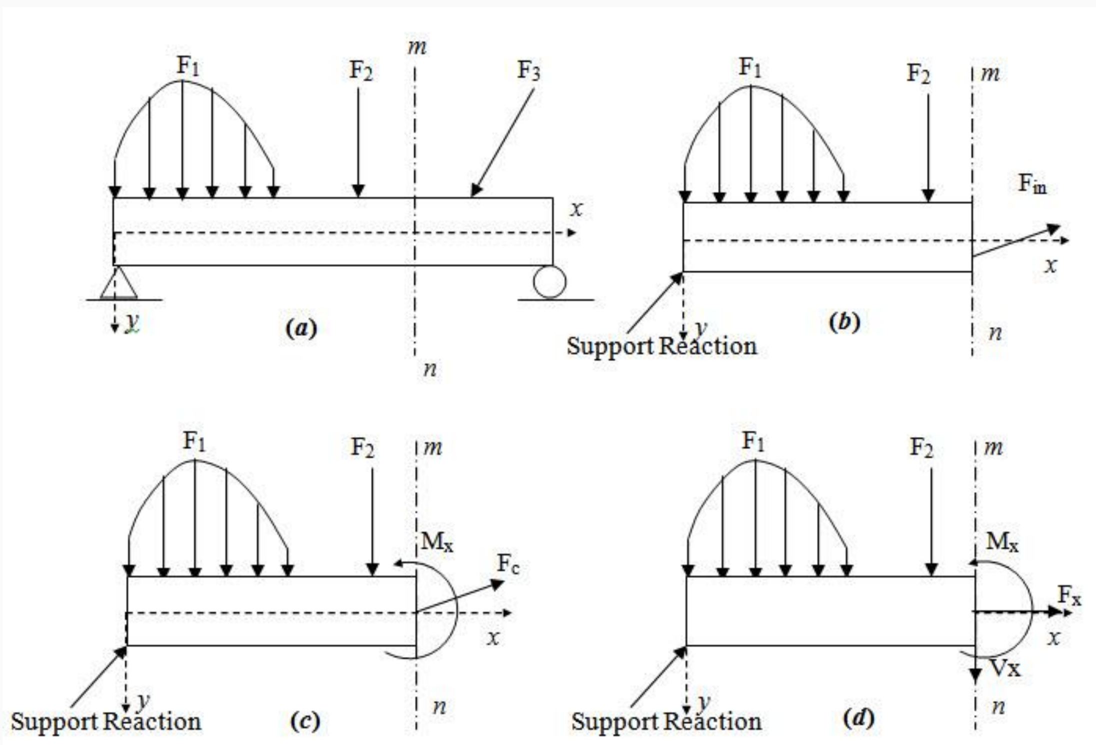


Fig. 2.1.

Similarly the internal force on any section may be represented by three quantities F_x, V_x and M_x called respectively as axial force, shear force and bending moment.

Sign Convention

Throughout the syllabus we will consistently use the following sign convention.

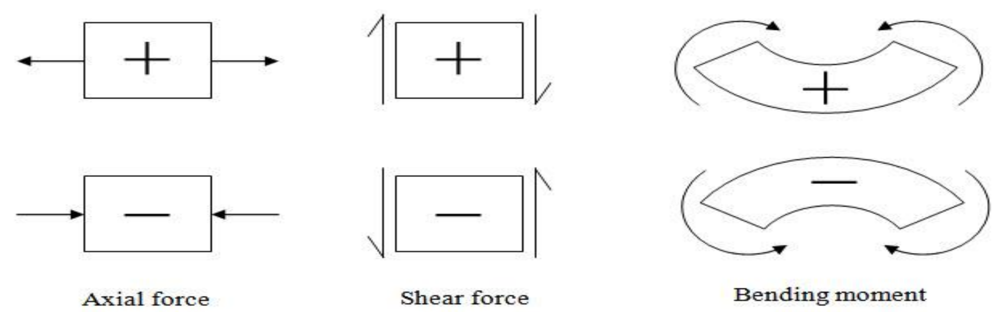


Fig. 2.2.

2.2 Computation of Support Reaction and Internal Forces

The following procedure may be followed in order to determine the support reaction of a beam.

(a) Draw the free body diagram of the entire structure. In this free body diagram only unknowns are the support reaction.

(b) Apply the static equilibrium condition to determine the unknowns.

The following procedure may be followed in order to determine the internal forces at any section of a beam.

- At the desired location take a section which cut the beam into two parts.
- Isolate any part and draw the free body diagram. In this diagram only unknowns are the internal forces.
- Apply the static equilibrium conditions to determine the internal forces.

This is demonstrated via the following example.

Example 1

A simply supported beam AB is subjected to a concentrated load P as shown in Figure 2.3. Calculate reactions at A and B. Calculate shear force and bending moment at a distance x from A.

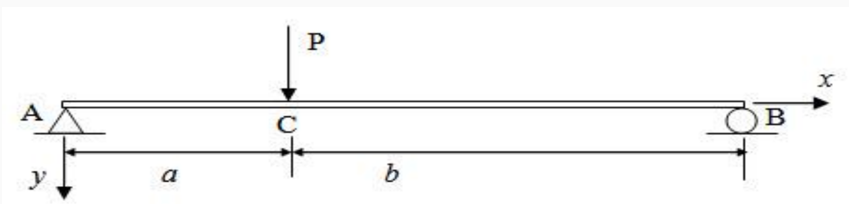


Fig. 2.3.

Calculation of support reactions

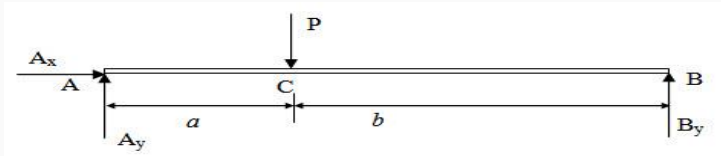


Fig. 2.4.

Figure 2.4 shows the free body diagram of the entire structure. Here support reactions A_x, A_y and B_y are unknowns. Now applying the static equilibrium conditions we have,

$$\sum {{F_x}}=0 \Rightarrow {A_x}=0$$ (2.1)

$$\sum {{F_y}}=0 \Rightarrow {A_y} + {A_y} - P=0 \Rightarrow {A_y} + {A_y}=P$$ (2.2)

$$\sum {{M_B}}=0 \Rightarrow {A_y}(a + b) - Pb=0 \Rightarrow {A_y}={{Pb} \over {\left({a + b} \right)}}$$ (2.3)

Substituting in equation (2.2) we have,

$${B_y}={{Pa} \over {\left({a + b} \right)}}$$ (2.4)

Calculation of shear force and bending moment

Two sections viz. and are considered. Corresponding free body diagrams are shown in Figure 2.5.

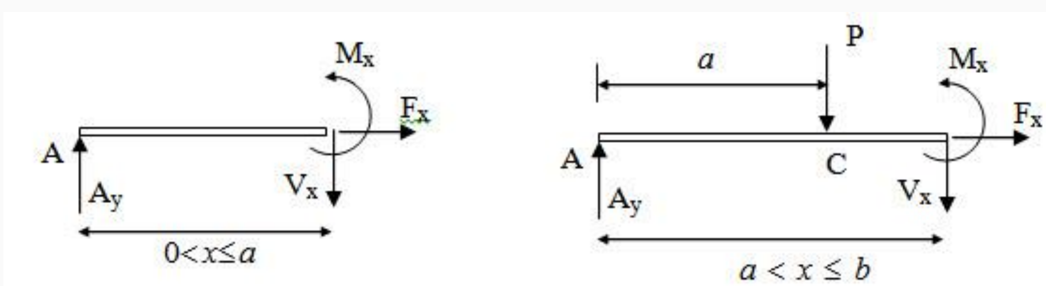


Fig. 2.5.

<u>**For $0 < x \leq a$**</u>

$$\sum {{F_x}}=0 \Rightarrow {F_x}=0$$ (2.4)

$$\sum {{F_y}}=0 \Rightarrow {A_y} - {V_x}=0 \Rightarrow {V_x}={{Pb} \over {\left({a + b} \right)}}$$ (2.5)

$$\sum {M_B}=0 \Rightarrow {A_y}x - {M_x} = 0 \Rightarrow {M_x}={{Pb} \over {\left({a + b} \right)}}x \qquad (2.6)$$

<u>For a ≤ x < b</u>

$$\sum {F_x}=0 \Rightarrow {F_x}=0 \qquad (2.7)$$

$$\sum {F_y}= 0 \Rightarrow {A_y} - P - {V_x} = 0 \Rightarrow {V_x}= - {{Pa} \over {\left({a + b} \right)}} \qquad (2.8)$$

$$\sum {M_B}=0 \Rightarrow {A_y}x - P(x - a) - {M_x}=0 \Rightarrow {M_x} = Pa\left({1 - {x \over {a + b}}} \right) \qquad (2.9)$$

2.3 Bending Moment and Shear Force Diagram

Shear force and bending moment diagrams are the graphical representation of variation of shear force and bending moment respectively along the axis of the beam. For illustration consider the previous example where the variation of shear force and bending moment may be summarized as,

location	Shear force	Bending moment
$0 < x \le a$	$\frac{Pb}{(a+b)}$	$\frac{Pb}{(a+b)}x$
$a \le x < b$	$-\frac{Pa}{(a+b)}$	$Pa\left(1-\frac{x}{a+b}\right)$

Corresponding diagrams are shown in Figure 2.6.

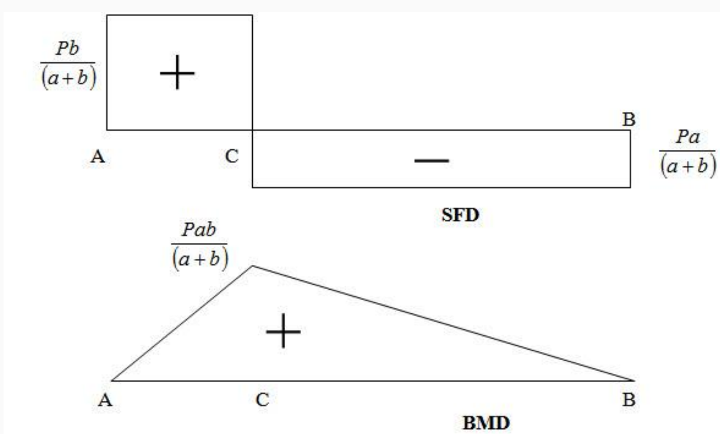


Fig. 2.6.

Example 2

A simply supported beam AB is subjected to a uniformly distributed load of intensity of q as shown in Figure 2.7. Calculate support reactions and draw shear force and bending moment diagram..

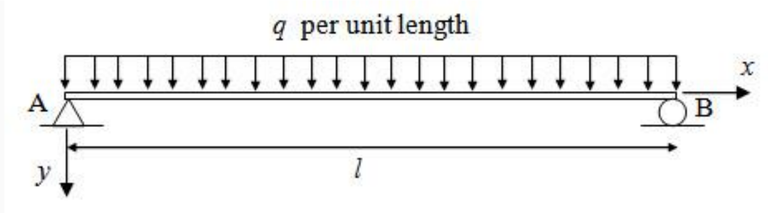


Fig. 2.7.

Solution

The free body diagram of the entire structure is shown in Figure 4.2.

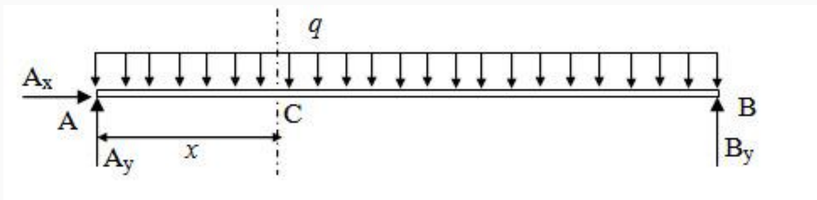


Fig. 2.8.

Applying equilibrium conditions we have,

$$\sum {{F_x}}=0 \Rightarrow {A_x}=0$$

$$\sum {{M_B}}=0 \Rightarrow {A_y}l - ql{l \over 2}=0 \Rightarrow {A_y}={{ql} \over 2}$$

$$\sum {{F_y}}=0 \Rightarrow {A_y} + {B_y} - ql = 0 \Rightarrow {B_y}={{ql} \over 2}$$

Take a section at C which is at a distance *x* from A. Figure 2.9 shows the free body diagram of the portion of the beam to the left part of the section.

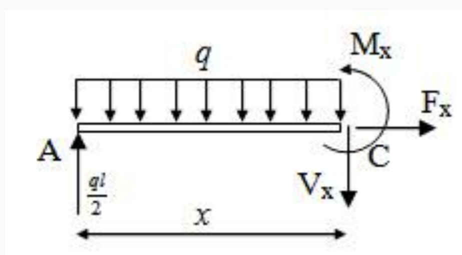


Fig. 2.9.

Taking force equilibrium in vertical direction, we have,

$$\sum {{F_y}}=0 \Rightarrow {{ql} \over 2} - qx - {V_x} = 0 \Rightarrow {V_x}=qx - {{ql} \over 2} \qquad (2.10)$$

Taking moment about C we have,

$$\sum {{M_C}}=0 \Rightarrow {{ql} \over 2}x - qx{x \over 2} - {M_x}=0 \Rightarrow {M_x}={{ql} \over 2}x - {{q{x^2}} \over 2} \qquad (2.11)$$

Shear force and bending moment diagram (graphical representations of equations (2.10) and (2.11)) are shown in Figure 2.10.

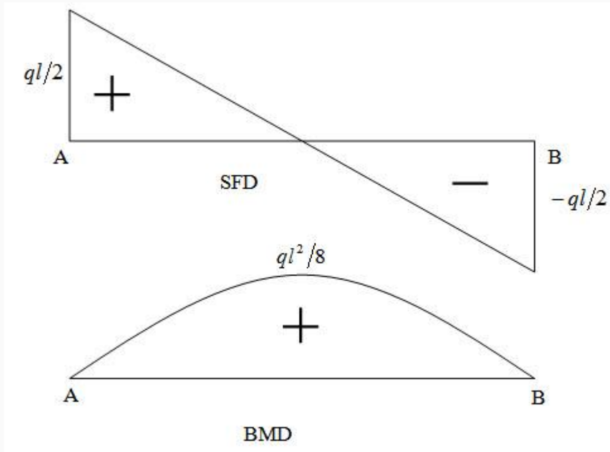


Fig. 2.10.

LESSON 3. Deflection of Beam: Direct Integration Technique - 1

3.1 Introduction

As mentioned in the introductory lesson the second important aspect of any structural design is 'serviceability' which refers to the conditions (other than the strength) under which a structure is still considered useful. One of such serviceability criterion commonly used in limit state design of beam is deflection. Different methods for determination of transverse deflection of a statically determinate beam will be discussed in the subsequent lesson. In this lesson we will derive differential equation for the elastic line (also referred to as deflection curve).

3.2 Differential Equation for the Elastic Line

Consider a beam (Figure 3.1a) undergoes transverse deformation as shown in Figure 3.1b. As a result of this deformation fibers on the convex side of the beam are elongated while those on the concave side are shortened. Somewhere in between top and bottom of the beam, there is a layer of fibers which remain unchanged in length. This neutral layer is called neutral surface and intersection of the neutral surface with the axial plane of symmetry is called neutral axis.

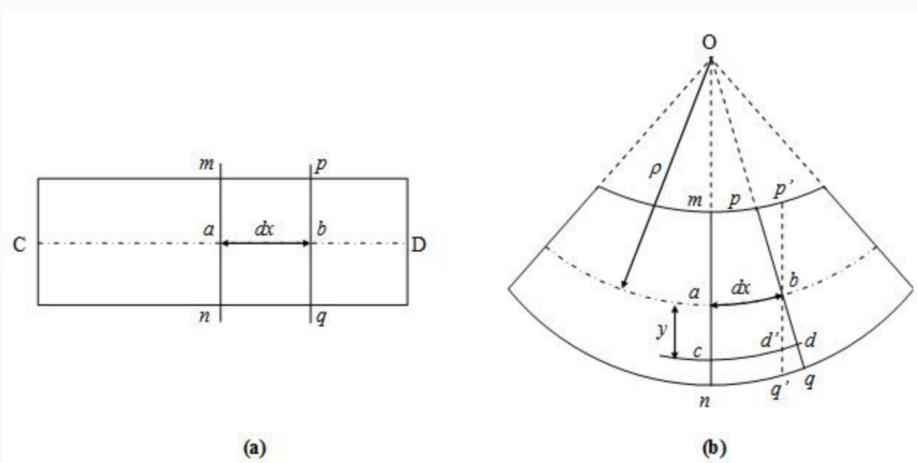


Fig. 3.1.

The deformed configuration of the neutral axis is called the elastic line or deflection curve. In this section we will derive the differential equation of this elastic line. The assumptions which constitute the basis of the derivation are given bellow.

- Material is homogeneous and obeys Hooke's law.
- The curvature is small.
- Any cross-section originally plane and normal to the neutral axis is assumed to remain plane and normal to the neutral axis during the deformation.

Consider an infinitesimal segment dx between two adjacent cross-section mn and pq in undeformed configuration as shown in Figure 3.1b. After deformation, mn and pq no longer remain parallel and let they intersect at O at an angle dq as shown in Figure 3.1b. Now the relation between dx and dq may be written as,

$$d\theta=dx{1 \over \rho } \qquad (3.1)$$

where ${1 /\rho }$ is the curvature of the neutral axis of the beam.

Now consider a fiber cd at a distance y from the neutral axis and having initial length dx. After deformation it elongates by amount. Therefore longitudinal strain in fiber cd may be expressed as,

$${\varepsilon _x}={{dd'} \over {dx}}={{yd\theta } \over {dx}}={y \over \rho } \qquad (3.2)$$

It is to be noted that if a fiber on the concave side of the neutral axis is considered, the distance y will be negative and consequently the strain is also negative. Now position and curvature of the neutral axis may be determined by using equilibrium condition as follows.

3.2.1 Position of the Neutral Axis

Following the Hooke's law the longitudinal stress (bending stress) may be written as,

$${\sigma _x}=E{\varepsilon _x}={E \over \rho }y \qquad (3.3)$$

where E is the Young's modulus. Equation (3.3) shows that the fiber stress varies linearly across the depth of the beam with maximum and minimum values at the two extreme (bottom and top) fibers and zero at neutral axis as shown in Figure 3.2.

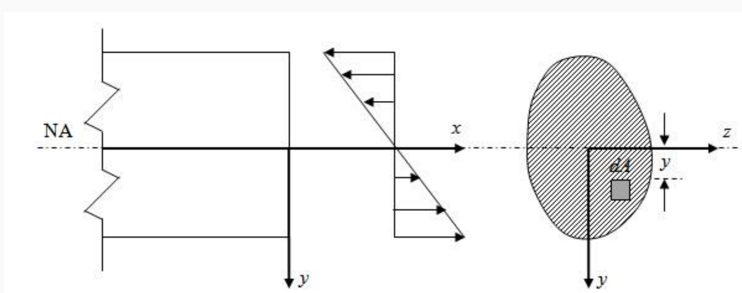


Fig. 3.2.

Now force on an infinitesimal area dA at a distance y from the neutral axis is. Since there is no normal force in the longitudinal direction, force equilibrium condition in the longitudinal direction may be written as,

$$\int\limits_A {{\sigma _x}dA}=0 \Rightarrow {E \over \rho }\int\limits_A {ydA}=0 \Rightarrow \int\limits_A {ydA}=0 \qquad (3.4)$$

In the above equation, $\int\limits_A {ydA}$ is the moment of area about z-axis (see Figure 3.2) which may also be expressed as $A{y_c}$ where ${y_c}$ is the distance of centroid from the neutral axis. Since $A \ne 0$, from equation (3.4) we have $y_c=0$. Therefore neutral axis of the cross-section passes through its centroid.

3.2.2 Curvature of Neutral Axis

Curvature of the neutral axis may be determined from the condition that the resulting couple induced by the longitudinal stress must be equal to the bending moment *M*. Therefore,

$$M=\int\limits_A {y{\sigma _x}dA}\Rightarrow M={E \over \rho }\int\limits_A {{y^2}dA}\Rightarrow {1 \over \rho }={M \over {EI}} \qquad (3.5)$$

where $I=\int\limits_A {{y^2}dA}$ is the second moment of area of the cross-section about *z*-axis.

If we consider the deflection curve as a smooth function of *x*, its curvature may be written as,

$$\frac{1}{\rho } = \frac{{\left| {\frac{{{d^2}y}}{{d{x^2}}}} \right|}}{{{{\left[{1 + {{\left({\frac{{dy}}{{dx}}} \right)}^2}} \right]}^{3/2}}}} \qquad (3.6)$$

For small deflection, slope ${{dy}/ {dx}}$ is very small as compared to unity and hence the curvature may be approximated as,

$${1 \over \rho }=\left| {{{d^2}y} \over {d{x^2}}} \right| \qquad (3.7)$$

Combining equation (3.5) and (3.7), we have,

$$\left| {{{d^2}y} \over {d{x^2}}} \right|={M \over {EI}} \qquad (3.8)$$

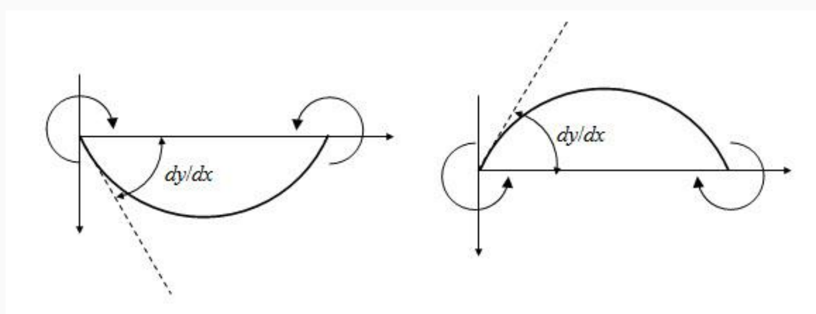


Fig. 3.3.

(a) Moment positive, curvature concave upward; Moment negetive, curvature concave downward.

As illustrated in Figure 3.3, when moment is positive, slope ${{dy}/ {dx}}$ algebraically decreases (${{{d^2}y}/ {d{x^2}}}$ is negative) with *x* and vice versa. Therefore equation (3.8) may be recast as,

$${{{d^2}y} \over {d{x^2}}}=- {M \over {EI}} \qquad (3.9)$$

Equation (3.9) is the differential equation of the elastic line for a beam. Following forms of equation (3.9) are also found useful especially when load has non-uniform distribution.

$${d \over {dx}}\left({EI{{{d^2}y} \over {d{x^2}}}} \right)=- {{dM} \over {dx}} \Rightarrow {d \over {dx}}\left({EI{{{d^2}y} \over {d{x^2}}}} \right)=- V \qquad (3.10)$$

$${{{d^2}} \over {d{x^2}}}\left({EI{{{d^2}y} \over {d{x^2}}}} \right)=- {{dV} \over {dx}} \Rightarrow {{{d^2}} \over {d{x^2}}}\left({EI{{{d^2}y} \over {d{x^2}}}} \right)=q \qquad (3.11)$$

where V is the shear force and q is the intensity of the distributed load at any cross-section. Deflection of beam may be determined by solving any of equation (3.9) – (3.11). This will be discussed in next lesson.

LESSON 4. Deflection of Beam: Direct Integration Technique - 2

4.1 Introduction

In the last lesson we derived the following differential equations.

$$ {{{d^2}y} \over {d{x^2}}}=- {M \over {EI}} \qquad (4.1)$$

$$ {{{d^2}} \over {d{x^2}}}\left({EI{{{d^2}y} \over {d{x^2}}}} \right)=-{{dV} \over {dx}} \Rightarrow {{{d^2}} \over {d{x^2}}}\left({EI{{{d^2}y} \over {d{x^2}}}} \right) = q \qquad (4.2)$$

In this lesson we will study how to determine the transverse deflection of a beam by solving the above equation by direct integration technique. The procedure is illustrated bellow via several examples.

Example 1

A simply supported beam AB is subjected to a uniformly distributed load of intensity of *q* as shown in Figure 4.1. Calculate the deflection at the midspan. Flexural rigidity of the beam is EI.

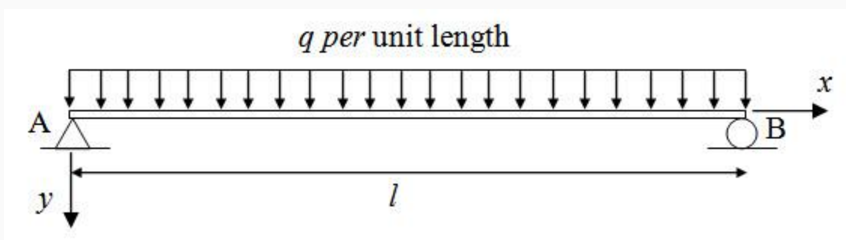


Fig. 4.1.

Solution

The free body diagram of the entire structure is shown in Figure 4.2.

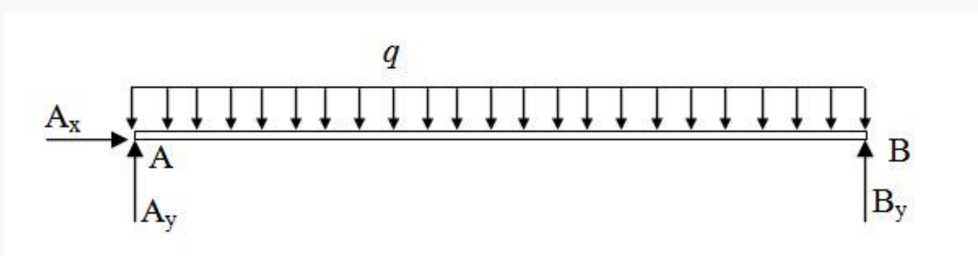


Fig. 4.2.

Applying equilibrium conditions we have,

$$\sum {{F_x}}=0 \Rightarrow {A_x}=0$$ (4.3a)

$$\sum {{M_B}}=0 \Rightarrow {A_y}l - ql{l \over 2}=0 \Rightarrow {A_y}={{ql} \over 2}$$ (4.3b)

$$\sum {{F_y}}=0 \Rightarrow {A_y} + {B_y} - ql=0 \Rightarrow {B_y}={{ql} \over 2}$$ (4.3c)

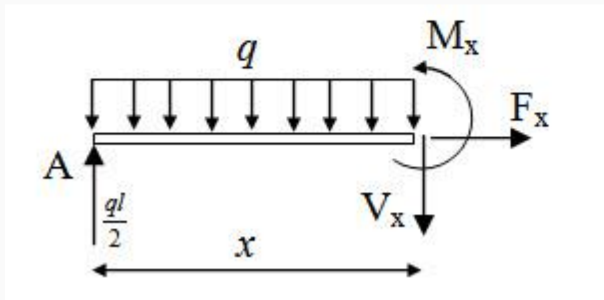


Fig. 4.3.

Using the FBD shown in Figure 4.3 bending moment at a distance *x* from A is,

$${M_x}={{ql} \over 2}x - {{q{x^2}} \over 2}$$ (4.4)

and equation (4.1) becomes

$$EI{{{d^2}y} \over {d{x^2}}}=- {{ql} \over 2}x + {{q{x^2}} \over 2}$$ (4.5)

$$\Rightarrow EI{{dy} \over {dx}}=-{{ql} \over 4}{x^2} + {{q{x^3}} \over 6} + {c_1}$$ (4.6)

$$\Rightarrow EIy=-{{ql} \over {12}}{x^3} + {{q{x^4}} \over {24}} + {c_1}x + {c_2}$$ (4.7)

where, c_1 and c_2 are integration constants. In order to evaluate these constants following boundary conditions are used.

y(x = 0) and y(x = l) = 0

Imposing the above boundary conditions we have, $${c_1}={{q{l^3}} \over {24}}$$ and $c_2=0$.

Substituting c_1 and c_2 in equation (4.7), we have the deflection curve as,

$$y(x)={{qx} \over {24EI}}\left({{l^3} - 2l{x^2} + {x^3}} \right)$$ (4.8)

Deflection at the midspan ($x=l/2$) is,

$$y(x = l/2)={{5q{l^4}} \over {384EI}}$$

Example 2

A simply supported beam AB is subjected to a linearly varying load as shown in Figure 4.4. Calculate the maximum deflection. Flexural rigidity of the beam is EI.

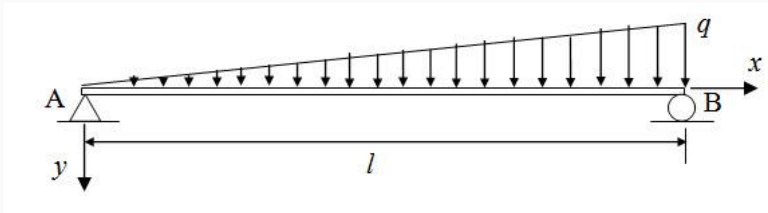


Fig. 4. 4.

Solution

The free body diagram of the entire structure is shown in Figure 4.5.

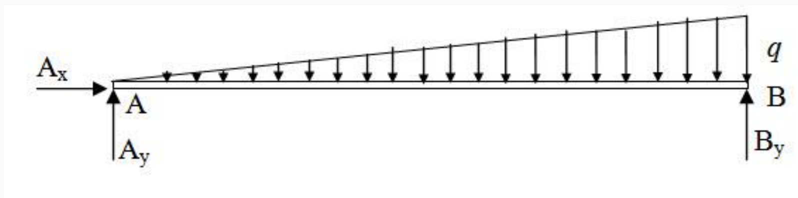


Fig.4.5

Applying equilibrium conditions we have,

$$\sum {{F_x}}=0 \Rightarrow {A_x}=0 \tag{4.3a}$$

$$\sum {{M_B}}=0 \Rightarrow {A_y}l - q{l \over 2}{l \over 3} = 0 \Rightarrow {A_y}={{ql} \over 6} \tag{4.3b}$$

$$\sum {{F_y}}= 0\Rightarrow {A_y} + {B_y} - q{l \over 2}=0 \Rightarrow {B_y}={{ql} \over 3} \tag{4.3c}$$

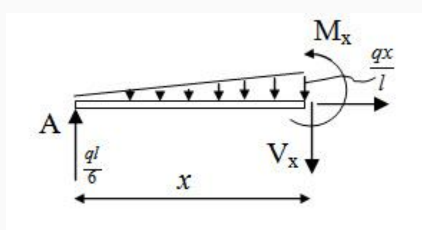


Fig.4.6

Using the FBD shown in Figure 4.3 bending moment at a distance *x* from A is,

$${M_x}={{ql} \over 6}x - {{qx} \over l}{x \over 2}{x \over 3} \Rightarrow {M_x}={{ql} \over 6}x - {{q{x^3}} \over {6l}} \tag{4.4}$$

and equation (4.1) becomes

$$EI{{{d^2}y} \over {d{x^2}}}=-{{ql} \over 6}x + {{q{x^3}} \over {6l}} \qquad (4.5)$$

$$\Rightarrow EI{{dy} \over {dx}}=-{{ql} \over {12}}{x^2} + {{q{x^4}} \over {24l}} + {c_1} \qquad (4.6)$$

$$\Rightarrow EIy=-{{ql} \over {36}}{x^3} + {{q{x^5}} \over {120l}} + {c_1}x + {c_2} \qquad (4.7)$$

where, c_1 and c_2 are integration constants. In order to evaluate these constants following boundary conditions are used.

$y(x = 0)$ and $y(x = l) = 0$

Imposing the above boundary conditions we have, ${c_1}={{7q{l^3}} \over {360}}$ and $c_2 = 0$.

Substituting c_1 and c_2 in equation (), we have the deflection curve as,

$$y(x) = {{qx} \over {360lEI}}\left({7{l^4} - 10{l^2}{x^2} + 3{x^4}} \right) \qquad (4.8)$$

Where deflection is maximum, ${{dy} / {dx}} = 0 \Rightarrow x=0.519l$. Substituting in equation (4.8), we have

$${\delta _{\max }}=y(x = 0.519l) = 0.00652{{q{l^4}} \over {EI}}$$

Alternative solution using Equation (4.2)

At any distance *x* from A the intensity of load is ${q_x} = {{qx} / l}$. Therefore equation (4.2) becomes,

$$EI{{{d^4}y} \over {d{x^4}}}={{qx} \over l} \qquad (4.9)$$

$$\Rightarrow EI{{{d^3}y} \over {d{x^3}}} = {{q{x^2}} \over {2l}} + {c_1} \qquad (4.10)$$

$$\Rightarrow EI{{{d^2}y} \over {d{x^2}}} = {{q{x^3}} \over {6l}} + {c_1}x + {c_2} \qquad (4.11)$$

$$\Rightarrow EI{{dy} \over {dx}} = {{q{x^4}} \over {24l}} + {c_1}{{{x^2}} \over 2} + {c_2}x + {c_3} \qquad (4.12)$$

$$\Rightarrow EIy = {{q{x^5}} \over {120l}} + {c_1}{{{x^3}} \over 6} + {c_2}{{{x^2}} \over 2} + {c_3}x + {c_4} \qquad (4.13)$$

Boundary conditions are,

$y(x = 0)\ 0$; $y(x = l) = 0$

${{{d^2}y} \over {d{x^2}}}(x = 0)=0$; ${{{d^2}y} \over {d{x^2}}}(x = l)=0$ $\left[{{\rm{Moment\ is\ zero\ at\ end}}} \right]$

Imposing the boundary conditions we have,

$${c_1}=-{{ql} \over 6}$$; $c_2 = 0$; $${c_3}={{7q{l^3}} \over {360}}$$; $c_4 = 0$.

Substituting c_1,c_2 ,c_3 and c_4 in equation (), we have the deflection curve as,

$$y(x)={{qx} \over {360lEI}}\left({7{l^4} - 10{l^2}{x^2} + 3{x^4}} \right)$$

Where deflection is maximum, $${{dy}/ {dx}} = 0 \Rightarrow x=0.519l$$. Substituting $x = 0.519l$ in equation (4.8), we have

$${\delta _{\max }} = y(x = 0.519l) = 0.00652{{q{l^4}} \over {EI}}$$

LESSON 5. Deflection of Beam: Moment-Area Method

5.1 Introduction

In this lesson we will study a semi-graphical method refer to as the Moment-Area method developed by Charles E. Greene for finding deflection of beam using moment curvature relation. The moment-curvature relation discussed in lesson 3 is rewritten as,

$${d \over {dx}}\left({{{dy} \over {dx}}} \right)=-{M \over {EI}} \qquad (5.1)$$

$$\Rightarrow {\left({{{dy} \over {dx}}} \right)_A}- {\left({{{dy} \over {dx}}} \right)_B}=\int\limits_{{x_A}}^{{x_B}} {{{Mdx} \over {EI}}} \qquad (5.2)$$

where, ${\left({{dy}/ {dx}}\right)_A}-{\left({{dy} / {dx}} \right)_B}$, hereafter referred to as ${\theta _{AB}}$ is the angle between tangents at A and B as illustrated in Figure 5.1a. Similarly the deflection at B with respect to tangent at A, may be written as,

$${\delta _{AB}}=\int\limits_{{x_A}}^{xB} {xd\theta }=\int\limits_{{x_A}}^{{x_B}} {{{Mxdx} \over {EI}}} \qquad (5.3)$$

It is to be noted that $\int\limits_{{x_A}}^{{x_B}} {{{Mxdx}/{EI}}}$ represents the statical moment with respect to B of the total bending moment area between A and B, divided by EI. Therefore equation (5.3) may also be written as,

$${\delta _{AB}}=\left[{\int\limits_{{x_A}}^{{x_B}} {{{Mdx} \over {EI}}} } \right]\bar x, \qquad (5.4)$$

$$\Rightarrow {\delta _{AB}}=\bar x{\theta _{AB}} \qquad (5.4)$$

where $\bar x$ is the centroidal distance as shown in Figure 5.1a.

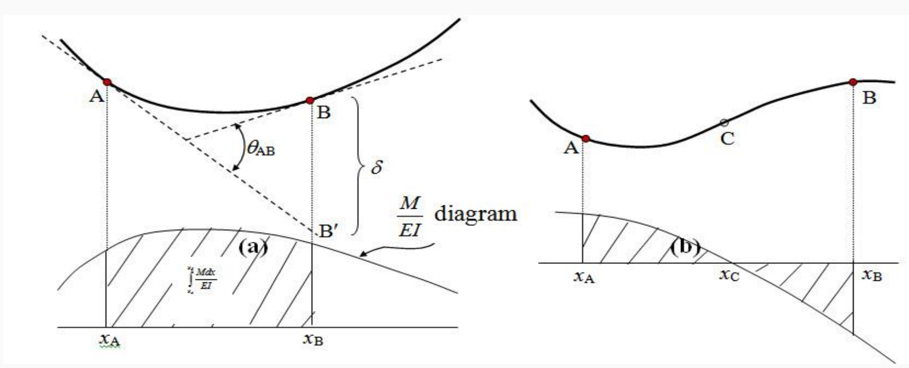


Fig. 5.1.

Based on equations (5.2) and (5.4) the moment-area theorem may be stated as,

Theorem 1

The change in slope between the tangents drawn to the elastic curve at any two points A and B is equal to the area of bending moment diagram between A and B, divided by EI.

Theorem 2

The deviation of any point B relative to the tangent drawn to the elastic curve at any other point A, in a direction perpendicular to the original position of the beam, is equal to the moment with respect to B of the area of bending moment diagram between A and B, divided by EI

Applications of the Moment-area theorem will now be demonstrated via several examples.

5.2 Example 1

A simply supported beam AB is subjected to a uniformly distributed load of intensity of q as shown in Figure 5.2. Calculate the deflection at the midspan. Flexural rigidity of the beam is EI.

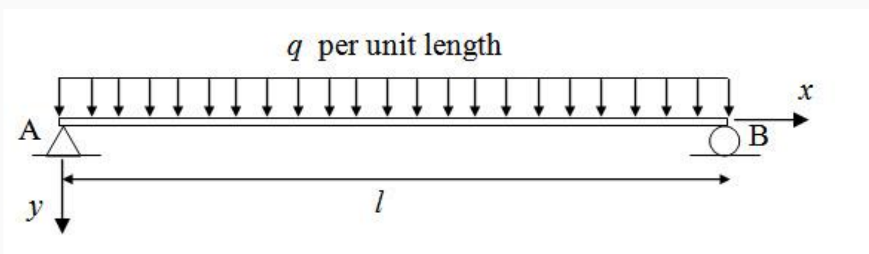
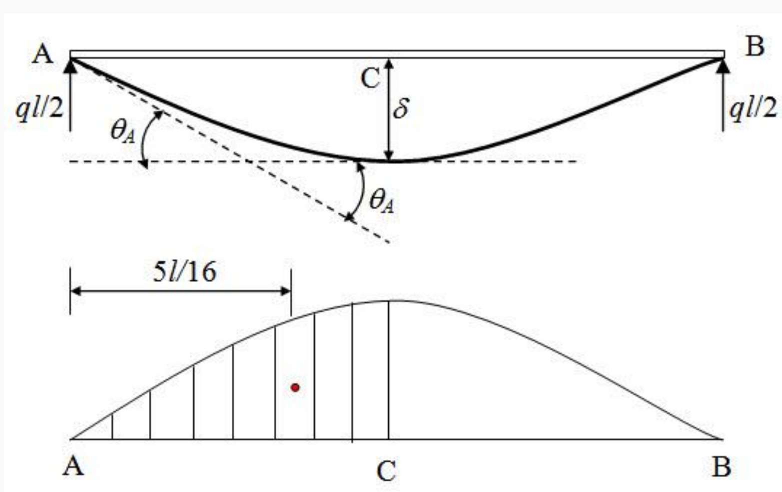


Fig. 5.2.

Solution

From Example 4.1, bending moment at a distance x from A is,

$$M_x = \frac{ql}{2}x - \frac{qx^2}{2} \qquad (4.4)$$



Due to symmetry slope of the elastic line at midspan is zero. Therefore

$${\theta _{AC}} = {\theta _A} = \int\limits_{{x_A}}^{{x_C}} {{{{M_x}dx} \over {EI}}}={1 \over {EI}}\int\limits_0^{{{l/2}}} 2{\left({{{ql} \over 2}x - {{q{x^2}} \over 2}} \right)dx}$$

$$\Rightarrow {\theta _A}={{q{l^3}} \over {24EI}}$$

Now since δ may be considered as the deflection at A with respect to tangent at C, we have,

$$\delta={\theta _A}\bar x = {{5q{l^3}} \over {384EI}}$$

5.3 Example 2

A cantilever beam AB is subjected to a concentrated load P at its tip as shown in Figure 5.3. Determine deflection and slope at B.

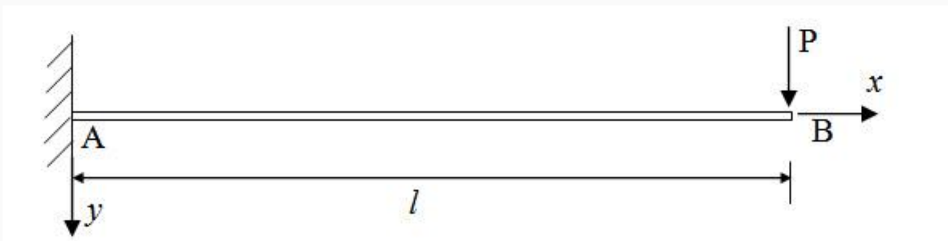
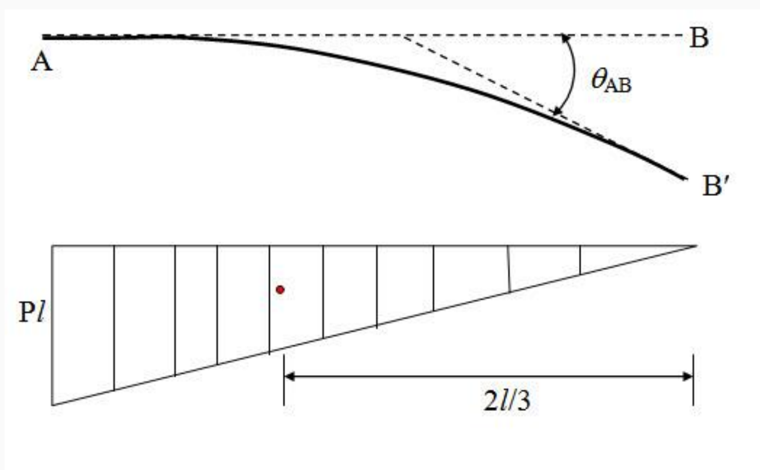


Fig. 5.3.

Solution



$${\theta _{AB}}=\int\limits_{{x_A}}^{{x_B}} {{{{M_x}dx} \over {EI}}}={{P{l^2}} \over {2EI}}$$

Since slope at A is zero,

$${\theta _B}={\theta _{AB}}=-{{P{l^2}} \over {2EI}}$$

$$\delta={\theta _B}\bar x={{P{l^2}} \over {2EI}}{{2l} \over 3}={{P{l^3}} \over {3EI}}$$

LESSON 6. Deflection of Beam: Conjugate Beam Theory

6.1 Introduction

A conjugate beam is a fictitious beam that corresponds to the real beam and loaded with M/EI diagram of the real beam. For instance consider a simply supported beam subjected to a uniformly distributed load as shown in Figure 5.1a. Figure 5.1b shows the bending moment diagram of the beam. Then the corresponding conjugate beam (Figure 5.1c) is a simply supported beam subjected to a distributed load equal to the M/EI diagram of the real beam.

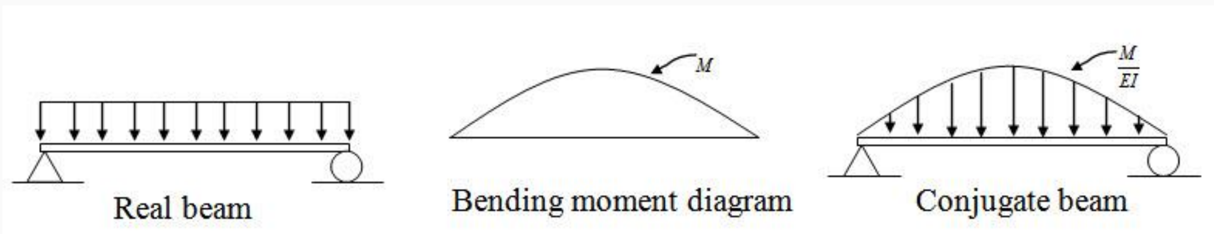
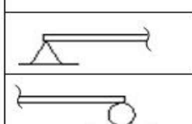
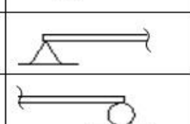
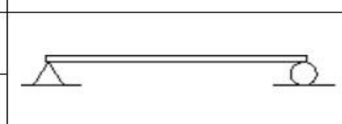
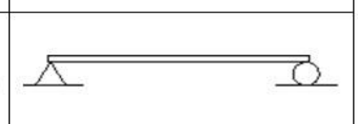
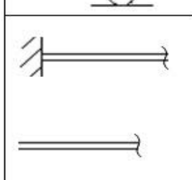
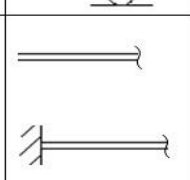
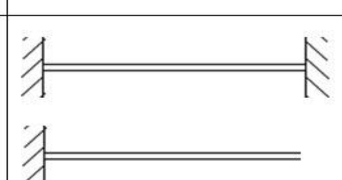
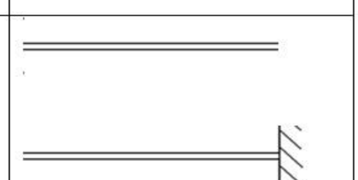
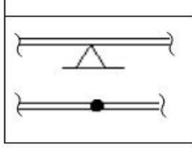
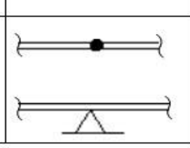
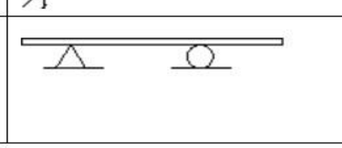
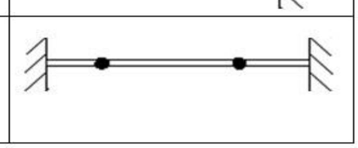


Fig. 6.1.

Supports of the conjugate beam may not necessarily be same as the real beam. Some examples of supports in real beam and their conjugate counterpart are given in Table 6.1.

Table 6.1: Real beam and it conjugate counterpart

Real beam support	Conjugate beam support	Real beam	Conjugate beam
			
			
			

Once the conjugate beam is formed, slope and deflection of the real beam may be obtained from the following relationship,

Slope on the real beam = Shear on the conjugate beam

Deflection on the real beam = Moment on the conjugate beam

6.2 Example 1

A cantilever beam AB is subjected to a concentrated load P at its tip as shown in Figure 6.2. Determine deflection and slope at B.

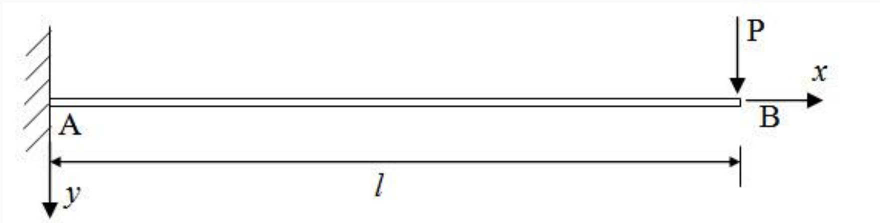


Fig. 6.2.

Solution

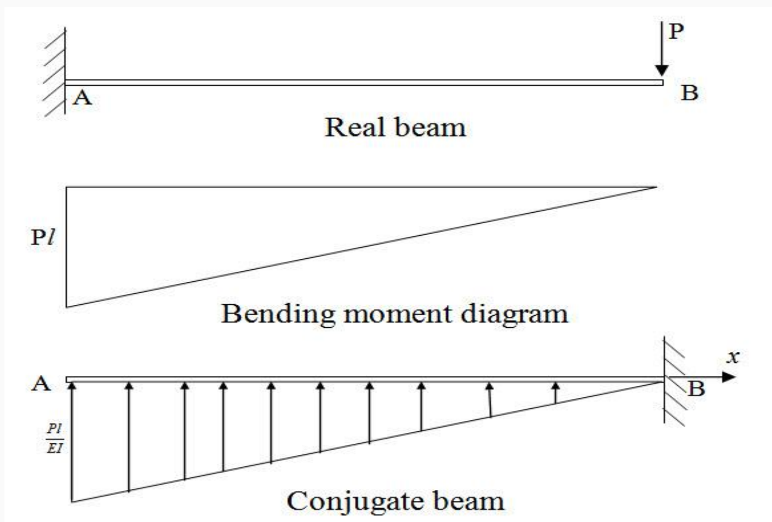


Fig. 6.2.

Real beam and corresponding conjugate beam are shown in Figure 6.2. Now, from the free body diagram of the entire structure (Figure 6.3), we have

$$B_y = -\frac{1}{2}\frac{Pl}{EI}l = -\frac{Pl^2}{2EI}$$

$$M_B = \frac{1}{2}\frac{Pl}{EI}l\frac{2l}{3} = \frac{Pl^3}{3EI}$$

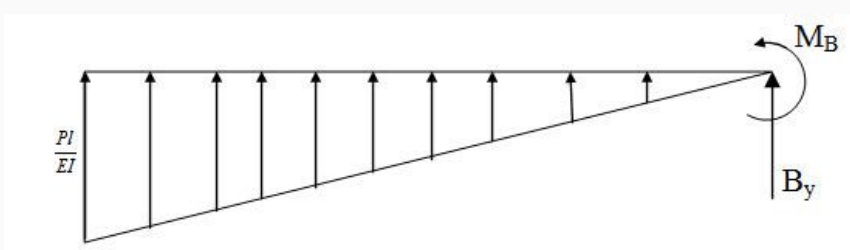


Fig. 6.3.

$${\theta _B}={B_y}=-{{P{l^2}} \over {2EI}}$$

$${\delta _B}={M_B}=-{{P{l^3}} \over {3EI}}$$

6.3 Example 2

A simply supported beam AB is subjected to a uniformly distributed load of intensity of q as shown in Figure 6.4. Calculate θ_A, θ_B and the deflection at the midspan. Flexural rigidity of the beam is EI.

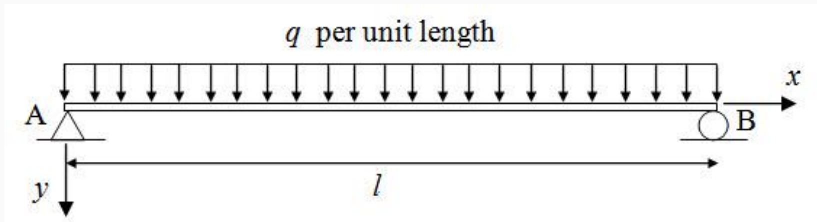


Fig. 6.4.

Solution

Bending moment and conjugate beam are shown in Figure 6.5.

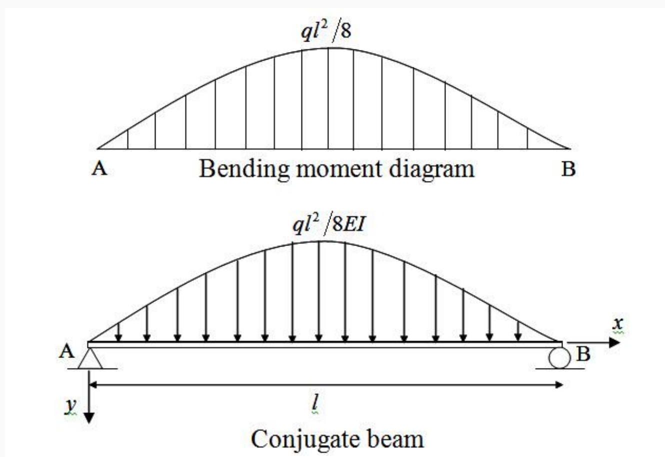


Fig. 6.5.

$${A_y}={B_y}={1 \over 2} \times {\rm{Area of the parabolic load distribution }}$$

$$\Rightarrow {A_y} = {B_y}={1 \over 2} \times {2 \over 3}l{{q{l^2}} \over 8}={{q{l^3}} \over {24}}$$

Shear force of the conjugate beam at A and B are respectively as A_y and B_y.

Therefore,

$${\theta _A}={A_y}={{q{l^3}} \over {24}}$$ and $${\theta _B}={B_y}={{q{l^3}} \over {24}}$$

Now in order to determine bending moment of the conjugate beam at the midspan the following free body diagram is considered.

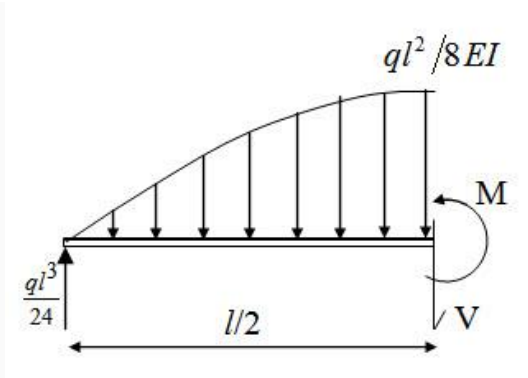


Fig. 6.6.

Applying equilibrium condition we have,

$$M={{q{l^3}} \over {24EI}}{l \over 2}-{{q{l^3}} \over {24EI}}{{3l} \over {16}}={{5q{l^4}} \over {384EI}}$$

Since deflection at the midspan of the real beam is equal to the bending moment at the midspan of the conjugate beam, we have,

$$\delta= M = {{5q{l^4}} \over {384EI}}$$

MODULE 2. Analysis of Statically Indeterminate Beams

LESSON 7.

Introduction

In all the examples considered in the previous module, the equations of static equilibrium (section 1.6) alone were sufficient to determine the unknown support reactions and internal forces. Such structures are called determinate structures. However in many practical structures, the number of unknown may exceed the number of equilibrium conditions and therefore the equations of statics alone cannot provide the solution. Such structures are called statically indeterminate or redundant structures. The extra unknowns are due to more number of external supports or internal members or both than that of actually required to maintain the static equilibrium configuration. In order to analyze indeterminate structures, additional equations considering the geometry of the deflected shape, also known as compatibility equations are required. In this lesson and the subsequent lessons in this module we will learn several methods to analyze statically indeterminate beams and rigidly jointed frames.

7.1 Static Indeterminacy

The degree of static indeterminacy or redundancy is defined as,

Degree of static indeterminacy = Total number of unknown (external and internal) - Number of independent equations of equilibrium

For instance, in the cantilever beam shown in Figure 7.1, the number of unknown reactions is three, viz, A_x, A_y and M_A. These three unknowns can be solved by three static equilibrium equations $\sum {F_x}=0$, $\sum {F_y}=0$, $\sum {M_A}=0$ and therefore this is a determinate structure. Now if end B is propped as shown in Figure 7.2, the number of unknown reactions becomes four (A_x, A_y, M_A and B_y) while the total number equations remain as three. Hence the beam now becomes statically indeterminate with degree of indeterminacy one.

Fig.7.1.

For statically indeterminate beams and rigidly jointed frames, there are two types of indeterminacy, *i*) external indeterminacy and *ii*) internal indeterminacy.

7.1.1 External Indeterminacy

The external indeterminacy is the excess of total number of support reactions over the static equilibrium equations. Some examples are given below,

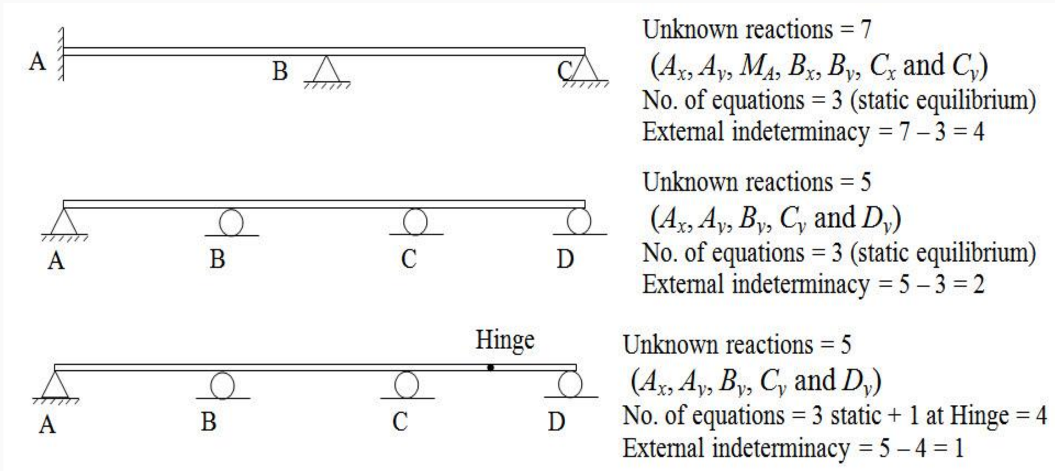


Fig. 7.2.

In the case of continuous beam shown above the internal forces (shear and moment) at any point in the beam can be determined by static equilibrium equations once the support reactions are known. Therefore, these beams are determinate internally but indeterminate externally.

7.1.2 Internal indeterminacy

Structures may also become indeterminate due to more number of members than that of actually required to maintain the static equilibrium configuration. Unlike externally indeterminate structures, here internal member forces may not be determined by static equilibrium equations even though the support reactions are known and therefore these structures are called internally indeterminate structures.

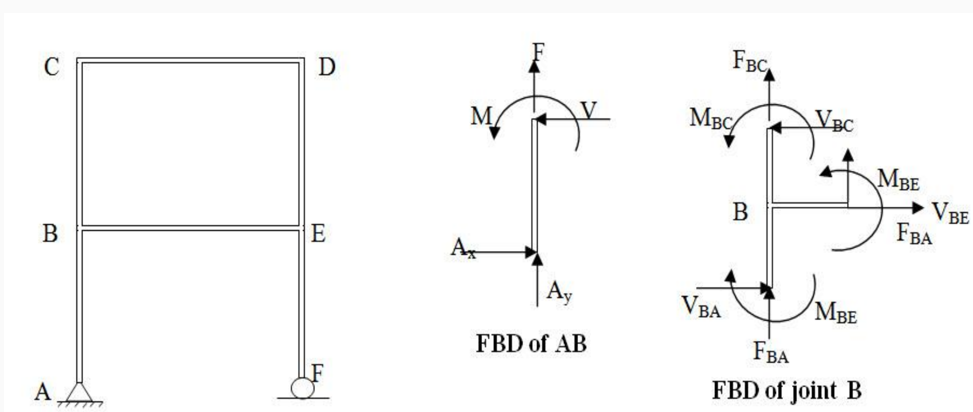


Fig. 7.3.

For illustration purpose, consider a rigid frame as shown in Figure 7.3. There are three unknown reactions (A_x, A_y and F_y) which can be determined by three equilibrium equations. Thus the structure is externally determinate. By considering free body diagram of member

AB, axial force (F), shear force (V) and moment (M) at any point in AB can be computed. Now consider joint B at which there are total nine internal forces (viz, axial, shear and moment each for BA, BC and BE). Since internal forces in member BA are already known, the number of unknown at B is six. Similarly by taking FBD of any joint one can see that the number of unknown internal forces is always six. As only three equilibrium equations are available, here internal indeterminacy is three.

In general the degree of internal indeterminacy of rigid frames is represented by,

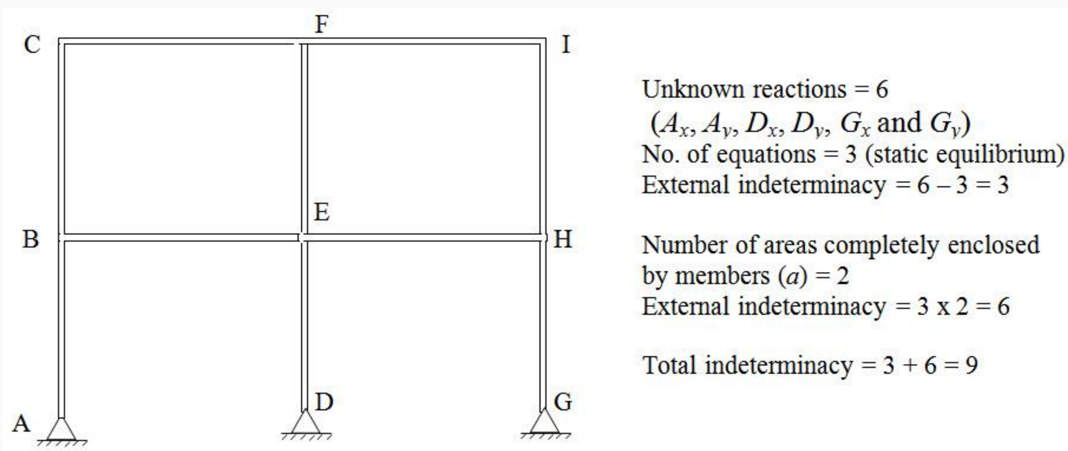
$$I=3a$$

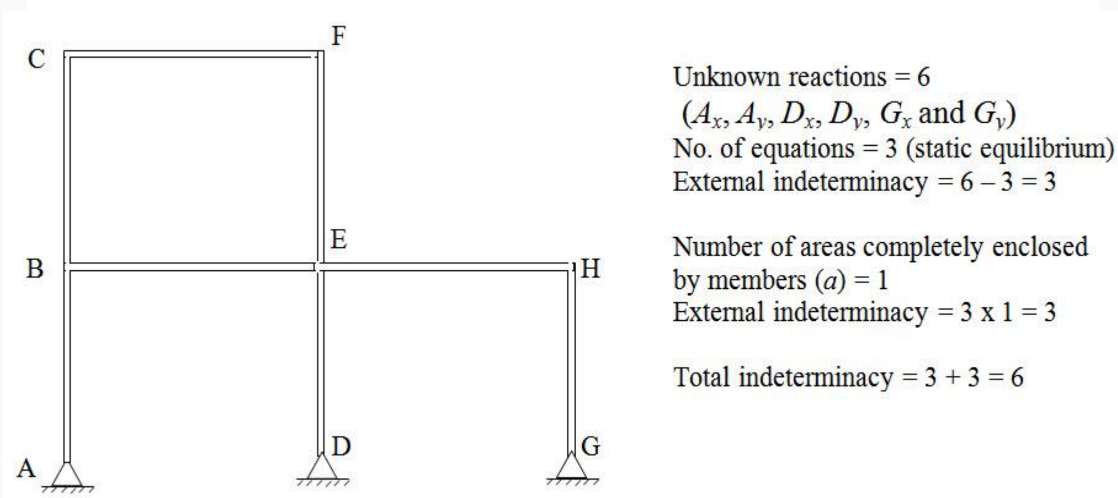
where, *a* is the number of areas completely enclosed by members of the frame. For the rigid frame shown in Figure 7.3, *a* = 1 and therefore degree of internal indeterminacy is $3 \times 1=3$.

7.1.3 Total Indeterminacy

Total indeterminacy = external indeterminacy + internal indeterminacy

Some examples are given bellow,





7.2 Method of Analysis of Statically Indeterminate Beams

The method of analysis of statically indeterminate beams may broadly be classified into two groups; (*i*) force method or flexibility method, and (*ii*) displacement method or stiffness method.

- **Force method:** In this method the redundant forces are taken as unknowns. Additional equations are obtained by considering displacement compatibility.
- **Displacement method:** In this method, displacements of joints are taken as unknowns. A set of algebraic equations in terms of unknown displacements is obtained by substituting the force-displacement relations into the equilibrium equations.

The choice between the force method and the displacement method mainly depends upon the type of structure and the support conditions. However these two methods are just the alternate procedure to solve the same basic equations.

LESSON 8. Force Method: Method of Consistent Deformation

8.1 Compatibility and Principle of Superposition

Displacement compatibility and principle of superpositon play an important role in the analysis of indeterminate structures.

8.1.1 Displacement compatibility

It is the condition which ensures the integrability or continuity of different members or components of a loaded structure while being deformed.

Example 1

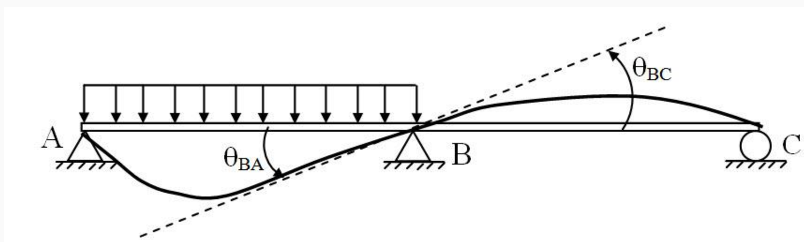


Fig. 8.1.

For the continuous beam shown in Figure 8.1, displacement compatibility conditions at B are,

$\delta_B = 0$ and $|\theta_{BA}| = |\theta_{BC}|$

The second condition implies that the relative rotation at B is zero.

Example 2

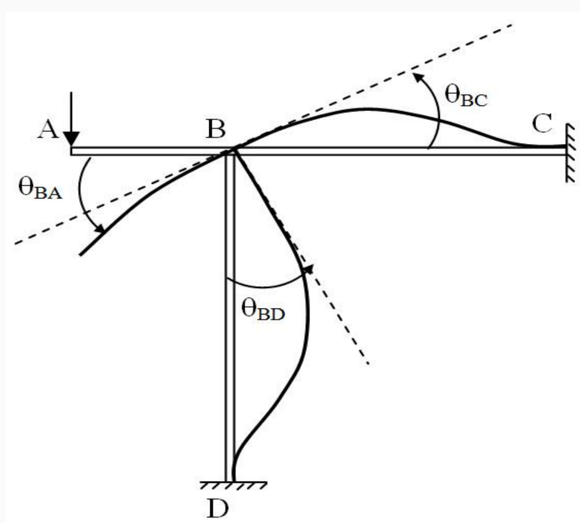


Fig. 8.2.

For the rigid frame shown in Figure 8.2, displacement compatibility conditions at B are,

$\theta_{BA} = \theta_{BC} = \theta_{BD}$

8.1.2 Principle of Superposition

For a linear elastic structure, the deflection caused by two or more loads acting simultaneously is the sum of deflections caused by each load separately. For instance, suppose d be the deflection at mid-span of a simply supported beam subjected to three concentrated load P_1, P_2 and P_3 (Figure 8.3a). If δ_1, δ_2, and δ_3 are the deflection at mid-span respectively due to P_1, P_2, and P_3 when they act separately, principle of superposition states,

$\delta = \delta_1 + \delta_2 + \delta_3$

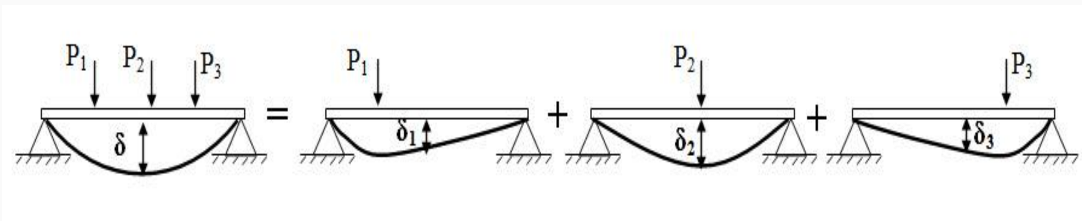


Fig. 8.3a.

For a linear elastic structure, load, P and deflection, δ , are related through stiffness, K, as P = Kδ. If δ_1 and δ_2 are the deflection due to P_1 and P_2 respectively, linearity implies,
$P_1/P_2 = \delta_1/\delta_2$.

8.2 General Procedure

Consider a propped cantilever beam subjected to a concentrated load at its mid-span as shown in Figure 8.4. It is an indeterminate structure with degree of static indeterminacy one.

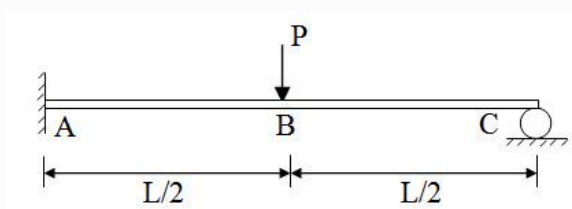


Fig. 8.4.

The steps involved in method of consistent deformation are as follows,

Step 1: Determine the Degree of Static Indeterminacy

The static indeterminacy of the propped cantilever beam is one.

Step 2: Redundant Force/Moment

A number of releases equal to the degree of indeterminacy is introduced. Each release is made by removing an external or an internal force/moment. This force/moment is called redundant force/moment. Redundant forces/moment should be chosen such that the remaining structure is stable and statically determinate.

For the given propped cantilever beam, the basic determinate structure may be obtained by removing the prop at C (Figure 8.5). Here vertical reaction C_y is the redundant force. The basic determinate structure becomes a cantilever beam with concentrated load at mid-span

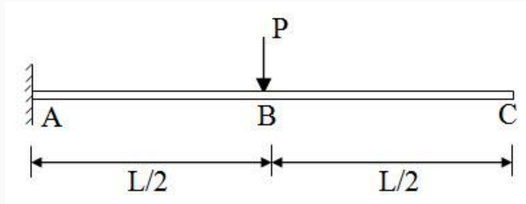


Fig. 8.5.

Alternatively moment constraint at A may also be taken as Redundant. This is explained in sub-section 1.2.1.

Step 3: Solution of Basis Determinate Structure

Calculate the magnitude of the displacement at the released end of the basic determinate structure. Any analysis procedure for determinate structure as discussed in Module I may be followed.

Fig. 8.6.

Here determine vertical displacement at C.

Step 4: Deflection due to Unknown Redundant Force/Moment

Remove all external loads on the basic determinate structure and apply an unknown value of redundant force/moment at the release end. Determine corresponding displacement at the release end in terms of the unknown value of redundant force/moment.

In this case apply vertical force at C and determine δ_C (Figure 8.7).

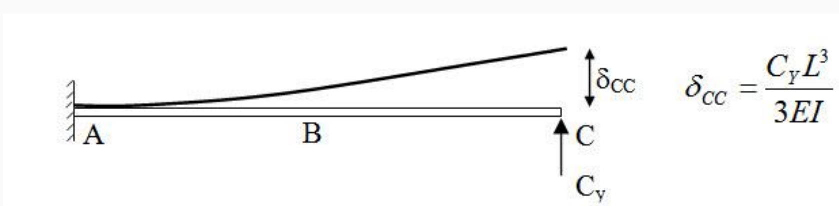


Fig. 8.7.

Step 5: Apply Compatibility Condition at the Release End

Apply displacement compatibility condition at the release end.

In this case vertical displacement at C is zero. Therefore,

$${\delta _C}+{\delta _{CC}}=0\Rightarrow-{{5P{L^3}}\over {48EI}}+{{{C_Y}{L^3}}\over {3EI}}=0\Rightarrow{C_Y}={{5P}\over{16}}$$

Step 6: Solve for Other Unknown

Once the redundant force/moment is determined, other support reaction may be found by using equilibrium equations.

$$\sum M_A = 0 \Rightarrow -M_A + P\frac{L}{2} - C_Y L = 0 \Rightarrow M_A = \frac{PL}{2} - \frac{5PL}{16} = \frac{3PL}{16}$$

$$\sum F_Y = 0 \Rightarrow A_Y + C_Y - P = 0 \Rightarrow A_Y = P - \frac{5P}{16} = \frac{11P}{16}$$

$$\sum F_X = 0 \Rightarrow A_X = 0$$

Solution

$A_X = 0$

$A_Y = \frac{11P}{16}$

$C_Y = \frac{5P}{16}$

$M_A = \frac{3PL}{16}$

8.2.1 Alternative Solution with Different Redundant Force/Moment

Step 1:

Degree of Static Indeterminacy is one.

Step 2:

Moment constraint at A is taken as redundant. The basic determinate structure becomes a simply supported beam with concentrated load at mid-span (Figure 8.8).

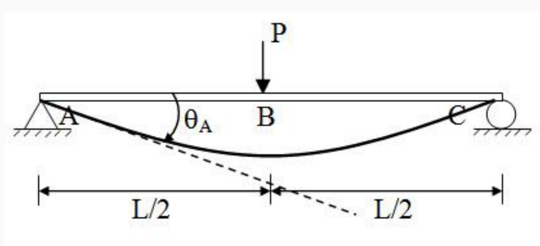


Fig. 8.8.

Step 3:

$${\theta _A}={{P{L^2}} \over {16EI}}$$

Step 4:

Apply moment M_A at A and calculate slope θ_{AA}.

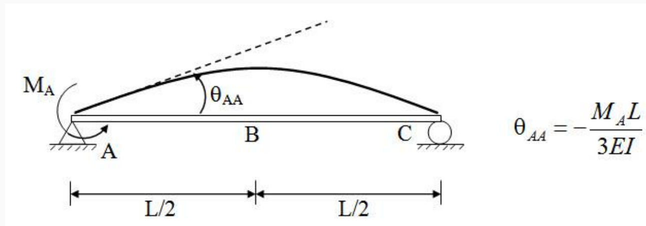


Fig. 8.9.

Step 5:

End A in the original structure is fixed and therefore slope at A is zero.

$${\theta _A} + {\theta _{AA}}=0 \Rightarrow {{P{L^2}} \over {16EI}}-{{{M_A}L} \over {3EI}}=0 \Rightarrow {M_A}={{3PL} \over {16}}$$

Step 6:

$$\sum M_A = 0 \Rightarrow -M_A + P\frac{L}{2} - C_Y L = 0 \Rightarrow C_Y = \frac{5P}{16}$$

$$\sum F_Y = 0 \Rightarrow A_Y + C_Y - P = 0 \Rightarrow A_Y = P - \frac{5P}{16} = \frac{11P}{16}$$

$$\sum F_X = 0 \Rightarrow A_X = 0$$

Solution

$A_X = 0$

$A_Y = \frac{11P}{16}$

$C_Y = \frac{5P}{16}$

$M_A = \frac{3PL}{16}$

8.3 Maxwell-Betti Reciprocal Theorem

Consider two points 1 and 2 in a simply supported beam as shown in Figure 8.10. The beam is separately subjected to two system of forces P_1 and P_2 at point 1 and 2 respectively. Let d_{12} be the deflection at point 2 due to P_1 acting at point 1 (Figure 8.10a) and d_{21} be the deflection at point 1 due to P_2 acting at point 2 Figure 8.10b).

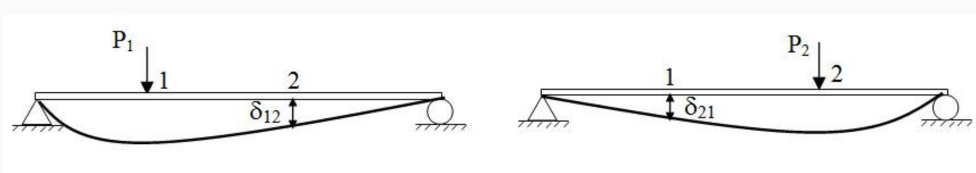


Fig.8.10 (a) **Fig.8.10 (b)**

The Reciprocal theorem states, the work done by the first system of forces acting through the displacement of second system is the same as the work done by the second system of forces acting through the displacement of first system. Therefore,

$${P_1}\times{\delta _{21}}={P_2}\times {\delta _{12}}$$

The reciprocal theorem is also valid for moment-rotation system. For instance, for the system shown in Figure 8.11, the reciprocal theorem gives,

$${M_1} \times {\theta _{21}}={M_2} \times {\theta _{12}}$$

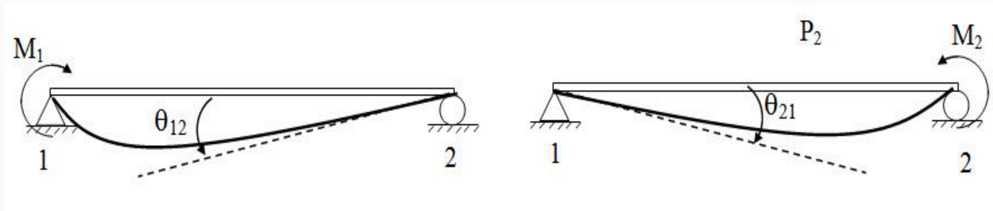


Fig. 8.11.

LESSON 9. Force Method: Three-Moment Equation

Beams that have more than one span are called continuous beam. In this lesson a general equation based on method consistent deformation (lesson 8) is derived and applied to solve continuous beam. This equation gives a relationship among the bending moments at three consecutive supports and hense often called as Three-Moment Equation.

9.1 Derivation of Three-Moment Equation

Consider a arbitrarily loaded continuous beam in which A, B and C are three consicutive support as shwon in Fgiure 9.1.

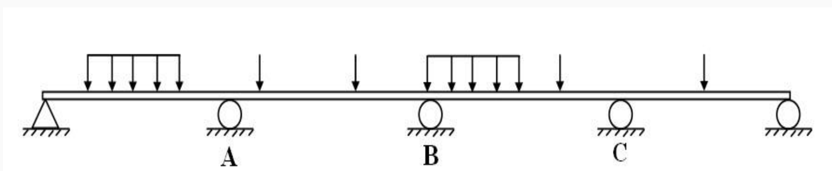


Fig. 9.1.

Let L_{AB}, I_{AB} and E_{AB}, L_{BC}, I_{BC} nd E_{BC} are span length, second moment of area and Young's modulus coresponding to span AB and BC respectively. This is an indeterminate beam which may be made statically determinate by releasing moment constraint (inserting hinge) at A, B and C. Therefore M_A, M_B and M_C are the redundatn moments. The basic determinate structure is shwon in Figure 9.2.

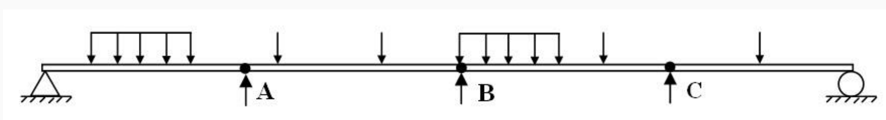


Fig. 9.2.

Deflected shape of the span AB and BC due to external loading and the redundant moments are shwon in Fgure 9.3a-b.

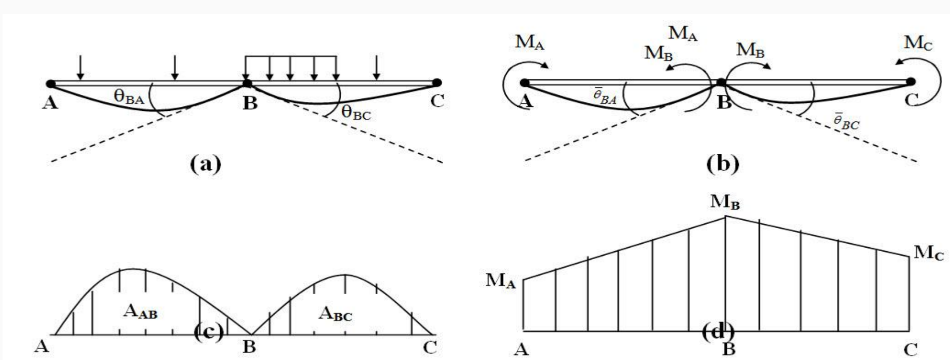


Fig. 9.3.

Slopes at θ_{BA}, θ_{BC} due to external loading (Figure 9.3a) and ${\bar \theta _{BA}}$, ${\bar \theta _{BC}}$ due to redundant momoment (Figure 9.3a) may be obtaind moment-area method discussed in lesson 5. Bending moment diagrams for each span are shwon in Figure 9.3c-d.

$${\theta _{BA}} = {{{\rm{Moment of }}M/EI{\rm{ diagram between A and B about A}}} \over {{\rm{L}}_{{\rm{AB}}}}} = {{{A_{AB}}{x_{AB}}} \over {{E_{AB}}{I_{AB}}{L_{AB}}}}$$

$${\theta _{BC}} = {{{\rm{Moment of }}M/EI{\rm{ diagram between B and C about C}}} \over {{\rm{L}}_{{\rm{AB}}}}} = {{{A_{BC}}{x_{CB}}} \over {{E_{BC}}{I_{BC}}{L_{BC}}}}$$

Similarly, ${\bar \theta _{BA}}$ and ${\bar \theta _{BC}}$ may be obtained as,

$${\bar \theta _{BA}} = {1 \over {{L_{AB}}}}\left[{{{{M_A}{L_{AB}}} \over {E{I_{AB}}}}{{{L_{AB}}} \over 2} + {1 \over 2}{{\left({{M_B} - {M_A}} \right){L_{AB}}} \over {E{I_{AB}}}}{{2{L_{AB}}} \over 3}} \right] = {{{M_A}{L_{AB}}} \over {6{E_{AB}}{I_{AB}}}} + {{{M_B}{L_{AB}}} \over {3{E_{AB}}{I_{AB}}}}$$

$${\bar \theta _{BC}} = {1 \over {{L_{BC}}}}\left[{{{{M_C}{L_{BC}}} \over {E{I_{BC}}}}{{{L_{BC}}} \over 2} + {1 \over 2}{{\left({{M_B} - {M_C}} \right){L_{BC}}} \over {E{I_{BC}}}}{{2{L_{AB}}} \over 3}} \right] = {{{M_C}{L_{BC}}} \over {6{E_{BC}}{I_{BC}}}} + {{{M_B}{L_{BC}}} \over {3{E_{BC}}{I_{BC}}}}$$

Now, compatibility condition at B says, the relative rotation at B is zero.

Therefore,

$${\theta _{BA}} + {\theta _{BC}} + {\bar \theta _{BA}} + {\bar \theta _{BC}}=0$$

$$\Rightarrow {{{A_{AB}}{x_{AB}}} \over {{E_{AB}}{I_{AB}}{L_{AB}}}} + {{{A_{BC}}{x_{CB}}} \over {{E_{BC}}{I_{BC}}{L_{BC}}}} + {{{M_A}{L_{AB}}} \over {6{E_{AB}}{I_{AB}}}} + {{{M_B}{L_{AB}}} \over {3{E_{AB}}{I_{AB}}}} + {{{M_C}{L_{BC}}} \over {6{E_{BC}}{I_{BC}}}} + {{{M_B}{L_{BC}}} \over {3{E_{BC}}{I_{BC}}}}=0$$

$$\Rightarrow {{{M_A}{L_{AB}}} \over {{E_{AB}}{I_{AB}}}} + 2{M_B}\left({{{{L_{AB}}} \over {{E_{AB}}{I_{AB}}}} + {{{L_{BC}}} \over {{E_{BC}}{I_{BC}}}}} \right) + {{{M_C}{L_{BC}}} \over {{E_{BC}}{I_{BC}}}}=-{{6{A_{AB}}{x_{AB}}} \over {{E_{AB}}{I_{AB}}{L_{AB}}}} - {{6{A_{BC}}{x_{CB}}} \over {{E_{BC}}{I_{BC}}{L_{BC}}}}$$

The above equation is known as the Three-Moment Equation.

9.1.2 Three-Moment Equation with Support Settlement

In the above form of Three-Moment Equation it was assumed that the support reactions and internal forces in the beam are induced only due to external load. However, sometime movement of joints, for example support settlement may take place and if it happens, its effect has to be considered in the analysis. In this section, a generalised Three-Moment Equation including the effect of support settlement is derived.

Consider the continuous beam given in Figure 9.1. Additionally suppose support A, B and C have settled to position A¢, B¢ and C¢ by amounts d_A, d_B and d_C respectively as shwon in Figure 9.4.

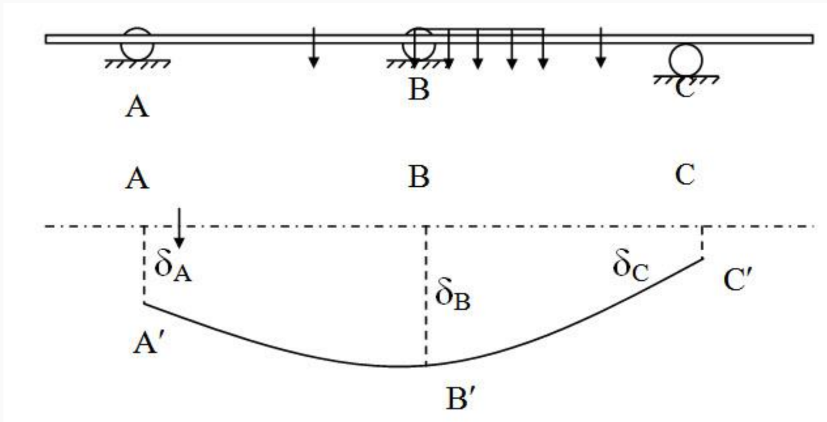


Fig. 9.4.

Deflection of B with respect to A, $d_{BA} = d_B - d_A$

Deflection of B with respect to C, $d_{BC} = d_B - d_C$

Relative deflection at B with respect to A and C cause rotation at B which may be obtained as,

$${\hat \theta _{BA}}=-{{{\delta _{BA}}} \over {{L_{AB}}}}$$ and $${\hat \theta _{BC}}=-{{{\delta _{BC}}} \over {{L_{BC}}}}$$

Now apply the compatibilty condition at B,

$${\theta _{BA}} + {\theta _{BC}} + {\bar \theta _{BA}} + {\bar \theta _{BC}} + {\hat \theta _{BA}} + {\hat \theta _{BC}}=0$$

$$\Rightarrow {{{M_A}{L_{AB}}} \over {{E_{AB}}{I_{AB}}}} + 2{M_B}\left({{{{L_{AB}}} \over {{E_{AB}}{I_{AB}}}} + {{{L_{BC}}} \over {{E_{BC}}{I_{BC}}}}} \right) + {{{M_C}{L_{BC}}} \over {{E_{BC}}{I_{BC}}}}=-{{6{A_{AB}}{x_{AB}}} \over {{E_{AB}}{I_{AB}}{L_{AB}}}} - {{6{A_{BC}}{x_{CB}}} \over {{E_{BC}}{I_{BC}}{L_{BC}}}} + 6\left({{{{\delta _{BA}}} \over {{L_{AB}}}} + {{{\delta _{BC}}} \over {{L_{BC}}}}} \right)$$

The generalised Three-Moment Equation may be simplified for special cases,

Case 1: E and I constant

$${M_A}{L_{AB}} + 2{M_B}\left({{L_{AB}} + {L_{BC}}} \right) + {M_C}{L_{BC}}=-{{6{A_{AB}}{x_{AB}}} \over {{L_{AB}}}} - {{6{A_{BC}}{x_{CB}}} \over {{L_{BC}}}} + 6EI\left({{{{\delta _{BA}}} \over {{L_{AB}}}} + {{{\delta _{BC}}} \over {{L_{BC}}}}} \right)$$

Case 1: E and I constant, no settlement

$${M_A}{L_{AB}} + 2{M_B}\left({{L_{AB}} + {L_{BC}}} \right) + {M_C}{L_{BC}}=-{{6{A_{AB}}{x_{AB}}} \over {{L_{AB}}}} - {{6{A_{BC}}{x_{CB}}} \over {{L_{BC}}}}$$

Example

A continuous beam ABCD is subjected to external load as shown bellow. Calculate support reactions.

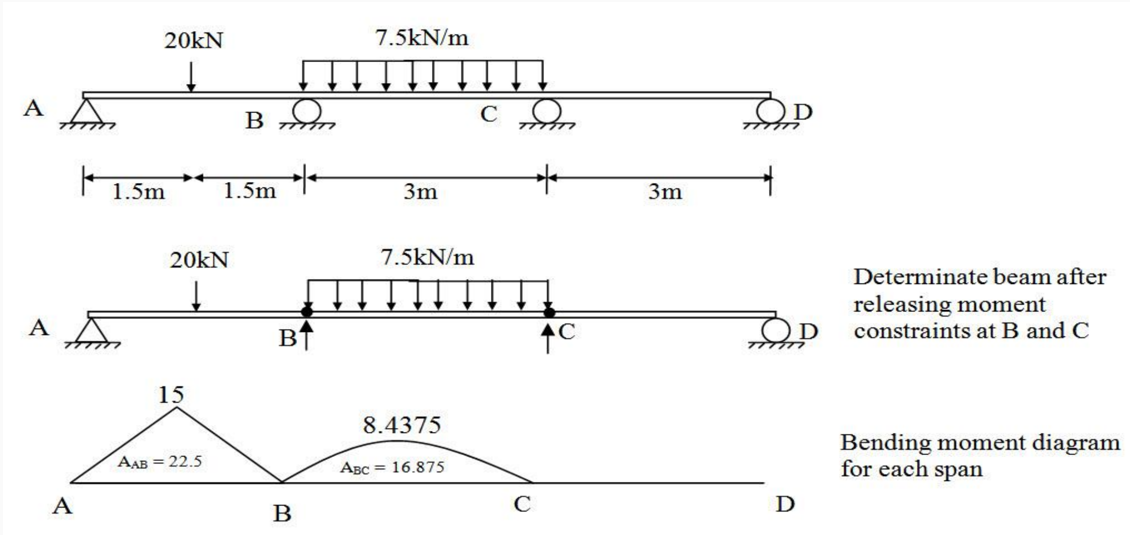


Fig.9.5.

Bending moment diagrams for each span due to external load are obtained using any method discussed in module I.

By inspection we have,

$M_A = 0$ and $M_D = 0$

$${A_{AB}}={1 \over 2} \times 15 \times 3=22.5$$ and $${x_{AB}}=1.5$$

$${A_{BC}}={2 \over 3} \times 8.4375 \times 3=16.875$$ and $${x_{CB}}=1.5$$

Now, applying Three-Moment Equation at B,

$$3{M_A} + 2{M_B}\left({3 + 3} \right) + 3{M_C}=-{{6 \times 22.5 \times 1.5} \over 3} - {{6 \times 16.875 \times 1.5} \over 3}$$

$$\Rightarrow 4{M_B} + {M_C}=-39.375 \qquad (1)$$

Now, applying Three-Moment Equation at C,

$$3{M_B} + 2{M_C}\left({3 + 3} \right) + 3{M_D}=-{{6 \times 16.875 \times 1.5} \over 3}$$

$$\Rightarrow {M_B} + 4{M_C}=-16.875 \qquad (2)$$

Solving (1) and (2), we have,

$${M_B}=- 9.375{\rm{kNm}}$$ and $${M_C}=-1.875{\rm{kNm}}$$

After determining the redundant moment, remaining support reaction may be obtained by using static equilibrium equations.

$$\sum {{M_B}}=0 \Rightarrow 3{A_y} - 20 \times 1.5 = {M_B} \Rightarrow 3{A_y} - 20 \times 1.5=-9.375 \Rightarrow {A_y} = 6.875{\rm{kN}}$$

$$\sum {{M_C}}=0 \Rightarrow 6{A_y} + 3{B_y} - 20 \times 4.5 - 7.5 \times 3 \times 1.5 = {M_C}$$

$$\Rightarrow 3{B_y}=-1.875 + 123.75 - 6 \times 6.875 \Rightarrow {B_y} = 26.875{\rm{kN}}$$

$$\sum {{M_C}}=0 \Rightarrow {M_C} - 3{D_y} = {D_y}=- 0.625{\rm{kN}}$$

$$\sum {{F_Y}}=0 \Rightarrow {A_y} + {B_y} + {C_y} + {D_y} = 20 + 7.5 \times 3 \Rightarrow {C_y} = 9.375{\rm{kN}}$$

LESSON 10. Force Method: Beams on Elastic Support

In many application, beams are required to be supported on a continous foundation. One such example is railway sleeper as shwon in Figure 10.1. If the reaction force offered by such continuous support is a function of the transverse deflection of the beam, the support is called elastic support. In this lesson we will learn analysis procedure of beam resting on elastic support.

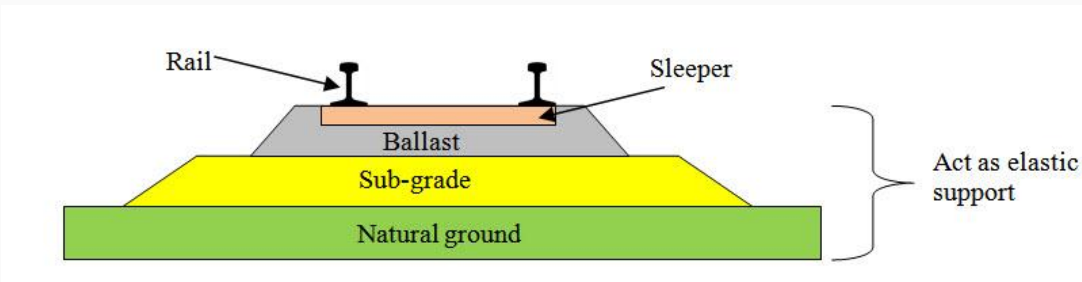


Fig. 10.1.

10.1 Formulation of Governing Equation

Consider a beam, resting on an elastic support, is subjected to any arbitrary load as shwon in Figure 10.2a. Support reaction which is a function of the transverse displacement is shwon in Figure 10.2b

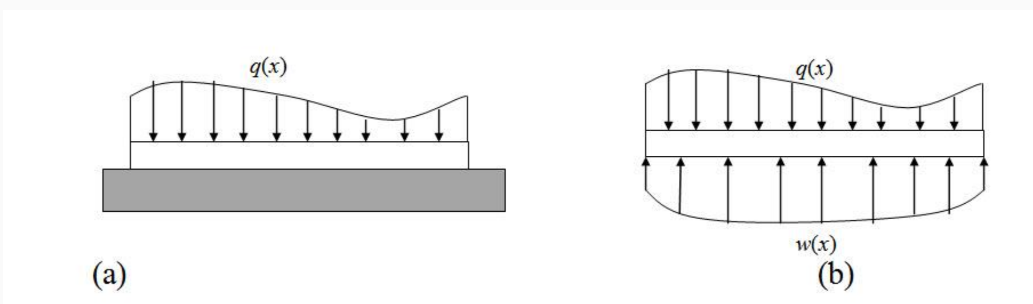


Fig. 10.2.

The reaction offered by the support is a linear funciton of displacement and may be written as,

$$w(x) = ky(x)$$

where, the constant k is the stiffness of the elastic foundation. Therefore at any location x, the net intensity of load is $q(x) - ky(x)$. Consequently the equation of elastic line, derived in lesson 3 (Equation 3.11) becomes,

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 y}{dx^2} \right) = q(x) - ky(x) \qquad (10.1)$$

For a beam with uniform cross-section and material property, taking *EI* out from the defferential operator, equation (10.1) becomes,

$$\frac{d^4 y}{dx^4} + 4\beta^4 y = q \quad (10.2)$$

Where, $\beta^4 = \frac{k}{4EI}$

The above equation is the differential equation for beam on elastic support.

In absence of any external load, the homogeneous form may be written as,

$$\frac{d^4 y}{dx^4} + 4\beta^4 y = 0 \quad (10.3)$$

The general solution of above homogeneous differential equation is,

$$y(x) = e^{\beta x}\left(C_1\sin \beta x + C_2\cos \beta x \right) + e^{-\beta x}\left(C_3\sin \beta x + C_4\cos \beta x \right)$$

10.1 Semi-infinite beam with concentrated load

Fig.10.3.

The general solution of above homogeneous differential equation is,

$$y(x) = e^{\beta x}\left(C_1\sin \beta x + C_2\cos \beta x \right) + e^{-\beta x}\left(C_3\sin \beta x + C_4\cos \beta x \right)$$

Now, for $x \to \infty$, y = 0 $\Rightarrow e^{\beta x}\left(C_1\sin \beta x + C_2\cos \beta x \right) = 0$

$$y(x) = e^{-\beta x}\left(C_3\sin \beta x + C_4\cos \beta x \right)$$

Boundary conditions,

$M(x = 0) = M_0$ and $V(x = 0) = -P_0$

From lesson 3, we have other forms of equations of elastic line as,

$\frac{d^2 y}{dx^2} = -\frac{M}{EI}$ and $\frac{d^3 y}{dx^3} = -\frac{V}{EI}$

Combining the above equations with the boundary conditions, we have,

$${\left. {{{{d^2}y} \over {d{x^2}}}} \right|_{x = 0}} =-{{{M_0}} \over {EI}}$$ and $${\left. {{{{d^3}y} \over {d{x^3}}}} \right|_{x = 0}} = {{{P_0}} \over {EI}}$$

$${{{d^2}y} \over {d{x^2}}} =-2{\beta ^2}{C_3}{e^{ - \beta x}}\cos \beta x + 2{\beta ^2}{C_4}{e^{ - \beta x}}\sin \beta x$$

$${{{d^3}y} \over {d{x^3}}} = 2{\beta ^3}{C_3}{e^{ - \beta x}}\cos \beta x - 2{\beta ^3}{C_4}{e^{ - \beta x}}\sin \beta x + 2{\beta ^3}{C_3}{e^{ - \beta x}}\sin \beta x + 2{\beta ^3}{C_4}{e^{ - \beta x}}\cos \beta x$$

$${\left. {{{{d^2}y} \over {d{x^2}}}} \right|_{x = 0}} =-2{\beta ^2}{C_3} =-{{{M_0}} \over {EI}} \Rightarrow {C_3} = {{{M_0}} \over {2EI{\beta ^2}}}$$

$${\left. {{{{d^3}y} \over {d{x^3}}}} \right|_{x = 0}} = 2{\beta ^3}{C_3} + 2{\beta ^3}{C_4} = {{{P_0}} \over {EI}} \Rightarrow {C_4} = {{{P_0}} \over {2EI{\beta ^3}}} - {{{M_0}} \over {2EI{\beta ^2}}}$$

Final solution

$$y(x) = {{{e^{ - \beta x}}} \over {2EI{\beta ^2}}}\left[{{M_0}\left({\sin \beta x - \cos \beta x} \right) + {{{P_0}} \over {2EI{\beta ^3}}}\cos \beta x} \right]$$

$$M(x) =-EI{{{d^2}y} \over {d{x^2}}}$$; $$V(x) =-EI{{{d^3}y} \over {d{x^3}}}$$

LESSON 11. Displacement Method: Slope Deflection Equation – 1

In the displacement method, unlike the force methods, displacements/rotations at joints are taken as unknowns. A set of algebraic equations in terms of unknown displacements/rotations is obtained by substituting the force-displacement relations into the equilibrium equations. Obtained equations are then solved for the unknown displacements/rotations.

In this lesson and the subsequent lessons we will learn two commonly used displacement method, *i*) Slope-deflection method, and *ii*) Moment distribution method.

11.1 Kinematic Unknowns

Since displacements/rotations are the primary unknown, identification of possible displacemnts/rotations at every joint is important. Total number of such unknowns is called kinematic unknowns. For instance, in the shown in Figure 11.1a, joints A and B can only undergo rotation θ_A and θ_B respectively and therefore the number of kinematic unknowns is 2. Now if end A is replaced by a fixed support as shown in Figure 11.1b, only possible displacement/roation is q_B and therefore the number of kinematic unknowns is reduced to 1. Similarly in Figure 11.1c the kinematic unknowns are θ_B and δ_C.

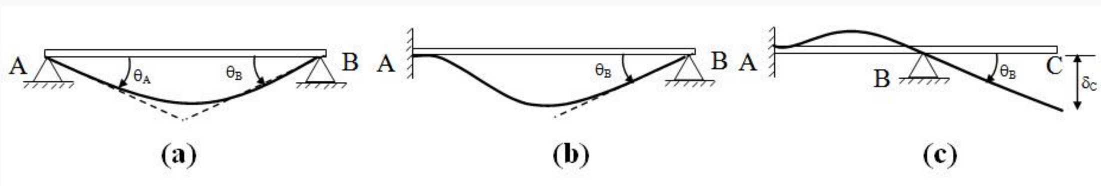


Fig. 11.1.

11.2 Slope-Deflection Method

11.2.1 Sign Convention

For the development and application of Slope-Deflection Method we will use a new sign convention which different from the sign convention discussed in lesson 2.

Moment: At the end of a member clockwise moment is positive.

Transverse displacement: Transverse displacement in upward direction is positive.

Rotation: Rotation in anti-clockwise direction is positive.

The above sign convention is depicted in Figure 11.2.

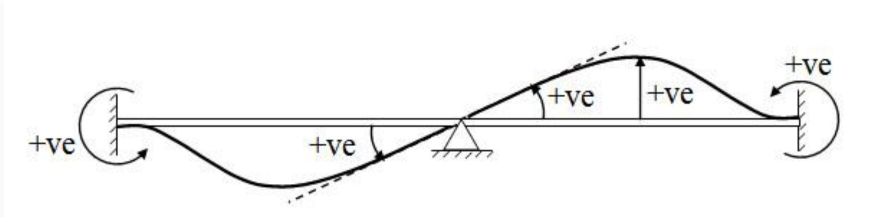


Fig. 11.2.

11.2.2 Basic Concept

Slope-deflection equations are obtained by expressing moment at the end of a member as the superposition of end moment due to external loads on the member assuming ends are restrained and end moments caused by the actual end displacements and rotations. If a structure is composed of several members, which is the case with continuous beam, the above concept is applied to each member seperately. For instance, consider member BC in an arbitrarily loaded continuous beam as shown in Figure 11.3.

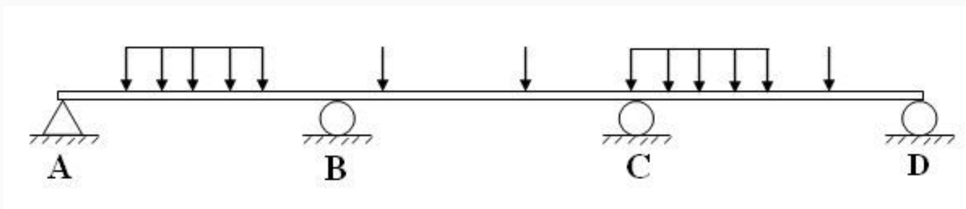


Fig. 11.3.

Let M_{FBC} and M_{FCB} are the end moments due to external loads assuming joint B and C are fixed as shown in Figure 11.4a. δ_B, θ_B and δ_C, θ_C are the displacement, rotation at joint B and C respectively as shown in Fgiure 11.4b.

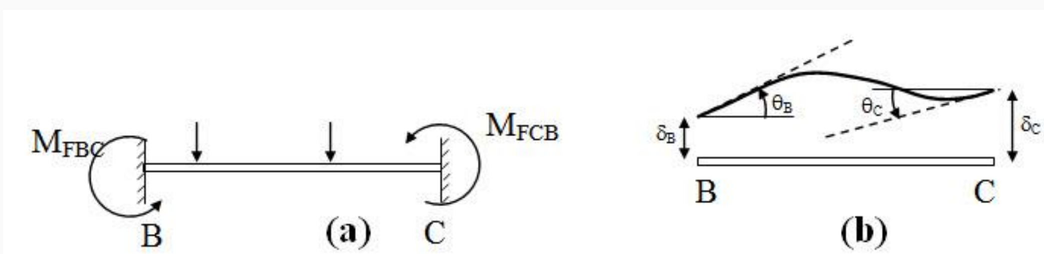


Fig. 11.4.

The end moments at B and C are expressed as:

$M_{BC} = M_{FBC}$ + Moment caused by δ_B, θ_B, δ_C, θ_C

$M_{CB} = M_{FCB}$ + Moment caused by δ_B, θ_B, δ_C, θ_C

M_{FBC}/ M_{FCB} are often called as *fixed end moment*. An illustration of the above concept is discussed in the next sub-section.

11.2.3 Derivation of Slope-Deflection Equations

As mentioned above, in slope-deflection equations, moment at any end is expressed as the sum of fixed end moment and moments due to deflection/rotation.

Fixed end moment

Fixed end moment may be determined by any standard method for solving fixed beam subjected to transverse loading. For ready reference, the fixed end moment for a number of common loading cases are given bellow.

Moment due to rotation at B (= q_B)

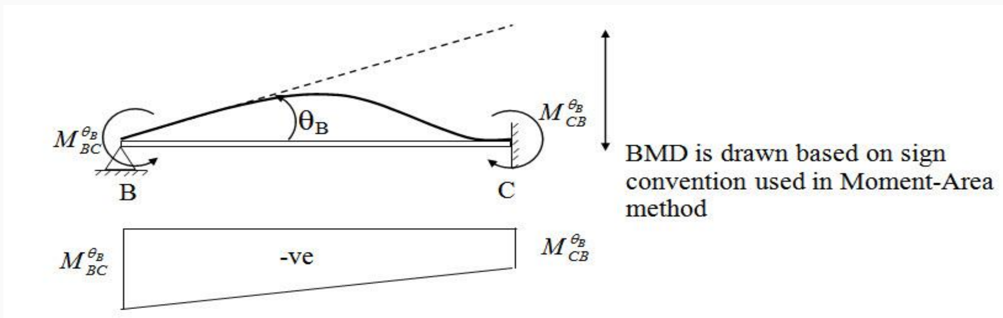


Fig. 11.5.

Let $M_{BC}^{\theta_B}$ and $M_{CC}^{\theta_B}$ are the moment respectively at B and C due to rotation θ_B at B. Applying Moment-Area method as discussed in Lesson 5, we have,

B. Applying Moment-Area method as discussed in Lesson 5, we have,

$$\theta_B = \mathrm{Area\ of\ M/EI\ diagram}$$

$$\Delta_B = \mathrm{Moment\ of\ M/EI\ diagram\ about\ C}$$

Therefore,

$$\theta_B = -{1 \over 2EI}\left(M_{BC}^{\theta_B} + M_{CB}^{\theta_B} \right) \times L_{BC} \qquad (1)$$

$$\Delta_B = \theta_B \times L_{BC} = -{1 \over 2EI}\left(M_{BC}^{\theta_B} + M_{CB}^{\theta_B} \right) \times L_{BC} \times \left({L_{BC} \over 2} + {L_{BC} \over 6}{M_{BC}^{\theta_B} - M_{CB}^{\theta_B} \over M_{BC}^{\theta_B} + M_{CB}^{\theta_B}} \right) \qquad (2)$$

$$\Rightarrow \theta_B \times L_{BC} = \theta_B \times L_{BC} \times \left({1 \over 2} + {1 \over 6}{M_{BC}^{\theta_B} - M_{CB}^{\theta_B} \over M_{BC}^{\theta_B} + M_{CB}^{\theta_B}} \right)$$

$$\Rightarrow M_{BC}^{\theta_B} = -2M_{CB}^{\theta_B} \qquad (2)$$

From Equation (2) and Equation (1), we have,

$M_{CB}^{\theta_B} = \frac{2EI\theta_B}{L_{BC}}$; and $M_{BC}^{\theta_B} = -\frac{4EI\theta_B}{L_{BC}}$

Moment due to rotation at C (= θ_C)

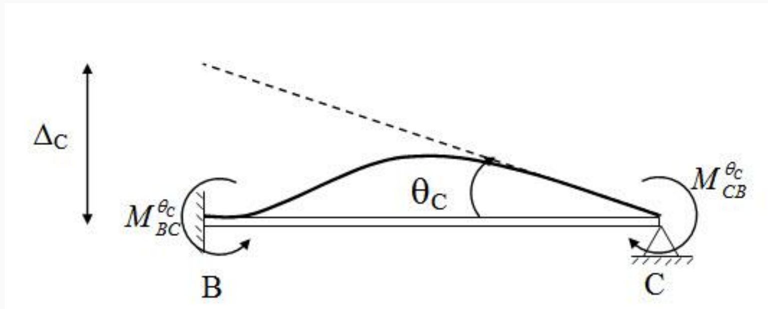


Fig. 11.6.

Following the similar procedure as in the previous case, $M_{BC}^{\theta_C}$ and $M_{CB}^{\theta_C}$ may be obtained as,

$M_{BC}^{\theta_B} = -\frac{2EI\theta_B}{L_{BC}}$ and $M_{CB}^{\theta_B} = \frac{4EI\theta_B}{L_{BC}}$.

Moment due to δ_B, δ_C

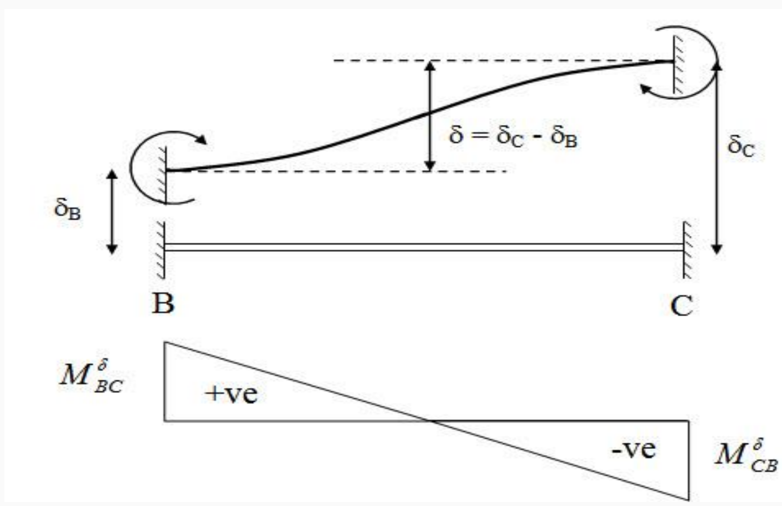


Fig. 11.7.

From the above figure we can write,

Change of slope between tangent at B and C = 0.

=>Area of M/EI diagram = 0

$\Rightarrow M_{BC}^{\delta} = M_{CB}^{\delta}$

Deviation of point C relative to the tangent drawn to B = Deviation of point Brelative to the tangent drawn to C = $\delta_B - \delta_C$.

$$\Rightarrow-{1 \over 2}M_{BC}^\delta {{{L_{BC}}} \over 2}{{{L_{BC}}} \over 6} + {1 \over 2}M_{CB}^\delta {{{L_{BC}}} \over 2}\left({{{{L_{BC}}} \over 2} + {{{L_{BC}}} \over 3}} \right) = \delta$$

$$\Rightarrow-M_{BC}^\delta {{L_{BC}^2} \over {24}} + M_{BC}^\delta {{5L_{BC}^2} \over {24}} = \delta$$

$$\Rightarrow M_{BC}^\delta=-{{6EI\delta } \over {L_{BC}^2}}$$

$$\Rightarrow M_{CB}^\delta=M_{BC}^\delta=-{{6EI\delta } \over {L_{BC}^2}}$$

Finally, the end moments at B and C are obtained by combining the contribution from fixed end moment and joint displacement/rotation as,

$${M_{BC}}={M_{FBC}} + \left({ - M_{BC}^{{\theta _B}}} \right) + \left({ - M_{BC}^{{\theta _C}}} \right) + M_{BC}^\delta$$

$${M_{CB}}={M_{FCB}} + M_{CB}^{{\theta _B}} + M_{CB}^{{\theta _C}} + M_{CB}^\delta$$

$$\left[{\matrix{\text{Moments are added according to the}\cr \text{sign convention mentioned in 11.2.1}\cr}} \right]$$

Substituting all the relevant terms, we have,

$${M_{BC}}={M_{FBC}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _B} + {\theta _C} - {{3\delta } \over {{L_{BC}}}}} \right)$$

$${M_{CB}}={M_{FCB}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _C} + {\theta _B} - {{3\delta } \over {{L_{BC}}}}} \right)$$

The abover two equations are the Slope-Deflection equation.

LESSON 12. Displacement Method: Slope Deflection Equation – 2

12.1 Introduction: In this lesson we will apply the slope -deflection equations, derived in the last lesson, to analyze continuous beams. The general steps are,

Step 1: Treat each span as fixed beam and calculate the fixed end moment. For a ready reference, fixed end moment for a number of common loading cases are summerized in lesson 11.

Step 2: Write slope-deflection equations for each span.

Step 3: Write equilibrium equations for each joint. This will give a set of algebraic equations in terms of unknown rotations. Solve it for the unknown rotations.

Step 4: Substitute the rotations back into the slope-deflection equations and solve for the end moments.

12.1 Example

Calculate the end momens and joint rotations for the continous beam shwon bellow. The beam has constant EI for both the span.

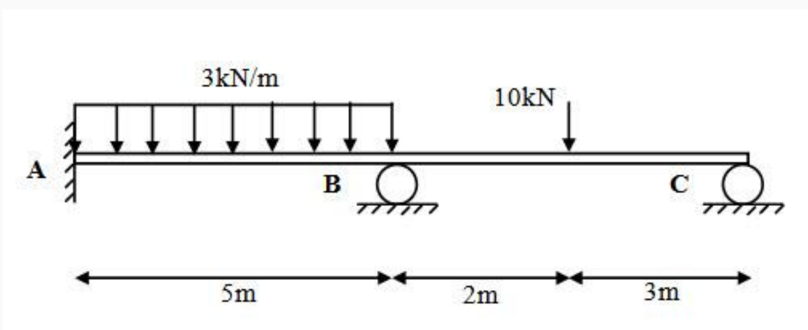


Fig.12.1

Step 1: Fixed end Moments

$$M{}_{FAB}=-{{3 \times {5^2}} \over {12}}=-6.25{\rm{ kNm}}$$; $$M{}_{FBA} = {{3 \times {5^2}} \over {12}} = 6.25{\rm{ kNm}}$$

$$M{}_{FBC}=-{{10 \times 2 \times {3^2}} \over {{5^2}}}=-7.2{\rm{ kNm}}$$;
$$M{}_{FCB}=-{{10 \times 3 \times {2^2}} \over {{5^2}}} = 4.8{\rm{ kNm}}$$

Step 2: Slope-Deflection Equaitons

For span AB,

$${M_{AB}} ={M_{FAB}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _A} + {\theta _B} - {{3\delta } \over {{L_{BC}}}}} \right)=-6.25{\rm{ }} + {{2EI} \over 5}{\theta _B} =-6.25{\rm{ }} + 0.4EI{\theta _B} \qquad (1)$$

$${M_{BA}} ={M_{FBA}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _B} + {\theta _C} - {{3\delta } \over {{L_{BC}}}}} \right) = 6.25 + {{4EI} \over 5}{\theta _B} = 6.25 + 0.8EI{\theta _B} \qquad (2)$$

For span BC,

$${M_{BC}} = {M_{FBC}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _B} + {\theta _C} - {{3\delta } \over {{L_{BC}}}}} \right)=-7.2 + {{2EI} \over 5}\left({2{\theta _B} + {\theta _C}} \right)=-7.2 + 0.4EI\left({2{\theta _B} + {\theta _C}} \right) \qquad (3)$$

$${M_{CB}} = {M_{FCB}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _C} + {\theta _B} - {{3\delta } \over {{L_{BC}}}}} \right) = 4.8 + {{2EI} \over 5}\left({2{\theta _C} + {\theta _B}} \right) = 4.8 + 0.4EI\left({2{\theta _C} + {\theta _B}} \right) \qquad (4)$$

Step 3: Equilibrium Equaitons

At B,

$${M_{BA}} + {M_{BC}} = 0 \Rightarrow 6.25 + 0.8EI{\theta _B} - 7.2 + 0.4EI\left({2{\theta _B} + {\theta _C}} \right)=0$$

$$\Rightarrow 1.6EI{\theta _B} + 0.4EI{\theta _C} - 0.95 = 0 \qquad (5)$$

At C,

$${M_{CB}} = 0 \Rightarrow 0.4EI{\theta _B} + 0.8EI{\theta _C} + 4.8 = 0 \qquad (6)$$

Solving (1) and (2), we have,

$${\theta _B} = {{2.3929} \over {EI}} ; \quad {\theta _C}=-{{7.1964} \over {EI}}$$

Step 4: End Moment calculation

Substituting, θ_B and θ_C into equations (1) – (4), we have,

$${M_{AB}}=-6.25{\rm{ }} + 0.4EI{\theta _B} =-5.29$$

$${M_{BA}} = 6.25 + 0.8EI{\theta _B}=8.16$$

$${M_{BC}}=-7.2 + 0.4EI\left({2{\theta _B} + {\theta _C}} \right) =-8.16$$

$${M_{CB}} = 4.8 + 0.4EI\left({2{\theta _C} + {\theta _B}} \right)=0$$

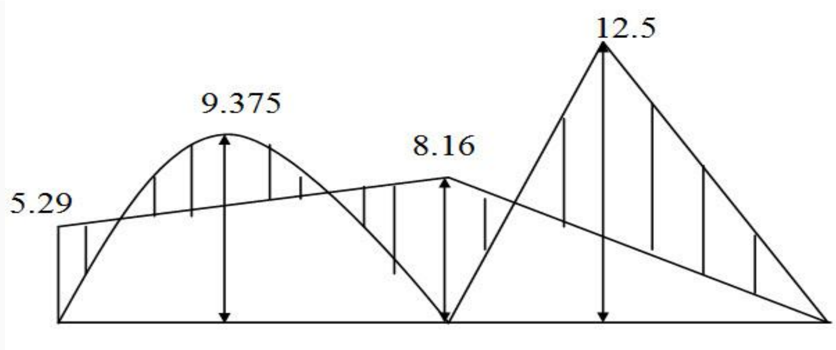


Fig.12.2. Bending moment diagram (kNm).

LESSON 13. Displacement Method: Slope Deflection Equation – 3

In this lesson we will apply the slope-deflection method for the analysis of rigid frames. Based on the nature of deformation, rigid frames are classified into two categories,

i) **Frames without sidesway:** lateral translation of joints are restrained

ii) **Frames with sidesway:** lateral translation of joints are not restrained

Few examples of frames with and without sidesway are depicted in Fig. 13.1.

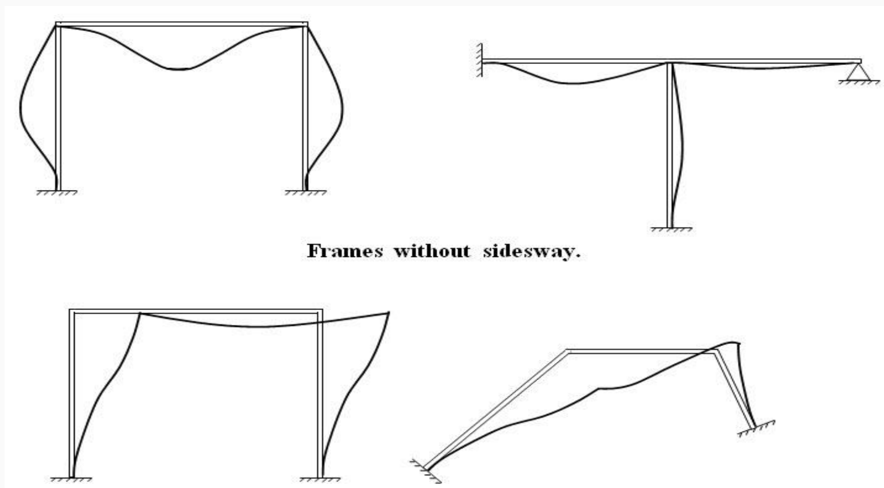


Fig. 13.1. Frames without sidesway.

13.1 Analysis of Frames Without Sidesway

The general procedure for analysis of frames without sidesway is same as for continous beams (lesson 12). This is illustrated in the following two examples.

Example 1

Draw the bending moment diagram for the follwing frame. EI is constant for all members.

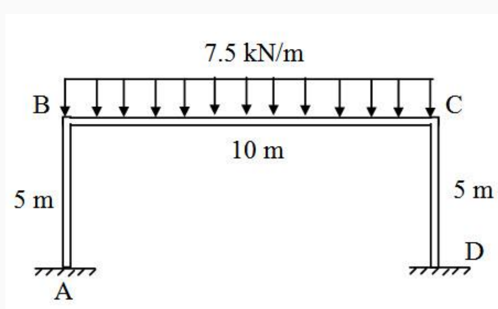


Fig. 13.2.

Step 1: Fixed end Moments

$$M{}_{FAB} =-{{5 \times 4} \over 8}=-2.5{\rm{kNm}}$$; $$M{}_{FCB} = {{7.5 \times {{10}^2}} \over {12}} = 62.5{\rm{kNm}}$$

$$M{}_{FAB} = M{}_{FBA} = M{}_{FCD} = M{}_{FDC}=0$$

Step 2: Slope-Deflection Equaitons

Since A and D are fixed ends, $\theta_A = \theta_D = 0$

Since there is no support settlement, $\delta = 0$

For span AB,

$${M_{AB}} = {M_{FAB}} + {{2EI} \over {{L_{AB}}}}\left({2{\theta _A} + {\theta _B} - {{3\delta } \over {{L_{AB}}}}} \right) = 0.4EI{\theta _B} \tag{13.1}$$

$${M_{BA}} = {M_{FBA}} + {{2EI} \over {{L_{AB}}}}\left({{\theta _A} + 2{\theta _B} - {{3\delta } \over {{L_{AB}}}}} \right) = 0.8EI{\theta _B} \tag{13.2}$$

For span BC,

$${M_{BC}} = {M_{FBC}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _B} + {\theta _C} - {{3\delta } \over {{L_{BC}}}}} \right)=-62.5 + 0.2EI\left({2{\theta _B} + {\theta _C}} \right) \tag{13.3}$$

$${M_{CB}} = {M_{FCB}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _C} + {\theta _B} - {{3\delta } \over {{L_{BC}}}}} \right) = 62.5 + 0.2EI\left({{\theta _B} + 2{\theta _C}} \right) \tag{13.4}$$

For span CD,

$${M_{CD}} = {M_{FCD}} + {{2EI} \over {{L_{CD}}}}\left({2{\theta _C} + {\theta _D} - {{3\delta } \over {{L_{CD}}}}} \right) = 0.8EI{\theta _C} \tag{13.5}$$

$${M_{DC}} = {M_{FDC}} + {{2EI} \over {{L_{CD}}}}\left({{\theta _C} + 2{\theta _D} - {{3\delta } \over {{L_{CD}}}}} \right) = 0.4EI{\theta _C} \tag{13.6}$$

Step 3: Equilibrium Equaitons

At B,

$${M_{BA}} + {M_{BC}} = 0 \Rightarrow 0.8EI{\theta _B} - 62.5 + 0.2EI\left({2{\theta _B} + {\theta _C}} \right)=0$$

$$\Rightarrow 1.2EI{\theta _B} + 0.2EI{\theta _C} - 62.5 = 0 \tag{13.7}$$

At C,

$${M_{CB}} + {M_{CD}} = 0 \Rightarrow 62.5 + 0.2EI\left({{\theta _B} + 2{\theta _C}} \right) + 0.8EI{\theta _C}=0$$

$$\Rightarrow 0.2EI{\theta _B} + 1.2EI{\theta _C} + 62.5 = 0 \qquad (13.8)$$

Solving equations (7) and (8),

$${\theta _B} = {{62.5} \over {EI}}$$ and $${\theta _C}=-{{62.5} \over {EI}}$$

Step 4: End Moment calculation

Substituting, θ_B and θ_C into equations (1) – (6), we have,

$${M_{AB}} = 0.4EI{\theta _B} = 25{\rm{ kNm}}$$

$${M_{BA}} = 0.8EI{\theta _B} = 50{\rm{ kNm}}$$

$${M_{BC}}=-62.5 + 0.2EI\left({2{\theta _B} + {\theta _C}} \right) =-50{\rm{ kNm}}$$

$${M_{CB}} = 62.5 + 0.2EI\left({{\theta _B} + 2{\theta _C}} \right) = 50{\rm{ kNm}}{M_{CB}} = 62.5 + 0.2EI\left({{\theta _B} + 2{\theta _C}} \right) = 50{\rm{ kNm}}$$

$${M_{CD}} = 0.8EI{\theta _C} =-50{\rm{ kNm}}$$

$${M_{DC}} = 0.4EI{\theta _C} =-25{\rm{ kNm}}$$

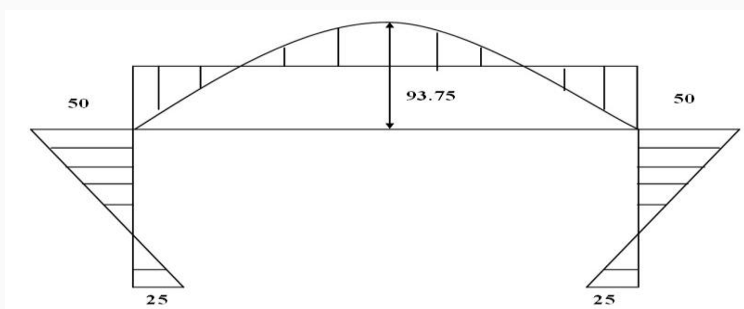


Fig. 13.3. Bending Moment Diagram.

Example 2

Draw the bending moment diagram for the follwing frame. EI is constant for all members.

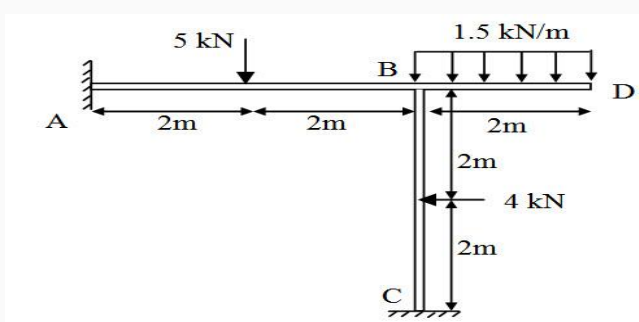


Fig.13.4.

Step 1: Fixed end Moments

$$M{}_{FAB}=-{{5 \times 4} \over 8}=-2.5{\rm{kNm}}$$; $$M{}_{FBA} = {{5 \times 4} \over 8} = 2.5{\rm{kNm}}$$

$$M{}_{FBC}=-{{4 \times 4} \over 8}=-2{\rm{kNm}}$$; $$M{}_{FCD}=-{{4 \times 4} \over 8}=-2{\rm{kNm}}$$

Step 2: Slope-Deflection Equaitons

Since A and C are fixed ends, $\theta_A = \theta_C = 0$

Since there is no support settlement, $\delta = 0$

For span AB,

$${M_{AB}} = {M_{FAB}} + {{2EI} \over {{L_{AB}}}}\left({2{\theta _A} + {\theta _B} - {{3\delta } \over {{L_{AB}}}}} \right) =-2.5 + 0.5EI{\theta _B} \qquad (13.9)$$

$${M_{BA}} = {M_{FBA}} + {{2EI} \over {{L_{AB}}}}\left({{\theta _A} + 2{\theta _B} - {{3\delta } \over {{L_{AB}}}}} \right) = 2.5 + EI{\theta _B} \qquad (13.10)$$

For span BC,

$${M_{BC}} = {M_{FBC}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _B} + {\theta _C} - {{3\delta } \over {{L_{BC}}}}} \right) =-2 + EI{\theta _B} \qquad (13.11)$$

$${M_{CB}} = {M_{FCB}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _C} + {\theta _B} - {{3\delta } \over {{L_{BC}}}}} \right) = 2 + 0.5EI{\theta _B} \qquad (13.12)$$

For span BD,

$${M_{BD}} =-{{1.5 \times {2^2}} \over 2} = - 3{\rm{ kNm}} \qquad (13.3)$$

Step 3: Equilibrium Equaitons

At B,

$${M_{BA}} + {M_{BC}} + {M_{BD}} = 0 \Rightarrow 2.5 + EI{\theta _B} - 2 + EI{\theta _B} - 3 = 0$$

$$\Rightarrow 2EI{\theta _B} - 19.5 = 0 \Rightarrow {\theta _B} = {{1.25} \over {EI}}$$

Step 4: End Moment calculation

Substituting, θ_B into equations (9) – (12), we have,

$${M_{AB}} =-2.5 + 0.5EI{\theta _B} =-1.875{\rm{ kNm}}$$

$${M_{BA}} = 2.5 + EI{\theta _B} = 3.75{\rm{ kNm}}$$

$${M_{BC}} =-2 + EI{\theta _B} =-0.75{\rm{kNm}}$$

$$M_{CB} = 2 + 0.5EI\theta_B = 2.265\ \mathrm{kNm}$$

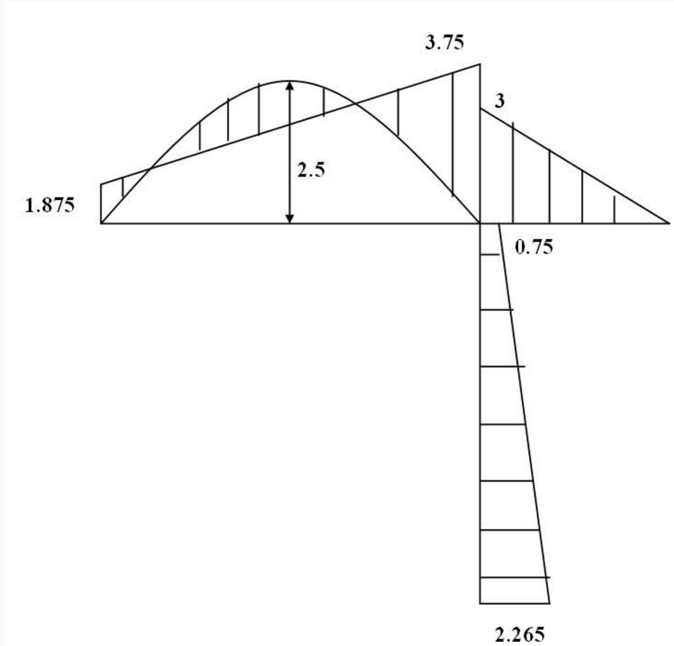


Fig.13.5. Bending Moment Diagram.

LESSON 14. Displacement Method: Slope Deflection Equation – 4

Introduction 14.1: In this lesson we will learn the steps involved in analyzing frames where joint translations are not restrained. In such cases joint translations are also unknown quantities. In addition to moment equilibrium (as discussed lesson 13), additional equations based on shear force in the member are formed. Obtained equations are then solved all the unknowns (rotations are translations). The method is illustrated via the following example.

14.1 Example

Draw the bending moment diagram for the follwing frame. EI is constant for all members.

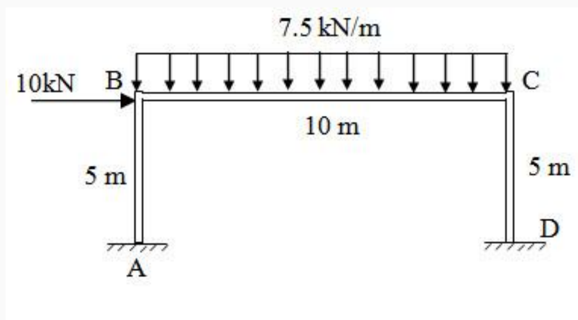


Fig.14.1

Step 1: Fixed end Moments

$$M{}_{FBC} =-{{7.5 \times {{10}^2}} \over {12}} =-62.5{\rm{kNm}}$$; $$M{}_{FCB} = {{7.5 \times {{10}^2}} \over {12}} = 62.5{\rm{kNm}}$$

$$M{}_{FAB} = M{}_{FBA} = M{}_{FCD} = M{}_{FDC} = 0$$

Step 2: Slope-Deflection Equaitons

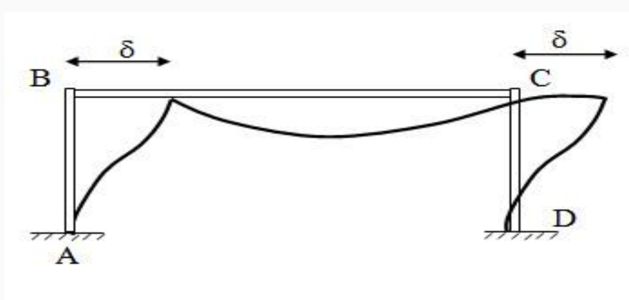


Fig.14.2

Since A and D are fixed ends, $\theta_A = \theta_D = 0$

Since axial deformation is neglected, $\delta_B = \delta_C = \delta$

For span AB,

$${M_{AB}} = {M_{FAB}} + {{2EI} \over {{L_{AB}}}}\left({2{\theta _A} + {\theta _B} - {{3\delta } \over {{L_{AB}}}}} \right) = 0.4EI{\theta _B} - 0.24EI\delta \qquad (14.1)$$

$${M_{BA}} = {M_{FBA}} + {{2EI} \over {{L_{AB}}}}\left({{\theta _A} + 2{\theta _B} - {{3\delta } \over {{L_{AB}}}}} \right) = 0.8EI{\theta _B} - 0.24EI\delta \qquad (14.2)$$

For span BC,

$${M_{BC}} = {M_{FBC}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _B} + {\theta _C} - {{3\delta } \over {{L_{BC}}}}} \right) =-62.5 + 0.2EI\left({2{\theta _B} + {\theta _C}} \right) \qquad (14.3)$$

$${M_{CB}} = {M_{FCB}} + {{2EI} \over {{L_{BC}}}}\left({2{\theta _C} + {\theta _B} - {{3\delta } \over {{L_{BC}}}}} \right) = 62.5 + 0.2EI\left({{\theta _B} + 2{\theta _C}} \right) \qquad (14.4)$$

For span CD,

$${M_{CD}} = {M_{FCD}} + {{2EI} \over {{L_{CD}}}}\left({2{\theta _C} + {\theta _D} - {{3\delta } \over {{L_{CD}}}}} \right) = 0.8EI{\theta _C} - 0.24EI\delta \qquad (14.5)$$

$${M_{DC}} = {M_{FDC}} + {{2EI} \over {{L_{CD}}}}\left({{\theta _C} + 2{\theta _D} - {{3\delta } \over {{L_{CD}}}}} \right) = 0.4EI{\theta _C} - 0.24EI\delta \qquad (14.6)$$

Step 3: Equilibrium Equaitons

At B,

$${M_{BA}} + {M_{BC}} = 0 \Rightarrow 0.8EI{\theta _B} - 0.24EI\delta-62.5+0.2EI\left({2{\theta _B} + {\theta _C}} \right) = 0$$

$$\Rightarrow 1.2EI{\theta _B} + 0.2EI{\theta _C}-0.24EI\delta-62.5=0 \qquad (14.7)$$

At C,

$${M_{CB}} + {M_{CD}} = 0 \Rightarrow 62.5 + 0.2EI\left({{\theta _B} + 2{\theta _C}} \right) + 0.8EI{\theta _C}--0.24EI\delta= 0$$

$$\Rightarrow 0.2EI{\theta _B} + 1.2EI{\theta _C}-0.24EI\delta+62.5=0 \qquad (14.8)$$

Step 3: Additional Shear Equation

Free body diagram of each member are shown bellow.

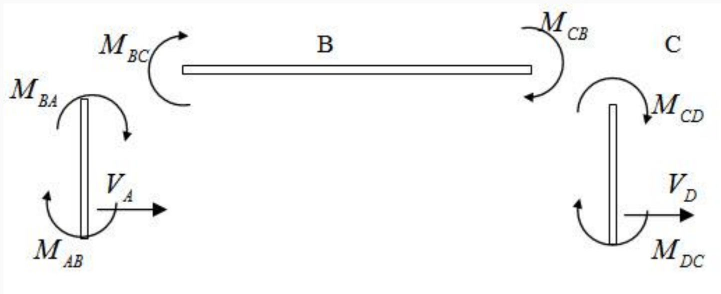


Fig.14.3

Summation of force in horizontal direction is zero $\Rightarrow \sum {F_x} = 0$

Here, horizontal forces are, external horizontal force of 10 kN and shear forces V_A and V_B respectively at support A and D.

From the above FBD, V_A and V_B may be expressed as,

$${V_A} = {{{M_{AB}} + {M_{BA}}} \over {{L_{AB}}}} = {{0.4EI{\theta _B} - 0.24EI\delta+0.8EI{\theta _B}-0.24EI\delta }\over 5}={{1.2EI{\theta _B}-0.48EI\delta } \over 5}$$

$${V_B} = {{{M_{AB}} + {M_{BA}}} \over {{L_{AB}}}} = {{0.8EI{\theta _C} - 0.24EI\delta+0.4EI{\theta _C}-0.24EI\delta }\over 5}={{1.2EI{\theta _C}-0.48EI\delta } \over 5}$$

Now,

$$\sum {{F_x} = 0}\Rightarrow {V_A} + {V_B} + 10 = 0$$

$${{1.2EI{\theta _B} - 0.48EI\delta } \over 5} + {{1.2EI{\theta _C} - 0.48EI\delta } \over 5} + 10 = 0$$

$$1.2EI{\theta _B} + 1.2EI{\theta _C}-0.96EI\delta+50= 0 \qquad (14.9)$$

Solving equations (7) – (9), we have,

$${\theta _B} = {{78.125} \over {EI}} \;,\quad {\theta _C} =-{{46.875} \over {EI}} \;,\quad \delta=-{{91.1458} \over {EI}}$$

Step 4: End Moment calculation

Substituting, θ_B, θ_C and δ into equations (1) – (6), we have,

$${M_{AB}} = 0.4EI{\theta _B} - 0.24EI\delta= 9.375{\rm{ kNm}}$$

$${M_{BA}} = 0.8EI{\theta _B} - 0.24EI\delta= 40.625{\rm{ kNm}}$$

$${M_{BC}} =-62.5 + 0.2EI\left({2{\theta _B} + {\theta _C}} \right) =-40.625{\rm{kNm}}$$

$${M_{CB}} = 62.5 + 0.2EI\left({{\theta _B} + 2{\theta _C}} \right) =59.375{\rm{ kNm}}$$

$${M_{CD}} = 0.8EI{\theta _C} - 0.24EI\delta=-59.375{\rm{kNm}}$$

$${M_{DC}} = 0.4EI{\theta _C} - 0.24EI\delta=-40.625{\rm{ kNm}}$$

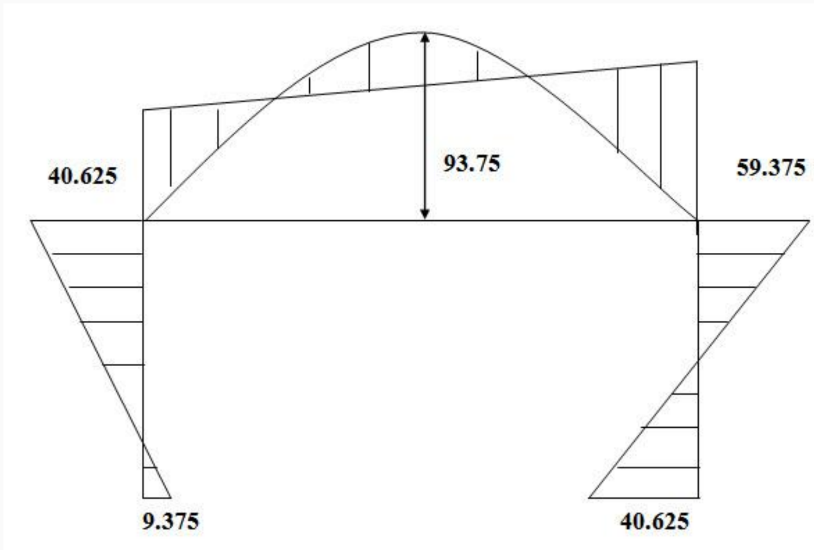


Fig. 14.4. Bending Moment Diagram.

LESSON 15. Displacement Method: Moment Distribution Method – 1

In slope-deflection method the unknown displacements/rotations are obtained by solving a set of algebraic equations. This becomes cumbersome for structures with large number of members. In such cases the Moment distribution method, also known as the Hardy Cross method (named after Prof. Hardy Cross), provides a convenient means for analyzing the structures in an iterative way. In this lesson we will formulate the basic ingredients of Moment distribution method. Illustration of the general procedure and examples will be discussed in the subsequent lessons.

Sign Convention

For the development and application of Moment Dsitribution Method we will use similar sign convention as in the case of Slope-Deflection Method.

Moment: At the end of a member clockwise moment is positive.

Transverse displacement: Transverse displacement in upward direction is positive.

Rotation: Rotation in anti-clockwise direction is positive.

The above sign convention is depicted in Figure 15.1.

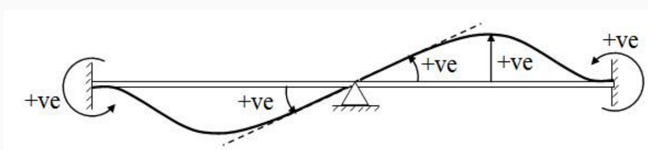


Fig. 15.1.

15.1 Absolute and Relative Stiffness

Stiffness of a member may be defined as the force/moment required to cause unit displacement/rotation. The central idea of Moment distribution method is to distribute moment at any joint, among the connenting members (members meeting at that joint) according to their rotational stiffnesses. In this section we will derive the expressions of rotational stifness of member with different support conditions.

15.1.1 Beam Hinged at Both Ends

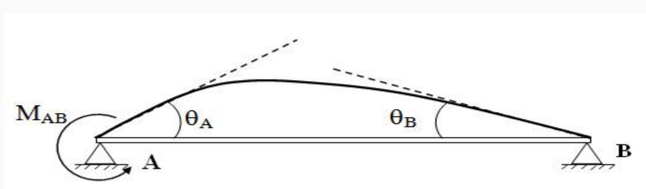


Fig. 15.2.

Slope deflection equation at A and B ($\delta = 0$),

$${M_{AB}} = {{2EI} \over L}\left({2{\theta _A} + {\theta _B}} \right)$$
(15.1)

$${M_{BA}} = {{2EI} \over L}\left({{\theta _A} + 2{\theta _B}} \right)$$
(15.2)

Now, at B, equilibrium equation is, $M_{BA} = 0$. Therefore form equaition (2), we have,

$${\theta _B}=-{1 \over 2}{\theta _A}$$.

Substituting, $\theta_B = -\theta_A/2$ in equation (1), we have,

$${M_{AB}} = {{2EI} \over L}\left({2{\theta _A} - {{{\theta _A}} \over 2}} \right)$$

$$\Rightarrow {M_{AB}} = {{3EI} \over L}{\theta _A}$$

Absolute Stiffness $$k = {{3EI} \over L}$$

15.1.2 Beam Hinged at one End and Fixed at other End

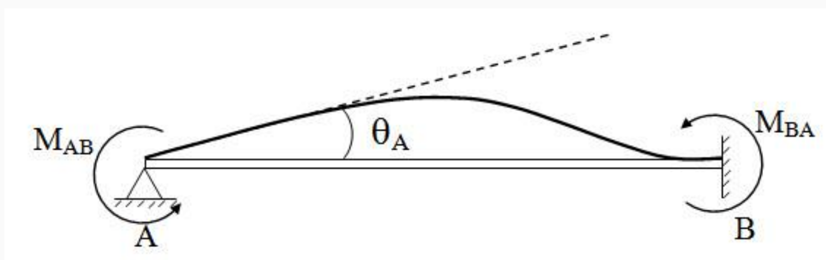


Fig. 15.3.

Slope deflection equation at A and B (q_B = 0, d = 0),

$${M_{AB}} = {{4EI} \over L}{\theta _A}$$
(15.3)

$${M_{BA}} = {{2EI} \over L}{\theta _A}$$
(15.4)

From equations (3) and (4), we have,

$${M_{BA}} = {1 \over 2}{M_{AB}}$$.
(15.5)

Absolute Stiffness $$k = {{4EI} \over L}$$

15.1.3 Several members meeting at a joint

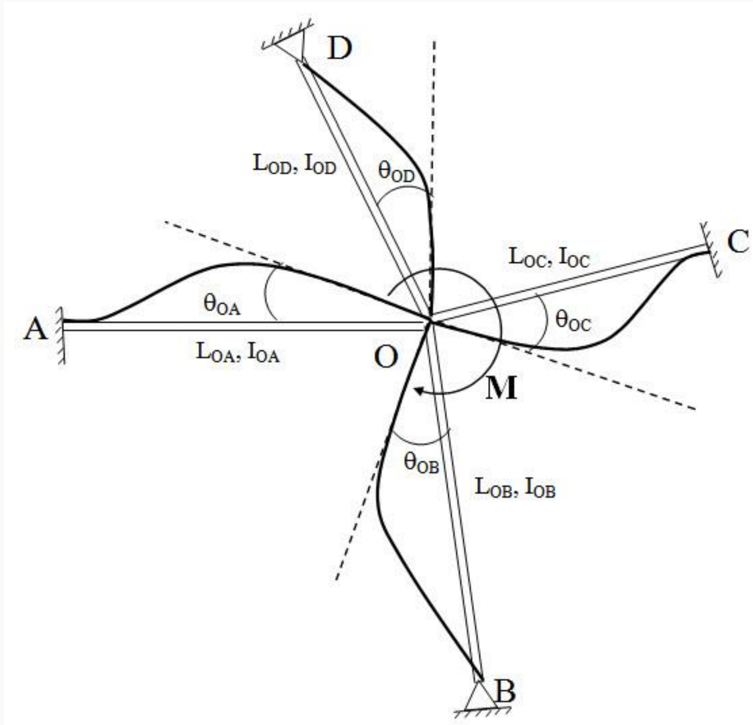


Fig. 15.4.

Figure 15.4 shows, four members (for illustration purpose only four members are taken, but the theory is applicable for any number of members), AO, BO, CO and DO meeting at O. L_{OA}, L_{OB}, L_{OC}, and L_{OD} are the length and I_{OA}, I_{OB}, I_{OC}, and I_{OD} are the second moment of area of the respective members. Support A, C are fixed and B, D are hinged. An external moment M is applied at O. The moment M will be distributed among all the members meeting at O. Suppose M_{OA}, M_{OB}, M_{OC}, and M_{OD} are the corresponding distribution.

Compatibility condition at O,

$$ {\theta _{OA}} = {\theta _{OB}} = {\theta _{OC}} = {\theta _{OD}} = \theta \qquad (15.6)$$

Equilibrium condition at O,

$$ {M_{OA}} = {M_{OB}} = {M_{OC}} = {M_{OD}} = M \qquad (15.7)$$

Now from the previous tow cases, we may express M_{OA}, M_{OB}, M_{OC}, and M_{OD} as,

$$ {M_{OA}} = {{4E{I_{OA}}} \over {{L_{OA}}}}\theta={k_{OA}}\theta \qquad (15.8)$$

$$ {M_{OB}} = {{3E{I_{OB}}} \over {{L_{OB}}}}\theta={k_{OB}}\theta \qquad (15.9)$$

From Equations (8a) – (8b),

$$M_{OA}:M_{OB}:M_{OC}:M_{OD}::k_{OA}:k_{OB}:k_{OC}:k_{OD}$$
(15.30)

From equations (7) and (9),

$$M_{OA} = {{k_{OA}} \over {k_{OA} + k_{OB} + k_{OC} + k_{OD}}}M = {{k_{OA}} \over {\sum k}}M$$

$$M_{OA} = {{k_{OB}} \over {k_{OA} + k_{OB} + k_{OC} + k_{OD}}}M = {{k_{OB}} \over {\sum k}}M$$

$$M_{OA} = {{k_{OC}} \over {k_{OA} + k_{OB} + k_{OC} + k_{OD}}}M = {{k_{OC}} \over {\sum k}}M$$

$$M_{OA} = {{k_{OD}} \over {k_{OA} + k_{OB} + k_{OC} + k_{OD}}}M = {{k_{OD}} \over {\sum k}}M$$

Therefore, moment acting at a joint will be divided amongst the connecting members in proportion to their stiffness.

The factors ${{k_{OA}} \over {\sum k}}$, ${{k_{OB}} \over {\sum k}}$, ${{k_{OC}} \over {\sum k}}$, and ${{k_{OD}} \over {\sum k}}$ are called distribution factor (DF) and moments M_{OA}, M_{OB}, M_{OC}, and M_{OD} are called distributed moments.

15.1.4 Carry Over Factor

Consider a fixed beam AB as shown bellow. Suppose the rotational constraint of joint A is released and a balancing moment M_{AB} is applied at A. Then M_{AB} will cause a moment M_{BA} at B.

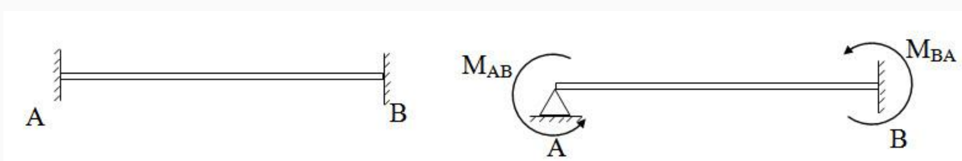


Fig. 15.5.

The carry over factor is defined as,

$$C_{AB} = {{M_{BA}} \over {M_{AB}}}$$

From 15.1.2 we have,

$$C_{AB} = {{M_{BA}} \over {M_{AB}}} = {1 \over 2}$$

Here, M_{BA} is called caried over momnet at B due to M_{AB} at A.

LESSON 16. Displacement Method: Moment Distribution Method – 2

16.1 Development of Moment Distribution Method

In this lesson, the generic procedure of the Moment Distribution Method is illustrated through an example. Consider a two span continuous beam ABC as shwon in Figure 16.1.

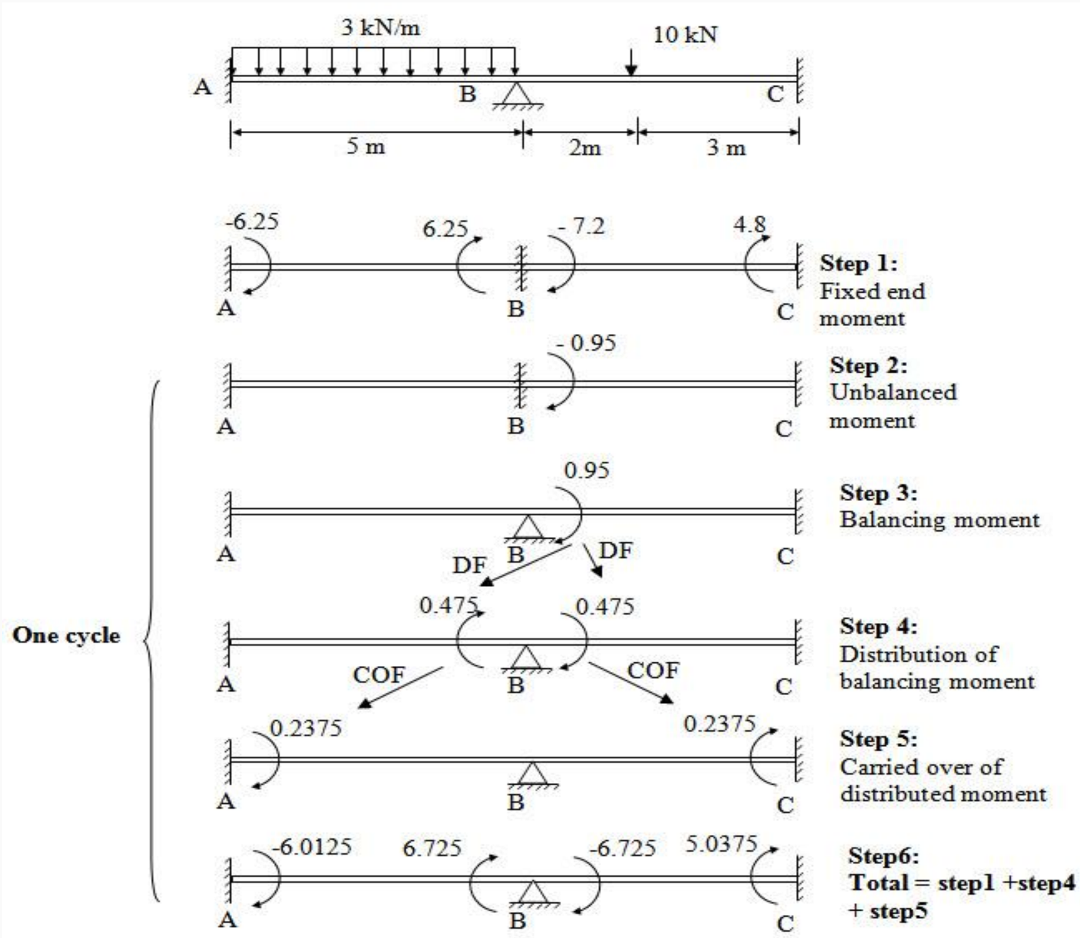


Fig.16.1.

The steps involved in moment distribution method are described bellow and also depicted in Figure 16.1.

Step 1: Fixed end Moments

First assume each span of the continuous beam as fixed beam and calculate the corresponding fixed end moments.

$M_{FAB}=-{{3 \times {5^2}} \over {12}}=-6.25{\rm{ kNm}}$; $M_{FBA} = {{3 \times {5^2}} \over {12}} = 6.25{\rm{ kNm}}$

$M_{FBC}=-{{10 \times 2 \times {3^2}} \over {{5^2}}}=-7.2{\rm{ kNm}}$; $M_{FCB}=-{{10 \times 3 \times {2^2}} \over {{5^2}}}=4.8{\rm{ kNm}}$

Assume $M_{AB}=M_{FAB}=-6.25{\rm{ kNm}}$, $M_{BA}=M_{FBA}=6.25{\rm{ kNm}}$, $M_{BC}=M_{FBC}=-7.2{\rm{ kNm}}$ AND $M_{CB}=M_{FCB}=4.8{\rm{ kNm}}$

Step 2: Unbalnced moment

Equilibriuam condition at B is $M_{BA} + M_{BC} = 0$. However initial assumption of M_{BA} and M_{BC} gives,

$M_{BA} + M_{BC} = 6.25 - 7.2 =-0.95$

Therefore at joint B there is an unbalanced moment of - 0.95 kNm.

Since ends A and C are fixed, there is no unbalanced moment at A and C.

Step 3: Balancing moment

Apply a balancing moment of 0.95 kNm at B.

Step 4: Distribution of balancing moment

Applied balancing momnet at B is then distributed among the connecting member i.e., BA and BC in porportion to their stiffness. The distribued moments are determined by multiplying the unbalanced moment by the distribution factor of the respective member.

From lesson 15.1.3 we have,.

${k_{BA}} = {{4E{I_{BA}}} \over {{L_{BA}}}} = {{4EI} \over 5}$ and ${k_{BC}} = {{4E{I_{BC}}} \over {{L_{BC}}}} = {{4EI} \over 5}$

Distribution factors for BA and BC are,

$$DF_{BA} = \frac{k_{BA}}{k_{BA} + k_{BC}} = \frac{4EI/5}{4EI/5 + 4EI/5} = \frac{1}{2}$$

$$DF_{BA} = \frac{k_{BC}}{k_{BA} + k_{BC}} = \frac{4EI/5}{4EI/5 + 4EI/5} = \frac{1}{2}$$

Therefore, distributed moment for BA and BC will be $0.5 \times 0.95 = 0.475{\rm{ kNm}}$.

Step 5: Carry over of distributed moment

For member AB, distributed moment at B will cause carried over moment at A. Simiarlary for member BC, distributed moment at B will cause carried over moment at C. These carried over moments are determined by multiplying the distributed moment by the carried over factor of the respective member.

$${C_{BA}} = {1 \over 2}$$ and $${C_{BC}} = {1 \over 2}$$

Therefore carried over moments are,

At A $0.5 \times 0.475 = 0.2375{\rm{ kNm}}$

At C $0.5 \times 0.475 = 0.2375{\rm{ kNm}}$

Step 6: Total moment at the end of cycle

Determine the total moment each end.

Total moment = Fixed end moment (Step 1) + Distributed unbalanced moment (Step 4) + Carried over moment (Step 5)

$$M{}_{AB}=-6.25 + 0.2375=-6.0125{\rm{ kNm}}$$

$$M{}_{BA}=6.250 + 0.475=6.725{\rm{ kNm}}$$

$$M{}_{BC}=-7.20 + 0.475=-6.725{\rm{ kNm}}$$

$$M{}_{AB}=4.8 + 0.2375=5.0375{\rm{ kNm}}$$

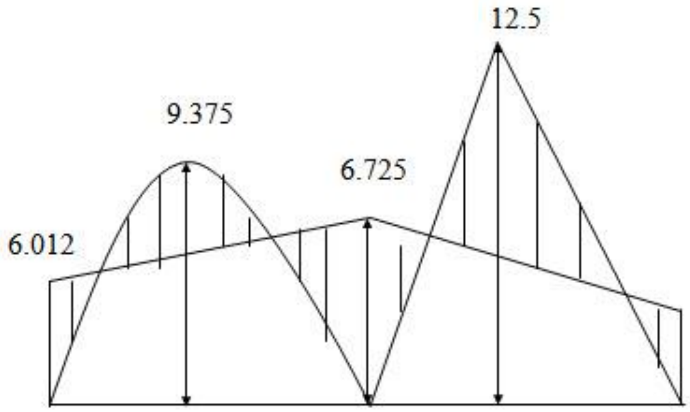
Step 2 to Step 6 is a complete cycle. At the end of each cycle check whether equilibrium equations are satisfed.

$M{}_{BA} + M{}_{BC} = 6.725 - 6.725 = 0$ => Equilibrium equation at B is satisfied.

If it is found that there are joints with unbalanced moment then again go to step 2 and repeat the cycle until all the unbalanced moments become zero.

Generally the entire calculation in moment distribution method is done in a tabular form as illustrated bellow.

A (0)	(0.5)B	B (0.5)	(0) C	DF shwon in parenthesis
-6.25	6.25	-7.2	4.8	Fixed end moment (Step 1)
0	0.475	0.475	0	distributd unbalanced moment (Step 2 – 4)
0.238	0	0	0.238	Carry over moments (Step 5)
-6.012	6.725	-6.725	5.038	Final moment at the end of cycle (Step 6)



Bending moment diagram (kNm)

LESSON 17. Displacement Method: Moment Distribution Method – 3

17.1 Introduction: In this lesson the application of the Moment Distribution Method in continusous beam is illustrated via two examples.

Example 1

Draw the bending moment diagram for the following continuous beam. All spans have constant EI.

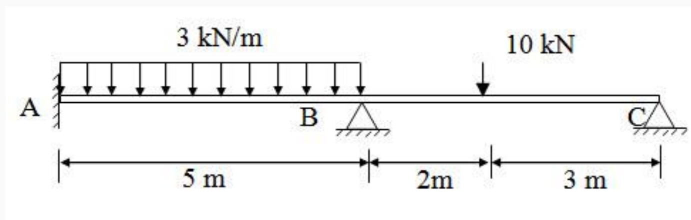


Fig. 17.1.

From lesson 15.1.3 we have,

$${k_{BA}} = {{4E{I_{BA}}} \over {{L_{BA}}}} = {{4EI} \over 5}$$ and $${k_{BC}} = {{3E{I_{BC}}} \over {{L_{BC}}}} = {{3EI} \over 5}$$

Distribution factors for BA and BC are,

$$D{F_{BA}} = {4 \over 7}$$ and $$D{F_{BC}} = {3 \over 7}$$

End A is fixed and therefore no moment will be carrid over to B from A. Carry over factors for other joints,

$${C_{BA}} = {1 \over 2}$$, $${C_{BC}} = {1 \over 2}$$, $${C_{CB}} = {1 \over 2}$$

Fixed end moments are,

$$M{}_{FAB}=-{{3 \times {5^2}} \over {12}}=-6.25{\rm{ kNm}}$$; $$M{}_{FBA} = {{3 \times {5^2}} \over {12}} = 6.25{\rm{ kNm}}$$

$$M{}_{FBC}=-{{10 \times 2 \times {3^2}} \over {{5^2}}}=-7.2{\rm{ kNm}}$$; $$M{}_{FCB} = {{10 \times 3 \times {2^2}} \over {{5^2}}} = 4.8{\rm{ kNm}}$$

Calculations are performed in the following table.

A	(4/7) B	B (3/7)	C	DF shwon in parenthesis
-6.25	6.25	-7.2	4.8	Fixed end moment (Step 1)
0	0.54	0.41	-4.8	distributd balancing moment
0.27	0	-2.4	0.21	Carry over moments
	1.37	1.03	-0.21	distributd balancing moment
0.68		-0.11	0.52	Carry over moments
	0.06	0.05	-0.52	distributd balancing moment
-5.29	8.22	-8.22	0	Final moment

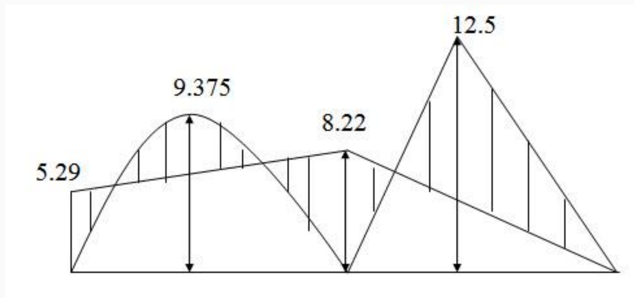


Fig. 17.2: Bending moment diagram (in kNm).

Example 2

Replace the fixed support at *A* by a hinge in the continuous beam shown in Example 1 and determine the bending moments.

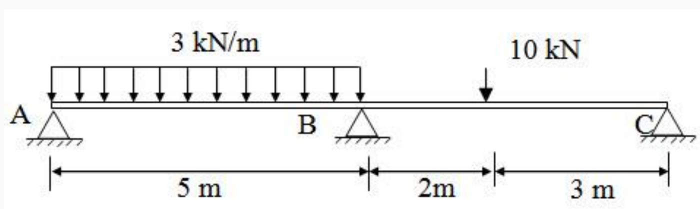


Fig. 17.3 .

From lesson 15.1.3 we have,

$$ {k_{BA}} = {{4E{I_{BA}}} \over {{L_{BA}}}} = {{4EI} \over 5} $$ and $$ {k_{BC}} = {{3E{I_{BC}}} \over {{L_{BC}}}} = {{3EI} \over 5} $$

Distribution factors for BA and BC are,

$$ D{F_{BA}} = {4 \over 7} $$ and $$ D{F_{BA}} = {3 \over 7} $$

Carry over factors,

$$ {C_{AB}} = {1 \over 2} $$, $$ {C_{BA}} = {1 \over 2} $$, $$ {C_{BC}} = {1 \over 2} $$, $$ {C_{CB}} = {1 \over 2} $$

Fixed end moments are,

$M_{FAB}=-\frac{3 \times 5^2}{12}=-6.25\,\mathrm{kNm}$; $M_{FBA} = \frac{3 \times 5^2}{12} = 6.25\,\mathrm{kNm}$

$M_{FBC}=-\frac{10 \times 2 \times 3^2}{5^2}=-7.2\,\mathrm{kNm}$; $M_{FCB}=-\frac{10 \times 3 \times 2^2}{5^2} = 4.8\,\mathrm{kNm}$

Calculations are performed in the following table.

A	(4/7) B	B (3/7)	C	DF shwon in parenthesis
-6.25	6.25	-7.2	4.8	Fixed end moment (Step 1)
6.25	0.54	0.41	-4.8	distributd balancing moment
0.27	3.12	-2.4	0.21	Carry over moments
-0.27	-0.41	-0.31	-0.21	distributd balancing moment
-0.21	-0.14	-0.11	0.16	Carry over moments
0.21	0.14	0.11	-0.16	distributd balancing moment
0.07	0.11	-0.08	0.06	Carry over moments
-0.07	-0.02	-0.01	-0.06	distributd balancing moment
0	9.59	-9.59	0	Final Moment

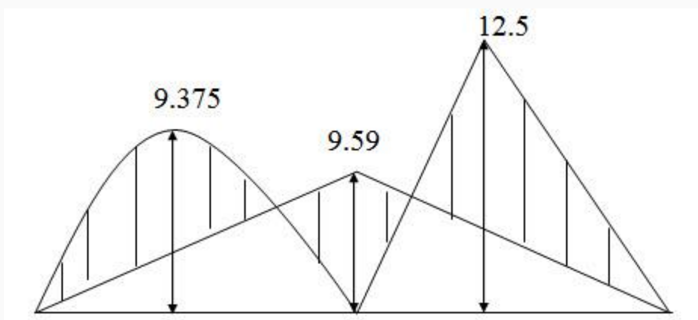


Fig. 17.4. Bending moment diagram (in kNm).

LESSON 18. Displacement Method: Moment Distribution Method – 4

18.1 Introduction : In this lesson the application of the Moment Distribution Method in frames where joint translations (side sway) are restrained is illustrated via two examples.

18.1.1 Example 1

Draw the bending moment diagram for the follwing frame. EI is constant for all members.

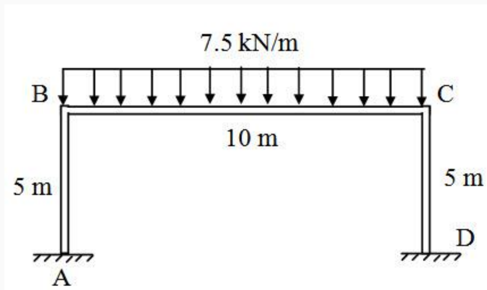


Fig. 18.1.

$${k_{BA}}={{4E{I_{BA}}} \over {{L_{BA}}}}={{4EI} \over 5}$$, $${k_{BC}}={{4E{I_{BC}}} \over {{L_{BC}}}}={{2EI} \over 5}$$ and $${k_{CD}}={{4E{I_{CD}}} \over {{L_{CD}}}}={{4EI} \over 5}$$

Distribution factors for BA and BC are,

$$D{F_{BA}}={2 \over 3}$$, $$D{F_{BC}}={1 \over 3}$$, $$D{F_{CB}}={1 \over 3}$$ and $$D{F_{CD}}={2 \over 3}$$

End A and D are fixed and therefore no moment will be carrid over to B and C from A and D respectively. Carry over factors for other joints,

$$C_{BA}={1 \over 2}$$, $$C_{BC}={1 \over 2}$$, $$C_{CB}={1 \over 2}$$ and $$C_{CD}={1 \over 2}$$

Fixed end moments are,

$$M{}_{FBC}=-{{7.5 \times {{10}^2}} \over {12}}=-62.5{\rm{kNm}}$$; $$M{}_{FCB}={{7.5 \times {{10}^2}} \over {12}}=62.5{\rm{kNm}}$$

$$M{}_{FAB} = M{}_{FBA} = M{}_{FCD} = M{}_{FDC}=0$$.

Calculations are performed in the following table.

A	B(2/3)	B(1/3)	C(1/3)	C(2/3)	D
0	0	-62.5	62.5	0	0
	41.67	20.83	-20.83	-41.67	
20.84		-10.42	10.42		-20.84
	6.95	3.47	-3.47	-6.95	
3.48		-1.74	1.74		-3.48
	1.16	0.58	-0.58	-1.16	
0.58		-0.29	0.29		-0.58
	0.19	0.10	-0.10	-0.19	
0.2		-0.05	0.05		-0.2
	0.03	0.02	-0.02	-0.03	
25.1	**50**	**-50**	**50**	**-50**	**25.1**

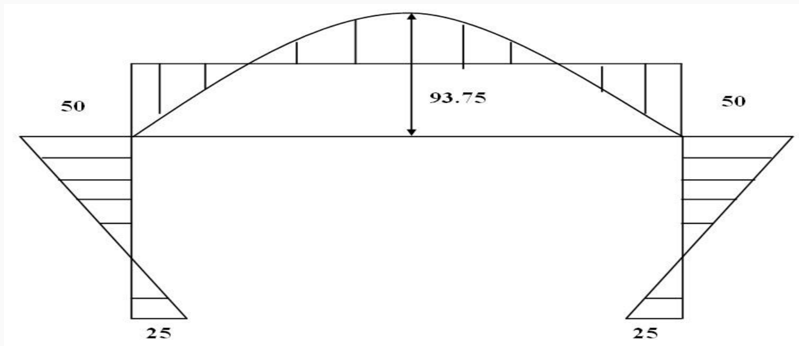


Fig. 18.2. Bending moment diagram (in kNm).

Example 2

Draw the bending moment diagram for the following rigid frame.

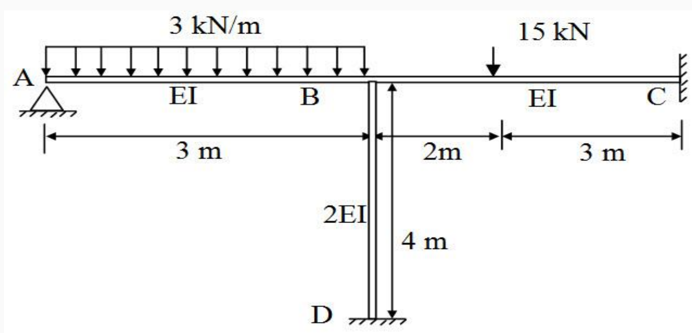


Fig. 18.3.

$k_{BA} = \frac{3EI_{BA}}{L_{BA}} = \frac{3EI}{3} = EI$, $k_{BC} = \frac{4EI_{BC}}{L_{BC}} = \frac{4EI}{5}$ and $k_{BD} = \frac{4EI_{BC}}{L_{BC}} = \frac{8EI}{4} = 2EI$

Distribution factors for BA and BC are,

$DF_{BA} = \frac{5}{19}$, $DF_{BC} = \frac{4}{19}$ and $DF_{BD} = \frac{10}{19}$

End C and D are fixed and therefore no moment will be carrid over to B from C and D. Carry over factors for other joints,

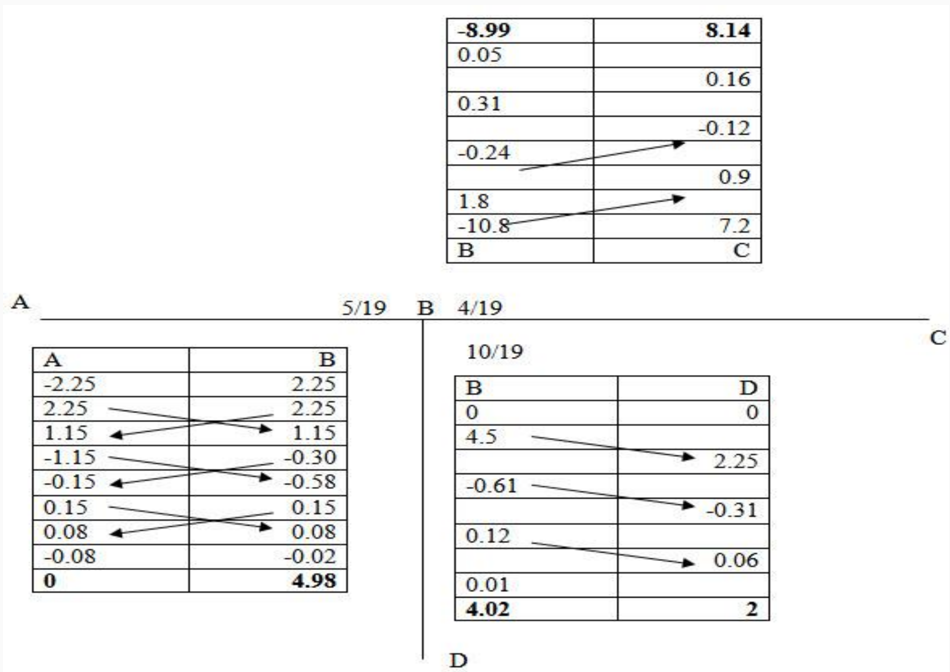
$C_{AB} = {1 \over 2}$, $C_{BA} = {1 \over 2}$, $C_{BC} = {1 \over 2}$, $C_{BD} = {1 \over 2}$

Fixed end moments are,

$M_{FAB}=-{{3 \times {3^2}} \over {12}}=-2.25{\rm{ kNm}}$; $M_{FBA} = {{3 \times {3^2}} \over {12}} = 2.25{\rm{ kNm}}$

$M_{FBC}=-{{15 \times 2 \times {3^2}} \over {{5^2}}}=-10.8{\rm{ kNm}}$; $M_{FCB} = {{15 \times 3 \times {2^2}} \over {{5^2}}} = 7.2{\rm{ kNm}}$

Calculations are performed in the following table.



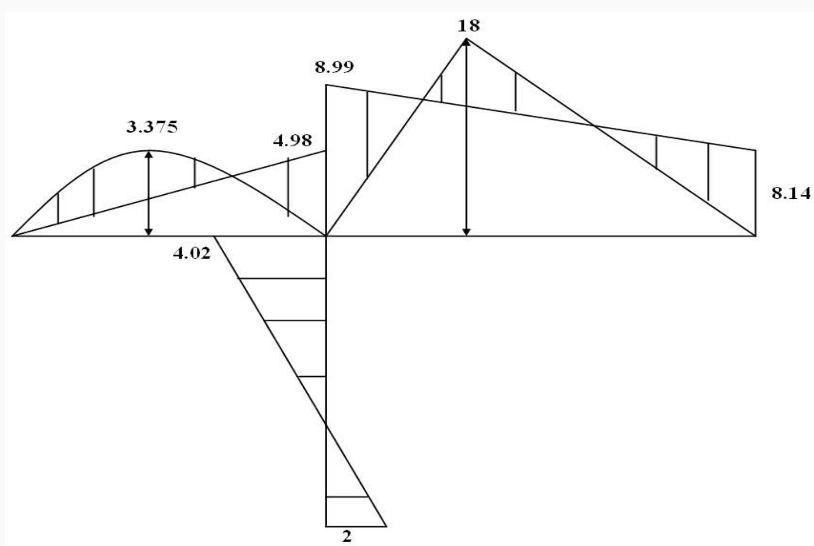


Fig. 18.4. Bending moment diagram.

LESSON 19. Displacement Method: Moment Distribution Method – 5

19.1 Introduction : In this lesson we will learn how to apply the Moment Distribution Method in frames where joint translation (sidesway) is allowed. The procedure is illustrated via the following example.

Example 1

Draw the bending moment diagram for the follwing frame. EI is constant for all members.

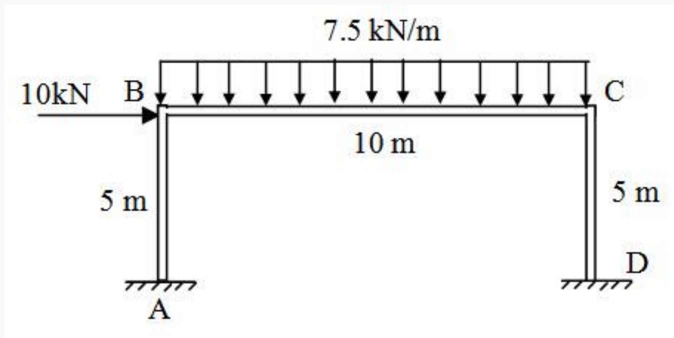


Fig. 19.1.

The above problem can be represented as the superposition of two sub-problems as shwon in Figure 18.2.

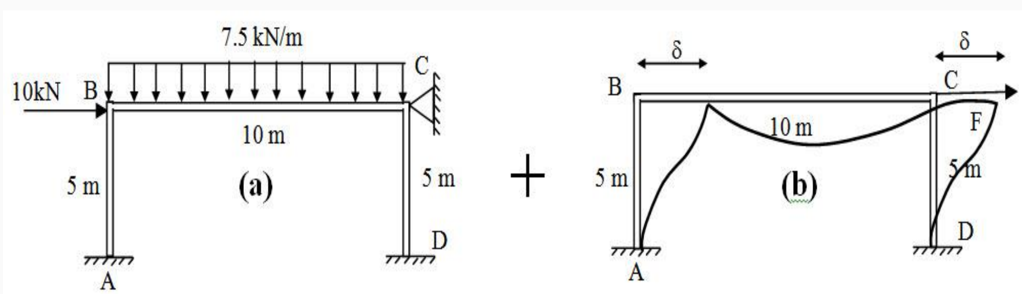


Fig. 19.2.

In the first sub-problem (Figure 18.2a), sidesway is prevented and therefore can be analyzed by moment distribtuin mehod as discussed in the previous lesson. Suppose C_x be the horizontal reaction at C. Now in the second sub-problem apply an arbitrary sidesway d as shown in Figure 18.2b. Calculate horizontal force F due to arbitrary sidesway d. Then the beam end moment in the orizinal structure is obtained as,

$$M_{original} = M_{sub - problem1} + kM_{sub - problem2}$$

Where, $k = {C_x \over F}$

Step1: Solution of sub-problem 1 (sidesway restrained)

${k_{BA}}={{4E{I_{BA}}} \over {{L_{BA}}}}={{4EI} \over 5}$, ${k_{BC}}={{4E{I_{BC}}} \over {{L_{BC}}}}={{2EI} \over 5}$ and ${k_{CD}}={{4E{I_{CD}}} \over {{L_{CD}}}} = {{4EI} \over 5}$

Distribution factors are,

$D{F_{BA}}={2 \over 3}$, $D{F_{BC}}={1 \over 3}$, $D{F_{CB}}={1 \over 3}$ and $D{F_{CD}}={2 \over 3}$

End A is fixed and therefore no moment will be carrid over to B from A. Carry over factors for other joints,

$C_{BA}={1 \over 2}$, $C_{BC}={1 \over 2}$, $C_{CB}={1 \over 2}$ and $C_{CD}={1 \over 2}$

Fixed end moments are,
$M{}_{FBC}=-{{7.5 \times {{10}^2}} \over {12}}=-62.5{\rm{kNm}}$; $M{}_{FCB}=-{{7.5 \times {{10}^2}} \over {12}}=62.5{\rm{kNm}}$

$M{}_{FAB} = M{}_{FBA} = M{}_{FCD} = M{}_{FDC}=0$

Calculations are performed in the following Table.

A	B(2/3)	B(1/3)	C(1/3)	C(2/3)	D
0	0	-62.5	62.5	0	0
	41.67	20.83	-20.83	-41.67	
20.84		-10.42	10.42		-20.84
	6.95	3.47	-3.47	-6.95	
3.48		-1.74	1.74		-3.48
	1.16	0.58	-0.58	-1.16	
0.58		-0.29	0.29		-0.58
	0.19	0.10	-0.10	-0.19	
0.2		-0.05	0.05		-0.2
	0.03	0.02	-0.02	-0.03	
25	**50**	**-50**	**50**	**-50**	**-25**

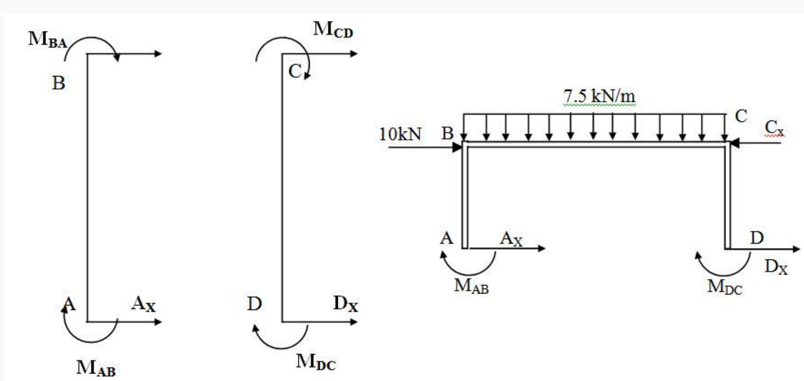


Fig. 19.3.

From the free body diagram of AB and CD (Figure 18.3),

$$\boldsymbol{{A_x}={{{M_{AB}} + {M_{BA}}} \over {{L_{AB}}}}={{25 + 50} \over 5}=75{\rm{ kN}}}$$

$$\boldsymbol{{D_x}={{{M_{CD}} + {M_{DC}}} \over {{L_{CD}}}}={{ - 25 - 50} \over 5}=75{\rm{ kN}}}$$

Also,

$${C_x}=10 + {A_x} + {B_x}=10{\rm{ - 75}} + 7{\rm{5}}={\rm{1}}0{\rm{ kN}}$$

Step 2: Solution of sub-problem 2 (for an arbitrary sidesway)

Let an arbitray sway $\delta = 100 / EI$ is applied at C. Fixed end moments due to δ are,

$$M{}_{FAB}=M{}_{FBA}=-{{6EI\delta } \over {{L_{AB}}}}=-{{6EI} \over 5} \times {{100} \over {EI}}=-120{\rm{ kNm}}$$

$$M{}_{FCD}=M{}_{FDC}=-{{6EI\delta } \over {{L_{CD}}}}=-{{6EI} \over 5} \times {{100} \over {EI}}=-120{\rm{ kNm}}$$

Moment distribution calculations,

A	B(2/3)	B(1/3)	C(1/3)	C(2/3)	D
-120	-120	0	0	-120	-120
	80	40	40	80	
40		20	20		40
	-13.33	-6.67	-6.67	-13.33	
-6.67		-3.34	-3.34		-6.67
	2.22	1.11	1.11	2.22	
1.11		0.56	0.56		1.11
	-0.37	-0.19	-0.19	-0.37	
-0.19		-0.1	-0.1		-0.19
	0.03	0.07	0.07	0.03	
-85.75	**-51.45**	**51.45**	**51.45**	**-51.45**	**-85.75**

Horizontal reactions at A and D are,

$${A_x}={{{M_{AB}} + {M_{BA}}} \over {{L_{AB}}}}={{ - 85.75 - 51.45} \over 5}=-27.44{\rm{ kN}}$$

$${D_x}={{{M_{CD}} + {M_{DC}}} \over {{L_{CD}}}}={{ - 85.75 - 51.45} \over 5}=-27.44{\rm{ kN}}$$

Therefore, total horizontal force due to lateral sway $\delta = 100 / EI$ an be ditermined by the following equilibrium equation,

$$F + {A_x} + {D_x}=0 \Rightarrow F = 54.88{\rm{ kN}}$$

Hence, $$k = {C_x}/F=10/54.88$$

Final moment now can be obtained as,

$$M_{original}=M_{sub - problem1} + kM_{sub - problem2}$$

Therefore,

$$M_{AB} = M_{AB1} + kM_{AB2}=25 + \left({10/54.88} \right) \times (- 85.75) = 9.375{\rm{ kNm}}$$

$$M_{BA} = M_{BA1} + kM_{BA2}=50 + \left({10/54.88} \right) \times (- 51.45)=40.625{\rm{ kNm}}$$

$$M_{BC}=M_{BC1} + kM_{BC2}=-50 + \left({10/54.88} \right) \times (51.45)=-40.625{\rm{ kNm}}$$

$$M_{CB}=M_{CB1} + kM_{CB2}=50 + \left({10/54.88} \right) \times (51.45)=59.375{\rm{ kNm}}$$

$$M_{CD}=M_{CD1} + kM_{CD2}=-50 + \left({10/54.88} \right) \times (- 51.45)=-59.375{\rm{ kNm}}$$

$$M_{DC}=M_{DC1} + kM_{DC2}=-25 + \left({10/54.88} \right) \times (- 85.75)=-40.625{\rm{ kNm}}$$

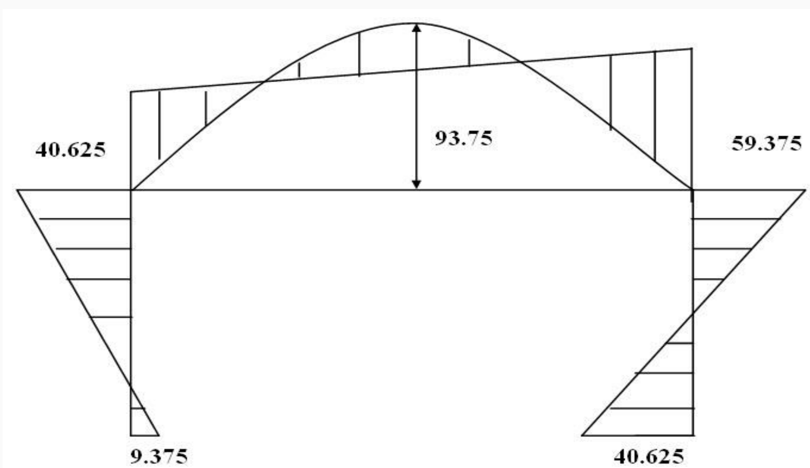


Figure 19.4: Bending moment Diagram.

LESSON 20. Approximate analysis of fixed and continuous beams - 1

Sometimes the configuration and complexity of the structures may be such that the exact method of analysis is either not available or unfeasible to apply. In such cases, an approximate methods my be constituted based on some simple and yet reasonable assumptions. In this lesson and the next lesson we will determine approximate solutions for some common types of statically indeterminate structures. .

20.1 Portal Method for Frames Subjected to Lateral Load

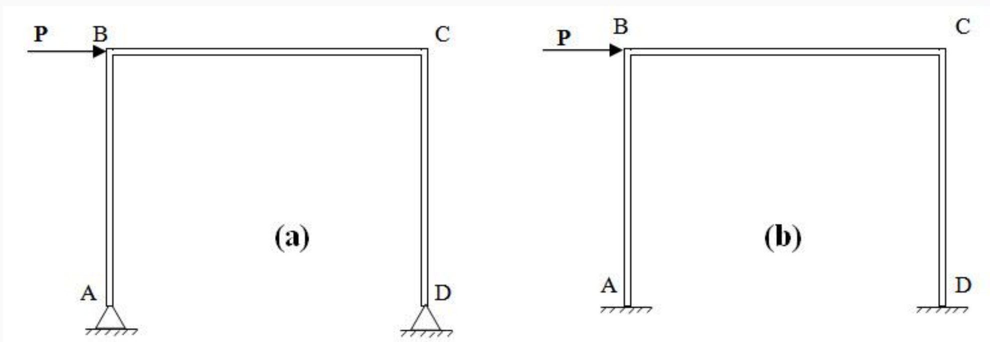


Fig. 20.1.

Consider a portal frame as shown in Fgiure 20.1a. The unknown reaction components are A_x, A_y, D_x and D_y which can not be determined by three equilibrium conditions. Therefore the structure is statically indeterminate with indeterminacy one. Analysis of this structure through the other methods (for statically indterminate structures) shows that . Now while analysing such frame if it is assumed *a priori* that the horizontal reactions at both legs are same i.e., , the number of unknown reaction components is reduced to three viz. A_y, D_y and H which can be determined through three equations of statics.

Consider another case of the same portal frame but legs are fixed as shwon in Figure 20.2b. The six unknown reaction components are A_x, A_y, M_A, D_x, D_y and M_D,. Hence the structure is indeterminate with indeterminacy three. In order to have a complete solution of the structure three assumptions must be made. Following are the observations based on analysis of this structure through the other methods (for statically indterminate structures),

- Horizontal reactions at both legs are same i.e., .
- Near the centre of each leg bending moment changes its sign (Figure 20.2a) and hence has zero value. These points are called points of inflection.

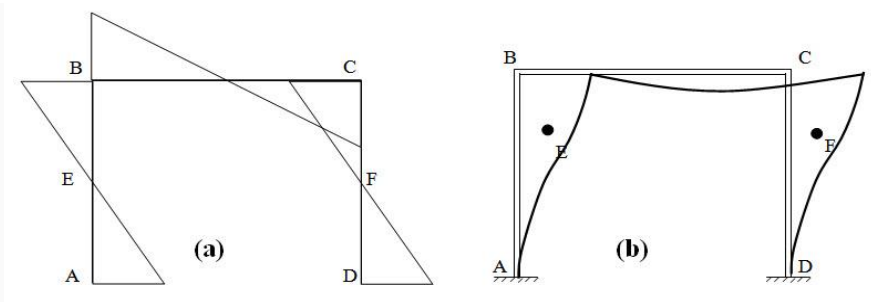


Fig.20.2

Based on the first observation it is assumed $A_x=D_x$. This reduces the degree of indeterminacy by one.

Based on the second observation it is assumed that there is a point of inflection at the centre of each leg. This is equivalent to assuming that hinges exist at the centre of each leg i.e, at E and F (Figure 20.2b). Each intermediate hinge gives one additional equation and therefore reduces indeterminacy by one.

Combining above two assumptions, now the degree of indeterminacy of the structure is,

3 – 1 (assumption 1) – 2´ 1(assumption 2) = 0

Therefore the structure now becomes determinate. The above method is illustrated in the following example.

Example 1: Single bay and single storey

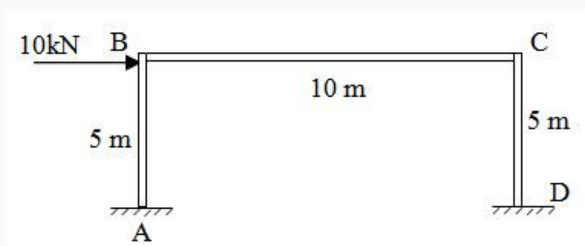


Fig. 20.3.

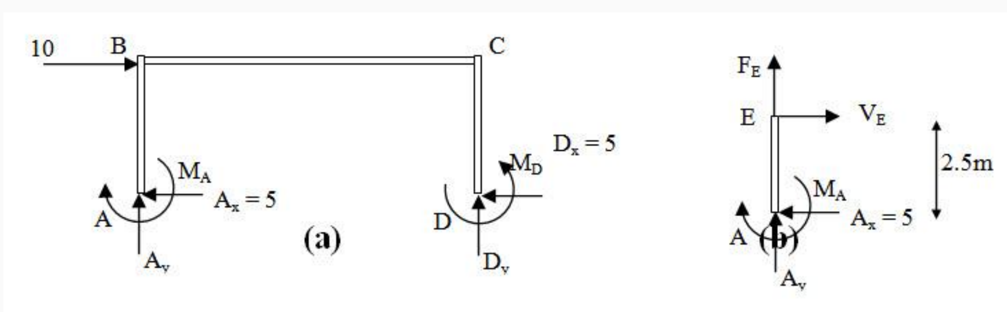


Fig. 20.4.

Assumption 1 gives,

$${A_x} = {D_x} = 10/2 = 5{\rm{ kN}}$$

According to assumption 2, moment at E is zero,

$$\sum {{M_A}}=0 \Rightarrow 2.5 \times 5 + {M_A}=0 \Rightarrow {M_A}=-12.5{\rm{ kNm}}$$ [Figure 20.4b]

Similarly $${M_D}=12.5{\rm{ kNm}}$$

Taking moment about A,

$$\sum {{M_A}}=0 \Rightarrow {M_A} - {M_D} + 10 \times 5 - 10 \times {D_y} = 0 \Rightarrow - 12.5 - 12.5 + 50 - 10 \times {D_y}= 0$$

$$\Rightarrow {D_y}=2.5{\rm{ kN}}$$

$$\sum {{F_y}}=0 \Rightarrow {A_y} + {D_y}=0 \Rightarrow {A_y}=-2.5{\rm{ kN}}$$

Final solution,

$${A_x}={D_x}=5{\rm{ kN}}$$; $$-{A_y} = {D_y}=2.5{\rm{ kN}}$$ and $$-{M_A}={M_D}=12.5{\rm{ kNm}}$$

Example 2: Multiple bays and multiple storeys

In practical situations building frames constitute multiple bays and multiple stories as shown in Figure 20.5. Here we will learn how to analyse such frames using portal method.

Assumptions

- Point of inflection exist at the mid-point of each girder and column.
- The total horizontal shear at each storey is divided between the columns of that storey such that the interior column carries twice the shear of exterior column.

The last assumption is arrived at by considering each bay as a portal as shown in Figure 20.6. Interior columns are composed of two columns and thus carries twice the shear of exterior column.

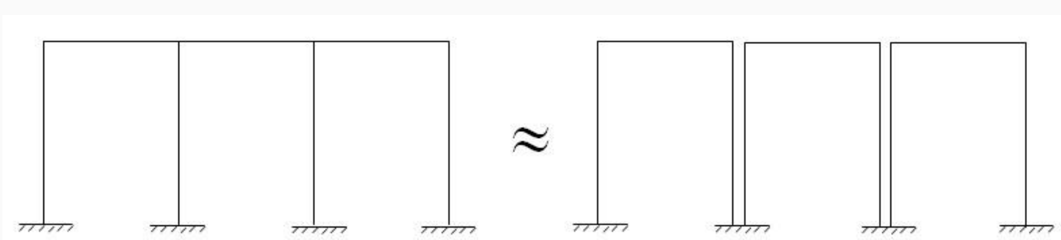


Fig. 20.5.

The portal method is illustrated via the following example.

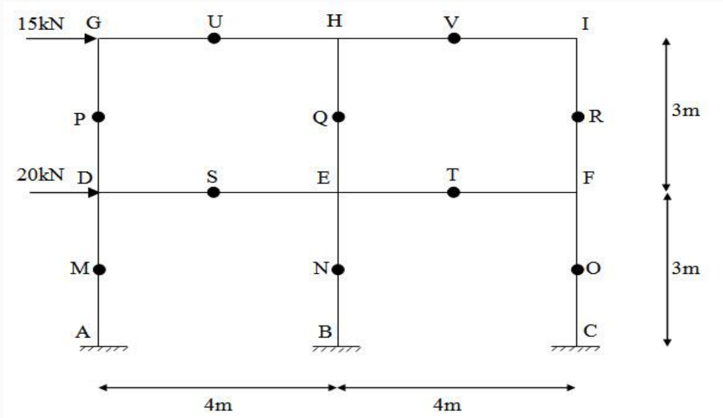


Fig. 20.6.

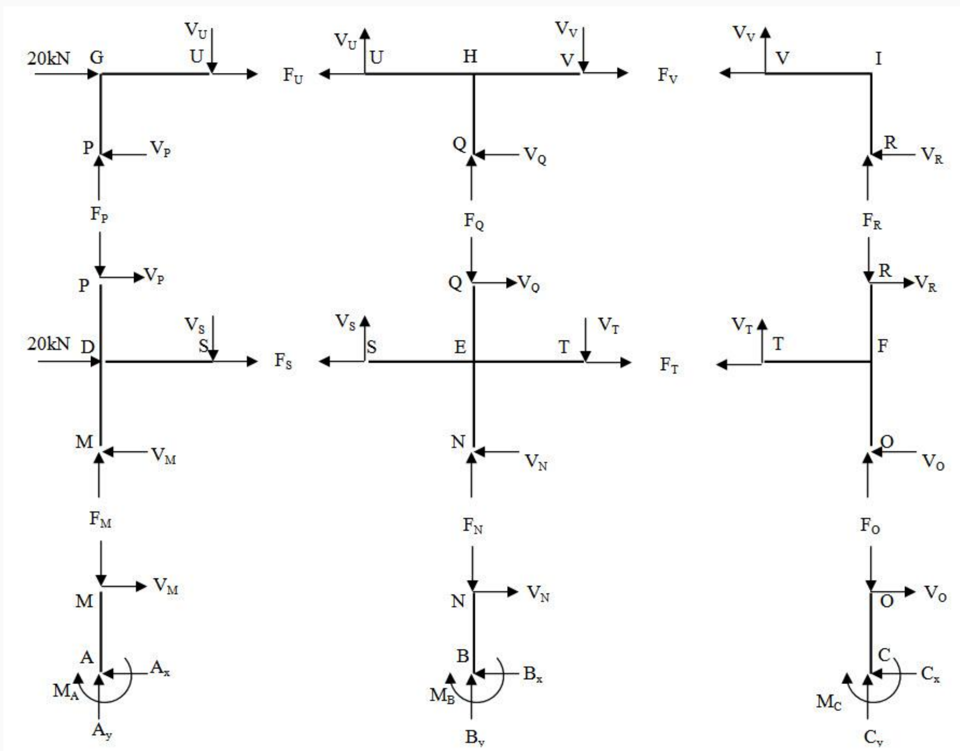


Fig. 20.7

Free body diagrams of different parts of the structures are shown in Fgiure 20.7.

From assumption 2, $2{A_x}= 2{C_x}={B_x}$ and $2{V_P}=2{V_R}={V_Q}$

For first storey, ${A_x} + {B_x} + {C_x}=20 + 20$ $\Rightarrow {A_x}=10{\rm{ kN}}$, ${B_x}=20{\rm{ kN}}$ and ${C_x}=10{\rm{ kN}}$

For second storey, ${V_A} + {V_C} + {V_B}=20$ $\Rightarrow {V_P}=5{\rm{ kN}}$, ${V_Q}=10{\rm{ kN}}$ and ${V_R}=5{\rm{ kN}}$

$\sum {{M_M}}=0 \Rightarrow {M_A} + 1.5 \times {V_A}=0 \Rightarrow {M_A}=-15{\rm{ kNm}}$

$$\sum {{M_N}}=0 \Rightarrow {M_A} + 1.5 \times {V_A}=0 \Rightarrow {M_A}=-15{\rm{ kNm}}$$

$$\sum {{M_O}}=0 \Rightarrow {M_A} + 1.5 \times {V_A}=0 \Rightarrow {M_A}=-15{\rm{ kNm}}$$

$$\sum {{M_U}}=0 \Rightarrow 1.5 \times {V_P} + 2 \times {F_P}=0 \Rightarrow {F_P}=-3.75{\rm{ kN}}$$

$$\sum {{M_V}}=0 \Rightarrow 1.5 \times {V_R} - 2 \times {F_R}=0 \Rightarrow {F_R}=3.75{\rm{ kN}}$$

$${F_P} + {F_Q} + {F_R}=0 \Rightarrow {F_Q}=0$$

$${A_y}={F_M}={F_P}=-3.75{\rm{ kN}}$$

$${B_y}={F_N}={F_Q}=0$$

$${C_y}={F_O}={F_R}=3.75{\rm{ kN}}$$

An illustration of determining unknown support reactions are given above. Similarly by considering free body diagram of different parts as shown in Figure 20.7 and applying equlibrium conditions, member forces can also be determined.

LESSON 21. Approximate Analysis of Fixed and Continuous Beams – 2

21.1 Introduction : In this lesson we will learn another method, called the Cantilever method for approximate analysis of rigid frames subjected to lateral load. Similar to the Portal method as discussed in the previous lecture, the *Cantilever method* is also based on few assumptions. These assumptions are as follows,

- There is a point of inflection at the mid-point of each girder and column.
- The axial load in each column of a storey is proportional to the horizontal distance of the that column from the centre of gravity of all the column of the storey under consideration.

The above two assumptions give additional equations required for solving unknown reaction components. The method is illustrated via the following examples.

Examples

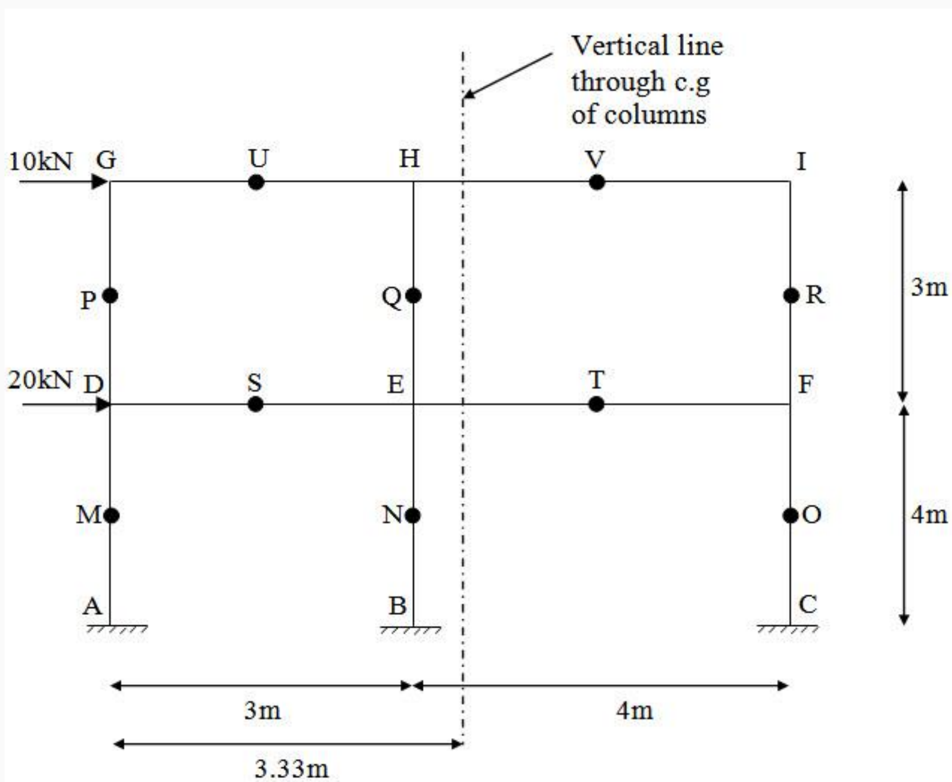


Fig. 21.1.

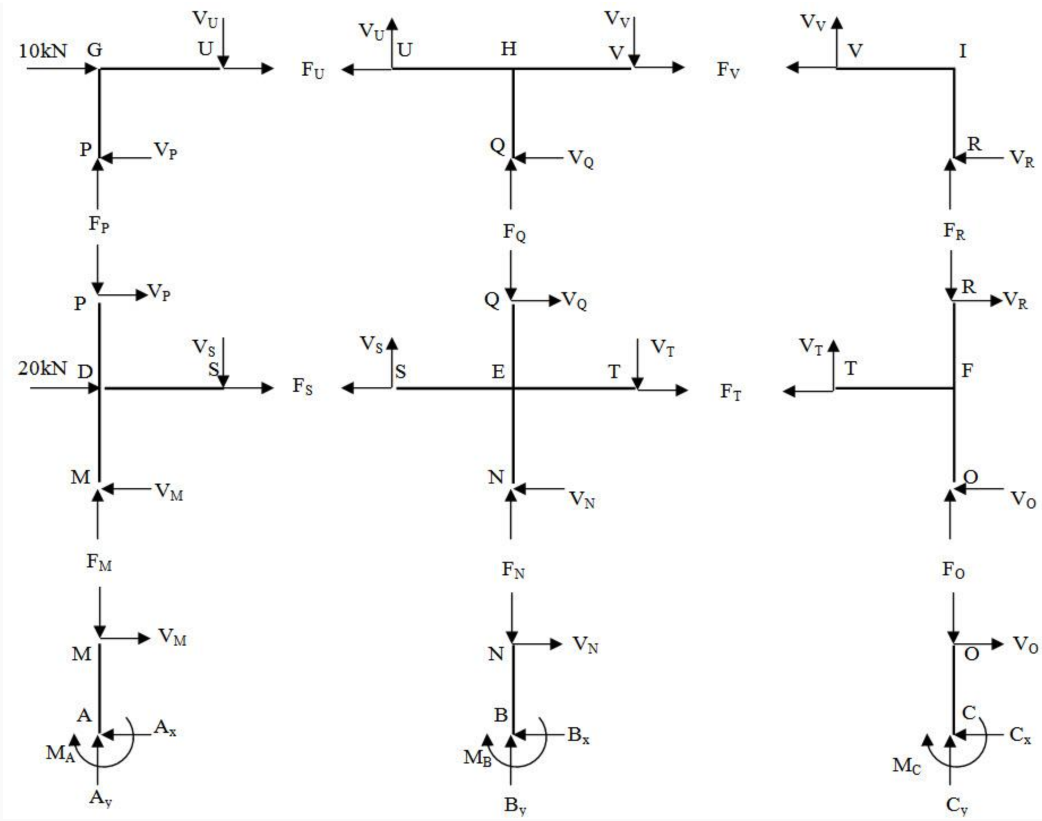


Fig. 21.2.

Free body diagrams of different parts of the structures are shown in Fgiure 20.7. Assuming all columns have the same cross-sectional area, the centre of gravity of the columns for each storey is, $(3 + 7)/3 = 3.33{\rm{m}}$ from A. Therefore, ${F_M}$, ${F_N}$ and ${F_O}$ are related as,

${F_N} = {{3.33 - 3} \over {3.33}}{F_M} = 0.1{F_M}$ and ${F_O} = {{3.33 - 7} \over {3.33}}{F_M} =-1.10{F_M}$

Taking moment about O of all the forces acting on the part above the horizontal plane passing through the points of inflection of the columns of the first storey,

we have,

$$\sum {{M_O}}=0 \Rightarrow 7 \times {F_M} + 4 \times {F_N} + 2 \times 20 + 5 \times 10 = 0$$

$$\Rightarrow 7 \times {F_M} + 4 \times 0.1{F_M} =-90 \Rightarrow-12.162{\rm{kN}}$$

Hence,

${F_N}=0.1{F_M}=-1.216{\rm{kN}}$ and ${F_O} =-1.10{F_M}=13.378{\rm{kN}}$

${F_P}$, ${F_Q}$ and ${F_R}$ will also follow the similar proportion. Therefore,

${F_Q}=0.1{F_P}$ and ${F_R}=-1.10{F_P}$

Now taking moment about R, we have,

$$\sum {{M_R}}=0 \Rightarrow 7 \times {F_P} + 4 \times {F_Q} + 2 \times 10 = 0$$

$$\Rightarrow 7 \times {F_P} + 4 \times 0.1 \times {F_P}=-20 \Rightarrow {F_P}=-2.70{\rm{kN}}$$

$${F_Q} = 0.1{F_P}=-0.27{\rm{kN}}$$ and $${F_R}=-1.10{F_P}=2.97{\rm{kN}}$$

$$\sum {{M_U}}=0 \Rightarrow 1.5 \times {V_P} + 1.5 \times {F_P} = 0 \Rightarrow {V_P} = 2.7{\rm{kN}}$$

$$\sum {{M_V}}=0 \Rightarrow 1.5 \times {V_R}-2 \times {F_R}=0 \Rightarrow {V_R}=3.96{\rm{ kN}}$$

$${V_P} + {V_Q} + {V_R}=10 \Rightarrow {V_Q}=3.07{\rm{kN}}$$

$${F_M} - {F_P} - {V_S}=0 \Rightarrow {V_S}=-9.462{\rm{kN}}$$

$$\sum {{M_D}}=1.5 \times {V_P} + 2 \times {V_M} + 1.5 \times {V_S}=0 \Rightarrow {V_M}=5.07{\rm{ kN}}$$

$${A_x}={V_M}=5.07{\rm{ kN}}$$

$${A_y}={F_M}=-12.162{\rm{ kN}}$$

$$\sum {{M_M}}=0 \Rightarrow {M_A} + 2 \times {A_x}=0 \Rightarrow {M_A}=-10.14{\rm{ kNm}}$$

An illustration of determining unknown support reactions at A is given above. Similarly by considering free body diagram of different parts as shown in Figure 21.2 and applying equlibrium conditions, other support reactions and member forces can also be determined.

MODULE 3. Columns and Struts

LESSON 22. Columns and Struts

22.1 Buckling and Stability

Lateral bending of a straight slender member from its longitudinal position due to compression is referred to as buckling. Buckling is encountered in many practical columns. Load at which buckling occurs depends on many factors such as material strength, geometry of the column, end conditions etc. In this module we will learn different methods for determining buckling load of slender columns.

22.2 Euler Load for Columns with Pinned End

Assumptions

- Member is prismatic and perfectly straight.
- The material is homogeneous and linear elastic.
- One end of the member is hinged and the other is restrained against horizontal movement as shown in Figure 22.2a.
- The compressive load is acting along the longitudinal axis of the member.
- Lateral deformation of the member is small.

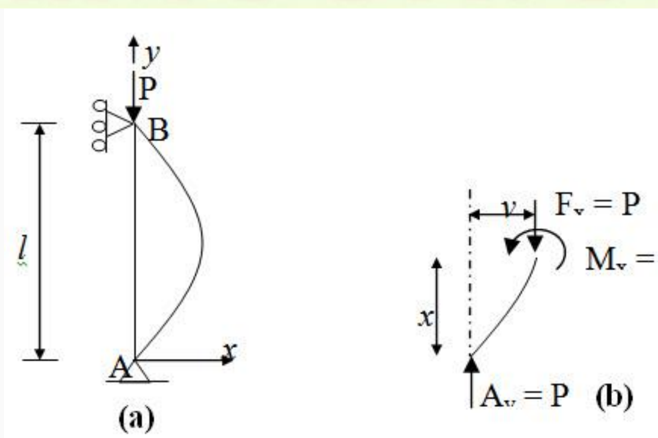


Fig. 22.2.

From equation of elastic line (lesson 3), we have,.

$$\frac{d^2y}{dx^2} = -\frac{M_x}{EI} \qquad (22.1)$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{P}{EI}y = 0 \qquad (22.2)$$

Equation (22.2) is a second order linear differential equation with constant coefficients. Boundary conditions are,

$$y(x = 0) = y(x = l) = 0 \qquad (22.3)$$

Equations (22.2) – (22.3) define a linear eigenvalue problem, whose solution may be written as,

$$y = A\cos kx + B\sin kx \qquad (22.4)$$

where, ${k^2} = {P / {EI}}$. Constants *A* and *B* may be determined from the boundary conditions (Equation 22.3),

Imposing y(x = 0) = 0 we have A = 0.

Imposing y(x = l) = 0 we have,

$$B\sin kl = 0 \qquad (22.5)$$

As $B \ne 0$, $\sin kl = 0 \Rightarrow kl = n\pi$

where,

Hence,

$${k^2} = {P \over {EI}} = {{{n^2}{\pi ^2}} \over {{l^2}}}$$

$$\Rightarrow {P_{crn}} = {{{n^2}{\pi ^2}} \over {{l^2}}}EI \qquad (22.6)$$

The eigenvalues ${P_{crn}}$ are the critical loads at which buckling takes place in different modes which are given by,

$$y = B\sin {{n\pi x} \over l} \qquad (22.7)$$

The smallest Euler buckling load is (n = 1),

$${P_E} = {{{\pi ^2}EI} \over {{l^2}}} \qquad (22.8)$$

22.3 Euler Load for Columns with Different End Conditions

Equation (22.8) may be recast as,

$${P_E} = {{{\pi ^2}EI} \over {l_{eff}^2}} \qquad (22.8)$$

where, l_{eff} is the effective length of the column. For column with both ends hinged, $l_{eff} = l$. Different end condition may increase or decrease the effective length and consequently change the critical buckling load. Effective lengths of column with different end conditions are given below.

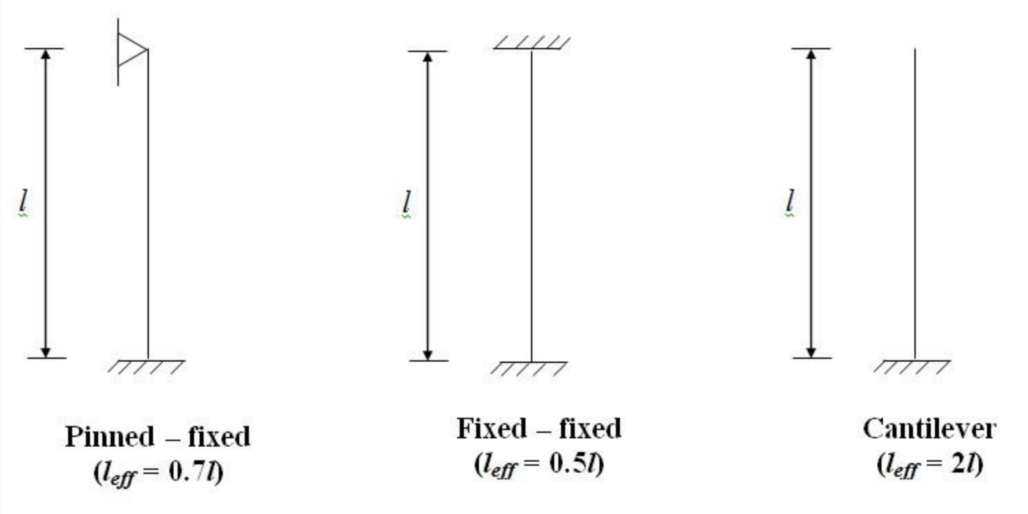

l
l
l
Pinned – fixed
(l_{eff} = 0.7l)
Fixed – fixed
(l_{eff} = 0.5l)
Cantilever
(l_{eff} = 2l)

Fig. 22.3.

LESSON 23. Columns and Struts

23.1 Limitation Of Euler's Theory of Buckling

Euler's formula for buckling load is,

$${P_E} = {{{\pi ^2}EI} \over {l_{eff}^2}} \qquad (23.1)$$

If the cross-section of the column is such that it has different *I* with respect to different axis, the buckling load is given by,

$${P_E} = {{{\pi ^2}E{I_{\min }}} \over {l_{eff}^2}} \qquad (23.2)$$

$$\Rightarrow {P_E} = {{{\pi ^2}AE} \over {l_{eff}^2}}{{{I_{\min }}} \over A} = {{{\pi ^2}AE} \over {l_{eff}^2}}k_{\min }^2$$ [${k_{\min }} = \sqrt {{{{I_{\min }}} \over A}}$ is minimum the radius of gyration]

$$\Rightarrow {{{P_E}} \over A} = {\pi ^2}E{\left({{{{k_{\min }}} \over {{l_{eff}}}}} \right)^2} \qquad (23.3)$$

Now ${{P_E}} /{E \le {\sigma _c}}$, where, σ_c is the crushing stress of a short column. Therefore,

$${\pi ^2}E{\left({{{{k_{\min }}} \over {{l_{eff}}}}} \right)^2} \le {\sigma _c}$$

$$\Rightarrow {{{l_{eff}}} \over {{k_{\min }}}} \ge \sqrt {{{{\pi ^2}E} \over {{\sigma _c}}}}$$

Therefore the Euler buckling theory is applicable only when the slenderness ratio ${{{l_{eff}}} /{{k_{\min }}}}$ has a minimum value. For instance, mild steel has the following properties,

E = 208GPa

σ_c = 320MPa

$${{{l_{eff}}} \over {{k_{\min }}}}=\ge \sqrt {{{{\pi ^2}208 \times {{10}^9}} \over {320 \times {{10}^6}}}}=80$$

Therefore, for mild steel column if the slenderness is greater than 80, then only the Euler theory can be applied in order to predict the buckling load.

Moreover, in Euler's theory it was assumed that the member is perfectly straight and homogeneous. Moreover the line of action of the applied load is assumed to be coincident with the centroidal axis of the column and therefore does not produce any moment. However in practical situations, column which satisfies all these idealization does not exist. This imposes further limitation on the direct application of Euler model in practical columns. In this lesson we will study the behavior of imperfect column and compare it with the Euler model derived in the last lesson.

23.2 Eccentrically Loaded Columns – Secant Formula

Consider an eccentrically loaded slender column as shown in Figure 23.1a.

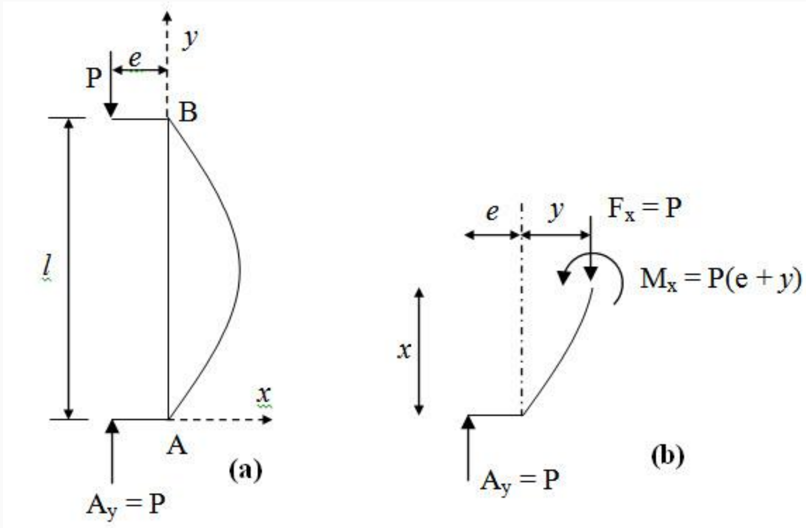


Fig. 23.1.

The equation of elastic line becomes,

$${{{d^2}y} \over {d{x^2}}}=-{{{M_x}} \over {EI}} \qquad (23.1)$$

$$\Rightarrow {{{d^2}y} \over {d{x^2}}} + {P \over {EI}}\left({e + y} \right) = 0 \qquad (23.2)$$

$$\Rightarrow {{{d^2}y} \over {d{x^2}}} + {k^2}y=-{k^2}e \qquad (23.3)$$

where, ${k^2} = {P / {EI}}$

The general solution of equation (23.3) is,

$$y=A\cos kx + B\sin kx - e \qquad (23.4)$$

Constants A and B are determined from the boundary conditions as follows,

$$y(x = 0)=0 \Rightarrow A=e$$

$$y(x = l)=0 \Rightarrow B={{1 - \cos kl} \over {\sin kl}}e$$

Substituting A and B in equation (23.4) yields,

$$y=\left({\cos kx + {{1 - \cos kl} \over {\sin kl}}\sin kx - 1} \right)e \qquad (23.5)$$

Lateral deflection at mid-height (y = l/2),

$${\left. y \right|_{x = l/2}}=\left({\cos {{kl} \over 2} + {{1 - \cos kl} \over {\sin kl}}\sin {{kl} \over 2} - 1} \right)e \qquad (23.6)$$

$$\Rightarrow {\left. y \right|_{x = l/2}}=\left({\sec {{kl} \over 2} - 1} \right)e \qquad (23.7)$$

Writing equation (23.7) in terms of Euler load ${P_E}={\pi ^2}EI/{l^2}$,

$${\left. y \right|_{x = l/2}}=\left[{\sec \left({{\pi\over 2}\sqrt {{P \over{{P_E}}}} } \right) - 1} \right]e \qquad (23.8)$$

LESSON 24. Beam-Column

A structural member subjected simultaneously to bending moment (like beam) and axial load (like column) is called Beam-Column. The bending moment in beam-column may be induced by transverse load (Figure 24.1a) or eccentricity of the axial load (Figure 24.1b).

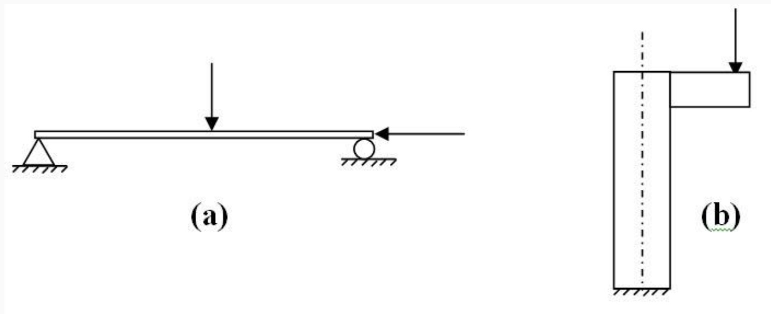


Fig. 24.1.

In this lesson we will derive the governing differential equation for beam-column and illustrate its application in different problems.

24.1 Differential Equation for Beam-Column

Consider a slender member subjected simultaneously to an arbitrary lateral load and an axial load as shown in Figure 24.2a. Free body diagram of an infinitesimal segment *dx*, in its deformed configuration is shown in Figure

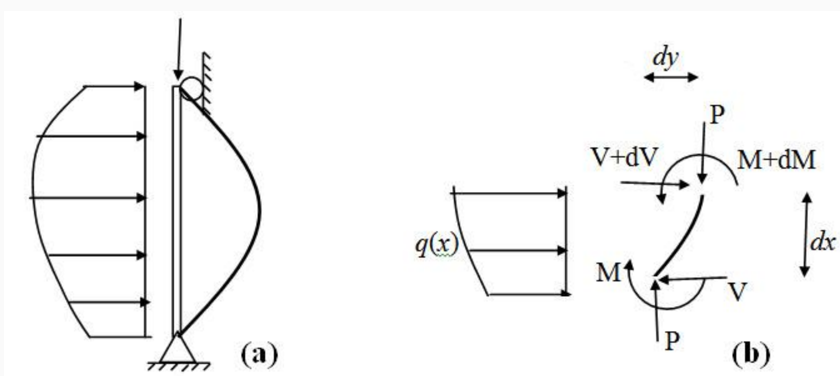


Fig. 24.2.

Applying the static equilibrium conditions,

$$\sum{{F_x}=0}\Rightarrow V + dV - V + q(x)dx$$

$$\frac{dV}{dx}=-q(x) \qquad (24.1)$$

$$\sum {M = 0}\Rightarrow-\left({M + dM} \right) + M + Vdx + Pdy + qdx\frac{dx}{2}=0$$

$$\Rightarrow-{{dM} \over {dx}} + P{{dy} \over {dx}}=-V$$ [Neglecting second order term $(dx)^2$] (24.2)

Differentiating Equation (24.2) with respect to *y*,

$$\Rightarrow-{{{d^2}M} \over {d{x^2}}} + {d \over {dx}}\left({P{{dy} \over {dx}}} \right)=-{{dV} \over {dx}} \qquad (24.3)$$

From the equation of elastic line, we have,

$${{{d^2}y} \over {d{x^2}}}=-{M \over {EI}} \qquad (24.4)$$

Substituting Equations (24.1) and (24.4), Equation (24.3) becomes,

$${{{d^4}y} \over {d{x^4}}} + {d \over {dx}}\left({P{{dy} \over {dx}}} \right)=q(x) \qquad (24.5)$$

Equation (24.5) is the beam-column governing differential equation.

Example

Consider a beam colum subjected simultaneously to a transverse load Q at its mid-span and axial compressive force P as shown in Figure 24.3a.

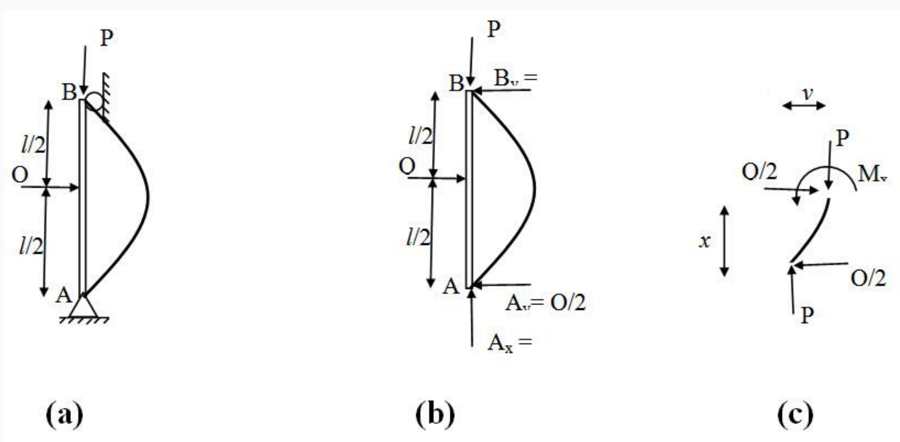


Fig. 24.3.

From Free body diagram of the whole structure (Figure 24.3b), support reactions are,

$${A_x} = P$$ and $${A_y} = {B_y} = Q/2$$

From Free body diagram Figure 24.3c,

$$\sum {{M_x} = 0}\Rightarrow-{M_x} + {Q \over 2}x + Py = 0$$ for $$0 \le x \le l/2 \qquad (24.6)$$

Substituting equation of elastic line (Equation 24.4) into Equation 24.6, we have,

$$EI{{{d^2}y} \over {d{x^2}}} + {Q \over 2}x + Py = 0 \qquad (24.7)$$

$$\Rightarrow {{{d^2}y} \over {d{x^2}}} + {k^2}y=-{{Qx} \over {2P}}{k^2}$$ $$\left[{{k^2}={P \over {EI}}} \right]$$ (24.8)

The general solution of the above differential equation,

$$y = {y_h} + {y_p}$$ (24.8)

Homogeneous solution y_h is,

$${y_h}=A\cos kx + B\sin kx$$ (24.10)

Particular solution y_p is,

$${y_p}=C + Dx$$ (24.11)

Substituting y_p into Equation (24.8),

$${{{d^2}{y_p}}\over{d{x^2}}}+{k^2}{y_p}=-{{Qx}\over{2P}}{k^2}$$

$$\Rightarrow{k^2}\left({C+Dx}\right)=-{{Qx}\over{2P}}{k^2}$$

$$\Rightarrow C=0$$ and $$D=-{Q \over {2P}}$$

Therefore,

$$y=A\cos kx + B\sin kx -{{Qx}\over{2P}}$$ (24.12)

Constants *A* and *B* are determined from the following boundary conditions,

y = 0 at x =0 $$\Rightarrow A=0$$

$${{dy} \over {dx}}=0$$ at x = l/2

$$Bk\cos \left({\frac{{kl}}{2}} \right) - \frac{Q}{{2P}} = 0 \Rightarrow B = \frac{Q}{{2Pk\cos \left({\frac{{kl}}{2}} \right)}}$$

Therefore,

$$y = \frac{{Q\sin kx}}{{2Pk\cos \left({\frac{{kl}}{2}} \right)}} - \frac{{Qx}}{{2P}}$$ for $$0 \le x \le l/2$$ (24.13)

In this case the maximum displacement occurs at the mid-span,

$${y_{\max }}=\frac{{Q\sin\left({\frac{{kl}}{2}}\right)}}{{2Pk\cos\left({\frac{{kl}}{2}}\right)}}-\frac{{Ql}}{{4P}}=\frac{Q}{{2Pk}}\left[{\tan\left({\frac{{kl}}{2}} \right)-\frac{{kl}}{2}}\right]$$

$$\Rightarrow {y_{\max }}=\frac{{Q{l^3}{k^2}}}{{2Pk{}^3{l^3}}}\left[{\tan\left({\frac{{kl}}{2}}\right) - \frac{{kl}}{2}}\right]$$

$$\Rightarrow {y_{\max }}=\frac{{Q{l^3}}}{{48EI}}\left[{\frac{{3\left({\tan\alpha-\alpha}\right)}}{{{\alpha^3}}}}\right]$$ $$\left[{\alpha=\frac{{kl}}{2}}\right]$$

$$\Rightarrow {y_{\max}}={y_0}\left[{\frac{{3\left({\tan\alpha-\alpha}\right)}}{{{\alpha^3}}}}\right]\qquad(24.14)$$

where, $${y_0}={{Q{l^3}} \over {48EI}}$$ is the deflection at the mid-span when $P = 0$.

Now, series expansion of $\tan \alpha$ is ,

$$\tan \alpha=\alpha+{{{\alpha ^3}} \over 3}+{{2{\alpha ^5}} \over {15}}+{{17{\alpha ^7}}\over {315}}+\cdots$$

$$\Rightarrow {{3\left({\tan \alpha-\alpha } \right)}\over {{\alpha ^3}}}=1+{{2{\alpha ^2}}\over 5}+{{17{\alpha ^4}}\over{105}}+\cdots$$

Therefore,

$${y_{\max }}={y_0}\left({1+{{2{\alpha ^2}} \over 5}+{{17{\alpha ^4}}\over{105}}+\cdots } \right)\qquad(24.15)$$

Now,

$${\alpha ^2}={{{k^2}{l^2}} \over 4}={{P{l^2}} \over {4EI}}=2.46{P \over {{P_E}}}$$ $$\left[{{P_E}={{{\pi ^2}EI} \over {{l^2}}}} \right]$$

Hence,

$${y_{\max }}={y_0}\left({1+0.984{P \over {{P_E}}}+0.998{{\left({{P \over {{P_E}}}}\right)}^2}+\cdots }\right)$$

$${y_{\max }}={y_0}\left({1+{P \over {{P_E}}}+{{\left({{P \over {{P_E}}}} \right)}^2}+\cdots } \right)$$

$$={y_{\max }}={y_0}{1 \over {1 - P/{P_E}}}$$ [From power series sum for] (24.16)

The factor $${1 \over {1 - P/{P_E}}}$$ is called the amplification factor or magnification factor.

The Maximum bending moment,

$${M_{\max }}={Q \over 2}{l \over 2} + P{y_{\max }}$$

$$\Rightarrow {M_{\max }}={{Ql} \over 4} + {{PQ{l^3}} \over {48EI}}{1 \over {1 - P/{P_E}}}$$

$$\Rightarrow {M_{\max }}={{Ql} \over 4}\left({1 + {{P{l^2}} \over {12EI}}{1 \over {1 - P/{P_E}}}} \right)$$

$$\Rightarrow {M_{\max }}={{Ql} \over 4}\left({1 + 0.82{P \over {{P_E}}}{1 \over {1 - P/{P_E}}}} \right)$$ $$\left[{{\rm{substituting }}{P_E}= {{{\pi ^2}EI} \over {{l^2}}}} \right]$$

$$\Rightarrow {M_{\max }}={{Ql} \over 4}\left({{{1 - 0.18P/{P_E}} \over {1 - P/{P_E}}}} \right)$$

The term $\left({{{1 - 0.18P/{P_E}} \over {1 - P/{P_E}}}} \right)$ is called amplification factor for bending moment due to axial load.

MODULE 4. Riveted and Welded Connections

LESSON 25. Rivet Joints: Basic Concept

Connections are the most important part of a structure. In this module we will learn analysis and design of two different types of connection commonly used in practical structures; (i) Riveted connection; and (ii) Welded connection.

In this lesson we will discuss different kinds of riveted joints and their failure mechanisms. Design of riveted connection will be discussed in the next lesson.

25.1 Types of Riveted Joints

Riveted joints are mainly of two types,

25.1.1 Lap joints

In lap joints two members which are to be connected are overlapped and rivets are inserted in the overlapping portion. Different types of riveted lap joints are illustrated in Figure 25.1.

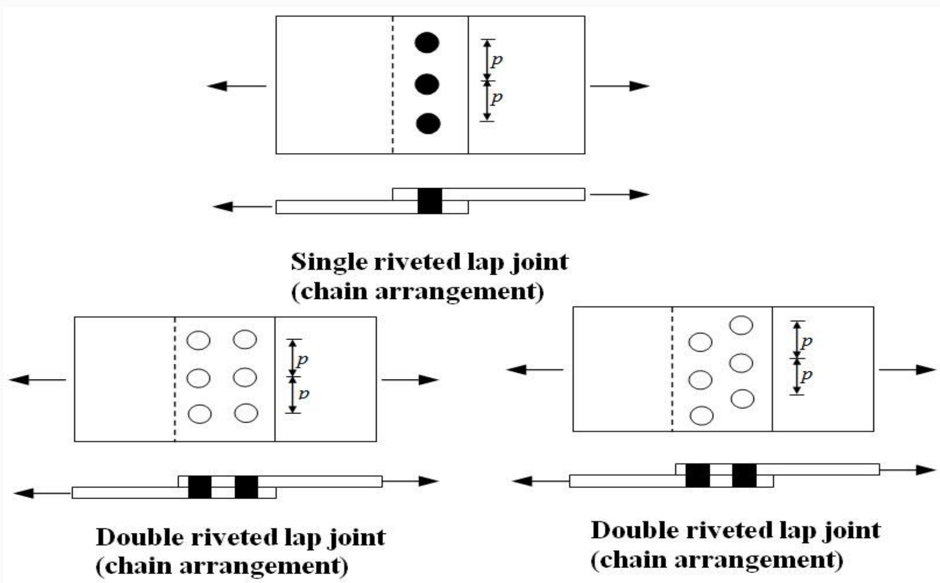


Fig. 25.1.

25.1.2 Butt joints

In butt joints, the members to be connected are placed against each other without forming any overlap and then connected together through one or more additional cover plates. When the cover plate is provided on one side of the joint it is called single cover butt joint and when provided on both sides of the joint, it is called double cover butt joint. Different types of riveted butt joints are illustrated in Figure 25.2.

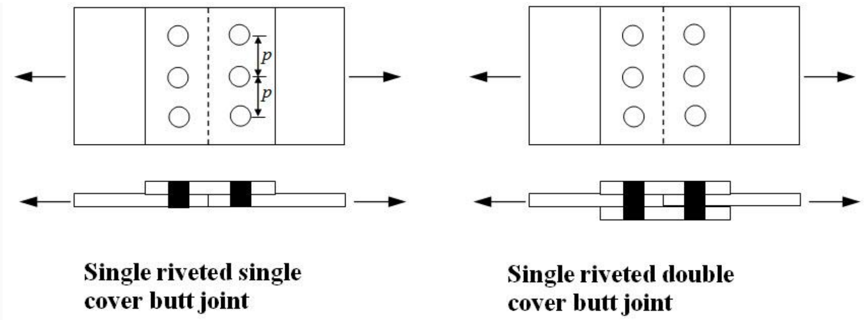


Fig. 25.2.

The distance between the centers of two adjacent rivets in a row is called pitch (p).

25.2 Failure Mechanism

A riveted joint is said to be failed when either of the rivets or the connected plates fail. Therefore strength of a riveted joint is determined by taking into account of all possible failure mechanisms. The possible failure mechanisms are illustrated below,

25.2.1 Tearing of plate

Due to the presence of holes the effective width of the plate decreases and consequently the tensile stress increases. If the induced tensile stress in the plate is more than the allowable value the plate fails in tension as shown in Figure 25.3..

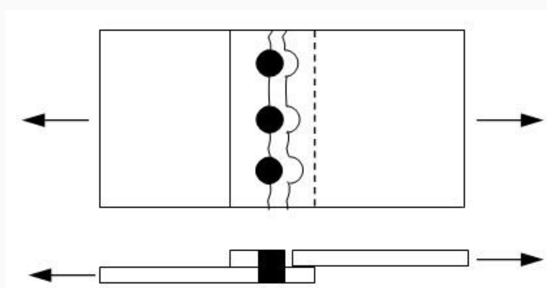


Fig. 25.3.

25.2.2 Tearing of plate at the edge

If the row of rivets is very close the edge of the plate then the plate may fail as shown in Figure 25.4.

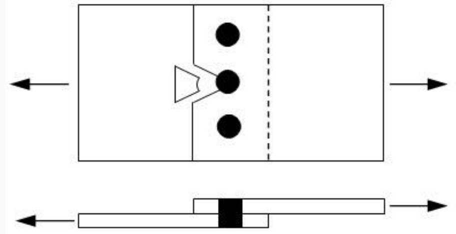


Fig. 25.4.

In order to prevent such failure a minimum distance (usually 1.5 times the diameter of the hole) of the row of rivets from the edge of the plate is maintained.

25.2.3 Shearing of rivet

The rivet may fail in shear as shown in Figure 25.5.

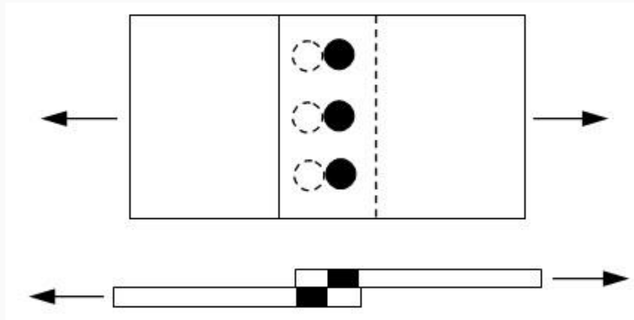


Fig. 25.5.

25.2.4 Bearing of rivet

If the stress at the contact surface between the rivet and the plate reaches the allowable bearing stress the rivet may fail in bearing as depicted in Figure 25.6.

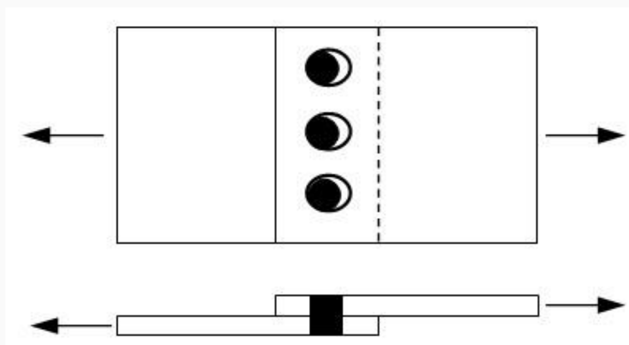


Fig. 25.6.

25.2.5 Failure mechanism in multiple riveted joints

In a multiple riveted joints, an individual row may fail in any mechanism as mentioned above. However failure of one row may not necessarily lead to complete joint failure. For instance, in the double riveted joint as shown in Figure 25.7, tearing of plate first occurs at row 1. However the joint as whole may not fail as the plates are still connected through the rivets at row 2. In order to have a complete failure, row 2 has to fail either by shear or by crushing and therefore strength of both the rows must be considered in determination of strength of joint. On the other hand if tearing of plate occurs as illustrated in Figure 25.8, the joint completely fails irrespective of row 1. In this case the strength of the joint will govern only by the strength of row 2.

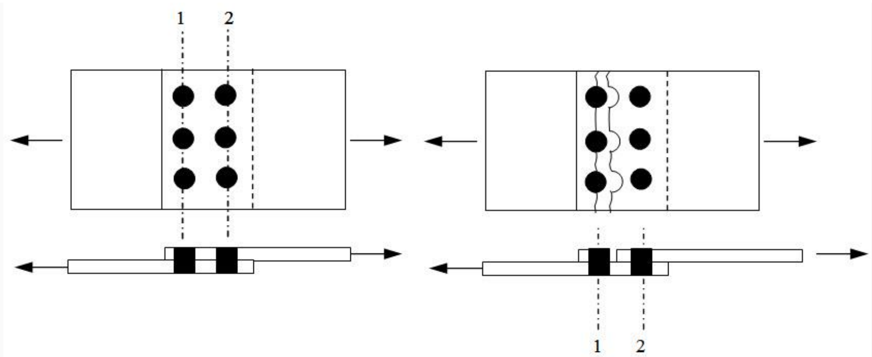


Fig. 25.7.

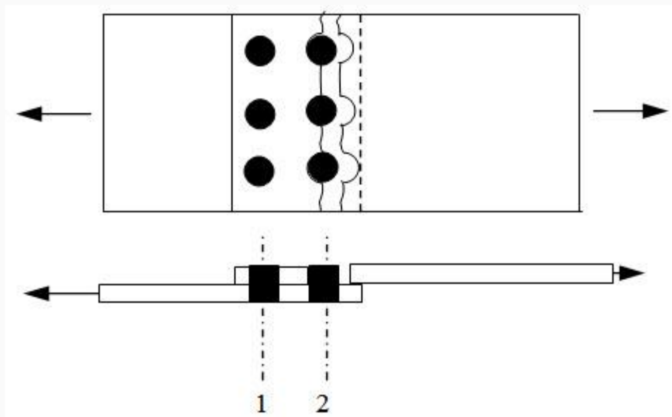


Fig. 25.8.

While determining the strength of a riveted joint (single or multiple) failure in all possible combination has to be considered.

25.3 Efficiency

In absence of rivet the maximum load that can be carried by the plate is $\sigma_t pt$. However presence of rivet reduces the strength. Efficiency of riveted joint is defined as,

$$\rm{Efficiency }(\eta)={{\rm{strength of joint}} \over {\sigma _t pt}}$$

where, t and σ_t are respectively the allowable tensile stress and thickness of the plate.

LESSON 26. Rivet Joints: Design

In this lesson first the design strength of a riveted joint will be determined. Then design of riveted joints will be illustrated via few examples.

26.1 Strength of Riveted Joints

Strength of a riveted joint in different failure mechanism is given bellow.

26.1.1 Tearing of plate

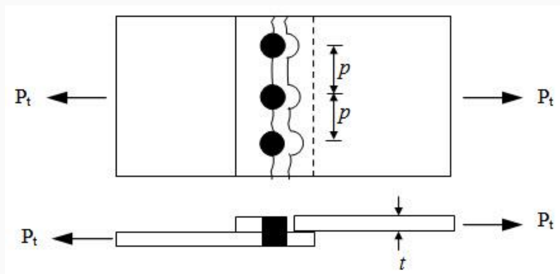


Fig. 26.1.

The maximum allowable force P_t can be determined as,

$$P_t=\sigma_t\left(b - nd \right)t$$

where,

$$\sigma_t={\rm allowable\ tensile\ stress\ of\ the\ plate\ material}$$

$$b={\rm width\ of\ the\ plate}$$

$$n={\rm number\ of\ rivets\ in\ the\ row\ where\ tearing\ takes\ place}$$

$$d={\rm Diameter\ of\ the\ hole}$$

26.1.2 Shearing of rivet

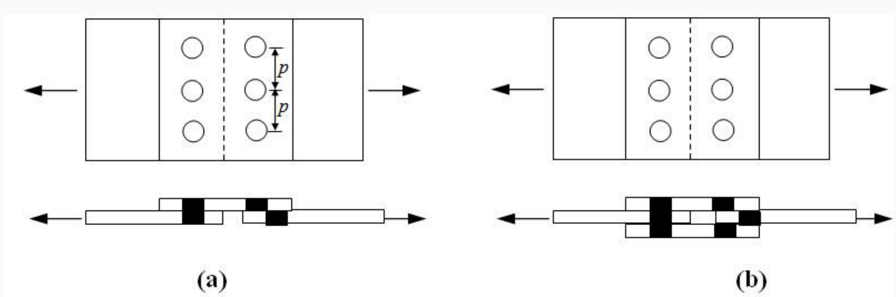


Fig. 26.2.

The maximum allowable force P_s can be determined as,

$${P_s}=n{\sigma _s}\left({{\pi\over 4}{d^2}} \right)$$ for lap joints / single cover butt joints (Figure 26.2a)

$${P_s}=2n{\sigma _s}\left({{\pi\over 4}{d^2}}\right)$$ for double cover butt joints (Figure 26.2b)

where,

$${\sigma _s}={\rm{ allowable shear stress of the rivet material}}$$

$$n={\rm{ number of rivets in the row where tearing takes place}}$$

$$d={\rm{ Diameter of the hole}}$$

26.1.3 Crushing/ Bearing of rivet

The maximum allowable force P_b can be determined as,

$${P_b}=n{\sigma _b}dt$$

where,

$${\sigma _b}={\rm{ allowable bearing stress between rivet and the plate material}}$$

$$n={\rm{ number of rivets in the row where tearing takes place}}$$

$$d={\rm{ Diameter of the hole}}$$

Design strength of a riveted joint = min (P_t,P_s,P_b)

A multiple riveted joint may fail either in a single or a combination of above three mechanisms. In such cases all possible combination has to be considered in order to determine the design strength of the joint.

26.2 Unwin's Formula

Unwin's formula gives a relation between hole diameter of rivet and thickness of the connected plates. The Unwin's formula is,

When t > 8mm, $$d=6\sqrt t$$

When t < 8mm the hole diameter is determined by equating bearing strength and shear strength of the joint as,

$${\sigma _b}t={\pi\over 4}d{\sigma _s} \Rightarrow d={{4t} \over \pi }{{{\sigma _b}} \over {{\sigma _s}}}$$

In any case d should not be less the thickness (t) of the plate.

Example

In a single riveted lap joint, the pitch of the rivet is 100mm, thickness of the plate is 15mm and rivet diameter is 30 mm. Allowable stresses are σ_t = 450 MPa , σ_s = 350MPa and σ_b = 600MPa. Determine the design strength of the joint per pitch length.

Tearing strength,

$$P_t = \sigma_t \left(b - nd \right) t$$

Since strength is to be determined per pitch length, b = p and n = 1 . Therefore,

$$P_t = \sigma_t \left(p - d \right) t = 450 \times \left(100-30 \right) \times 15 = 472.5 \ \mathrm{kN}$$

Shearing strength,

$$P_s = \sigma_s \left({\pi \over 4} d^2 \right) = 350 \times {\pi \over 4} \times 30^2 = 247.28 \ \mathrm{kN}$$

Crushing / Bearing strength,

$$P_b = \sigma_b dt = 600 \times 30 \times 15 = 270 \ \mathrm{kN}$$

Design strength = min (P_t,P_s,P_b) = 147.2 kN.

Efficiency of the joint $$\mathrm{Efficiency}\ (\eta) = {\mathrm{strength\ of\ joint} \over \sigma_t pt} = {247.28 \times 1000 \over 450 \times 100 \times 15} = 36.63\%$$

LESSON 27. Welded Joints: Basic Concept

In this lesson we will discuss different kinds of welded joints. Their design will be discussed in the next lesson.

27.1 Different Types of Welds

Welds may be broadly grouped into four types,

27.1.1 Groove welds

Groove welds are used to connect structural members that are aligned in the same plane and often used in butt joints. Various types of groove welds are depicted in Figure 27.1.

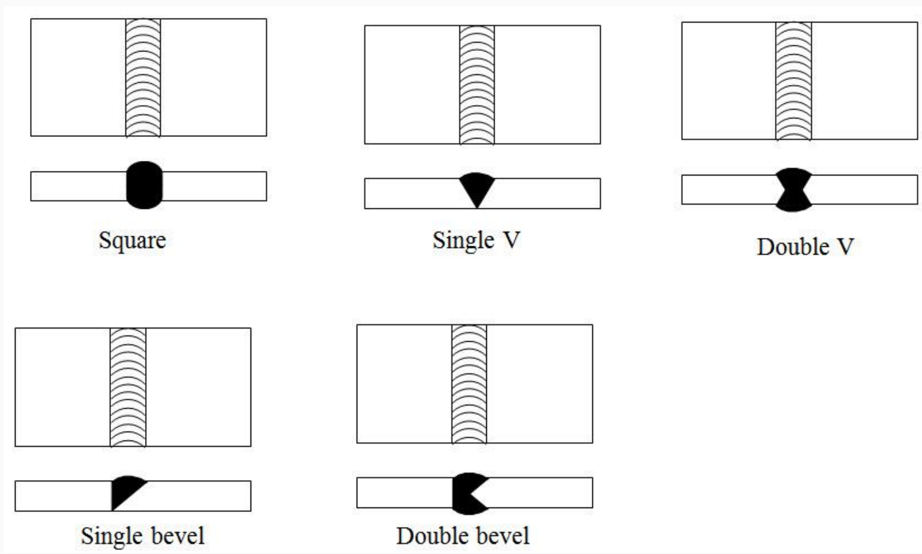


Fig. 27.1.

27.1.2 Fillet welds

Few examples of application of fillet welds are shown in Figure 27.2.

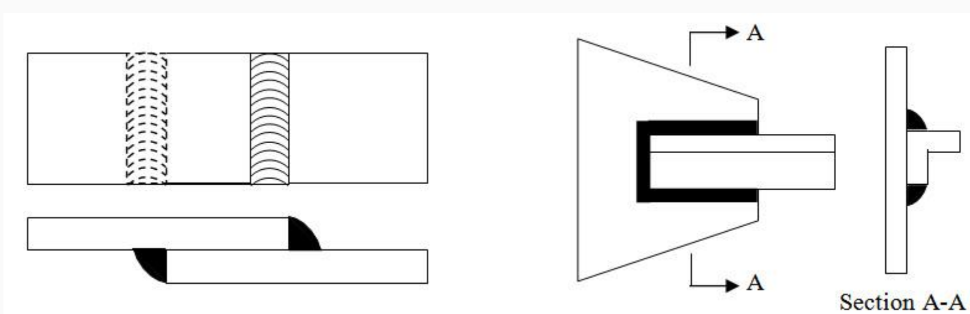


Fig. 27.2.

Fillets welds are mainly fail in shear.

27.1.3 Slot and Plug welds

There may be situations where fillet welds cannot be used due to unavailability of sufficient length. In such cases slot and plug welds are used. They are also occasionally used to fill up holes in connections. Few examples are shown in Figure 27.3.

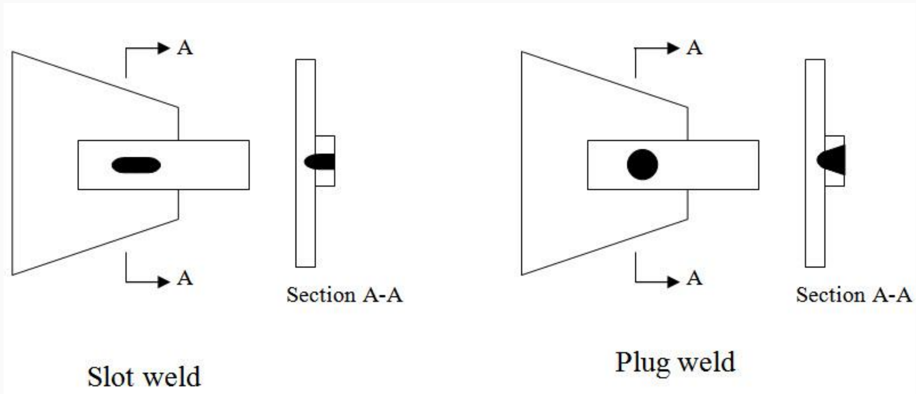


Fig. 27.3.

27.2 Types of Welded Joints

The types of welded joints depends on various factors such as size and shape of the member to be connected, area available for the joint, type of loading etc. The different types of welded joints commonly used are depicted in Figure 27.4.

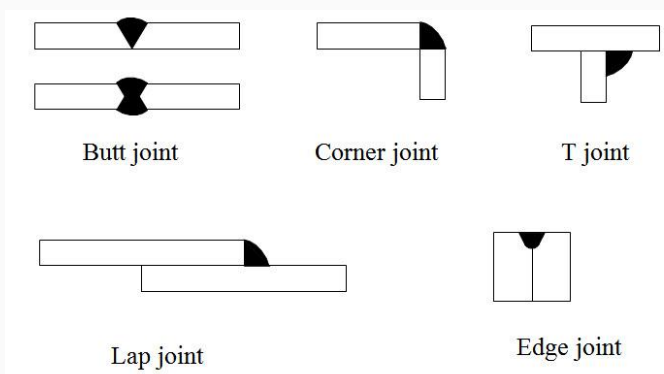


Fig. 27.4.

27.3 Effective Area of Welds

The effective area of groove or fillet weld is determined as,

$$\mathrm{A}_{\mathrm{eff}} = t_e \times l_e$$

where, t_e and l_e are the effective throat dimension and the effective length of the weld respectively.

Effective throat dimension (t_e) for different types of welds are given below.

27.3.1 Groove weld

$t_e = {5 \over 8}T$ [T is the thickness of the thinner member (Figure 27.5)

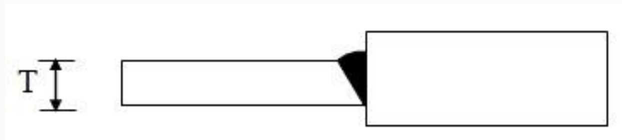


Fig. 27.5.

27.3.2 Fillet weld

The effective throat dimension of fillet weld is the shortest distance from the root of the face of the weld.

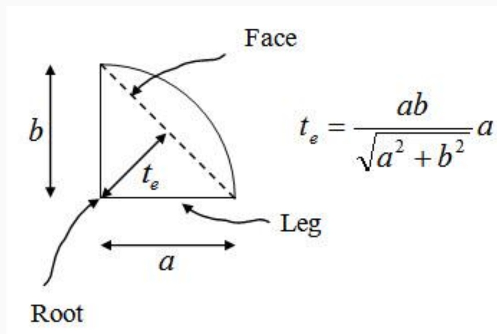


Fig. 27.6.

For slot and plug weld, the effective area is taken as the nominal area of the hole in the plane of the faying surface.

LESSON 28. Welded Joints: Design

In this lesson we will learn how to design various types of welds.

28.1 Design Strength

28.1.1 Butt Joints

Butt joints generally fail in tension.

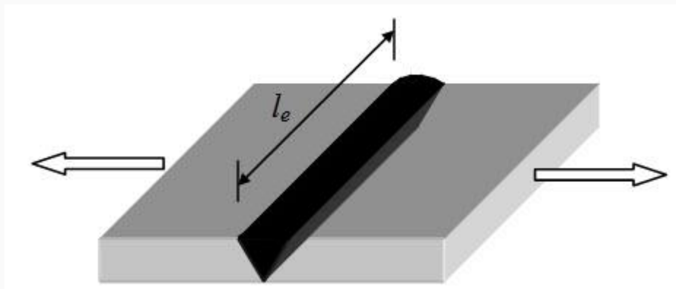


Fig.28.1.

The design strength of butt joint in tension is,

$$T_d = \sigma_t l_e t_e$$

where,

σ_t = minimum of the allowable tensile stress of the weld and the parent metal

t_e = effective throat dimension of the weld

l_e = effective length of the weld

28.1.2 Fillet Joints

Fillet joints generally fail in shear.

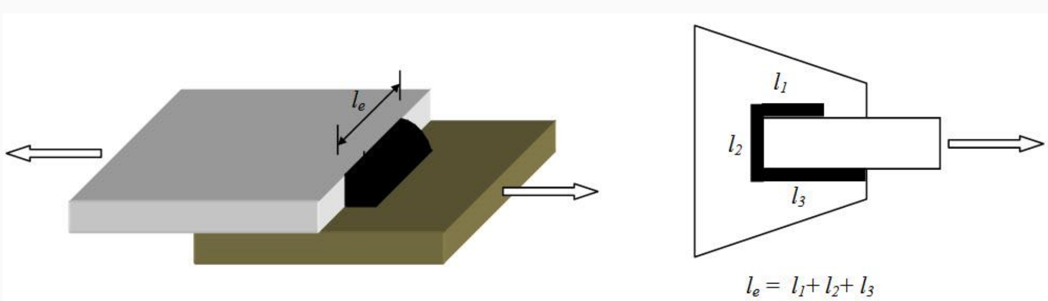


Fig. 28.2.

The design strength of fillet joint in shear is,

$${V_d}={\sigma _S}{l_e}{t_e}$$

where,

$${\sigma _S}={\rm{ allowable\ shear\ stress\ of\ the\ weld}}$$

$${t_e}={\rm{effective\ throat\ dimension\ of\ the\ weld}}$$

$${l_e}={\rm{effective\ length\ of\ the\ weld}}$$

28.1.3 Circular fillet weld subjected to torsion

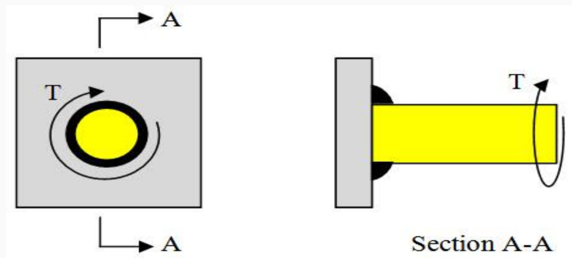


Fig. 28.3.

Maximum shear stress occurs at the throat area and is given by,

$${\tau _{\max }}={{T\left({0.5d + {t_e}} \right)} \over J}$$

where,

$$d={\rm{outer\ diameter\ of\ the\ shaft }}$$

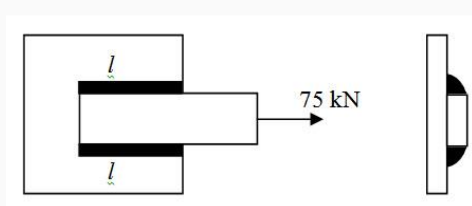
$$J={\rm{polar\ moment\ of\ area\ of\ the\ throat\ section}}={\pi\over {32}}\left[{{{\left({d + 2{t_e}} \right)}^4} - {d^4}} \right]$$

Now the maximum shear stress ${\tau _{\max }}$ should not be more than the allowable shear stress of the weld. Therefore at the limiting case,

$${\sigma _s}={\tau _{\max }}={{T\left({0.5d + {t_e}} \right)} \over J}$$

Example 1

Find the length of the fillet weld required for the following connection. Both plates are 10mm thick. Assume allowable shear stress of the weld is 70MPa.



Solution

Effective length of the fillet weld $${l_e}=2l$$

Effective throat dimension $${t_e}={t \over {\sqrt 2 }}$$ [t is the thickness of the plate]

Therefore, design strength of the weld,

$${V_d}={\sigma _S}{l_e}{t_e}=70 \times {10^6} \times 2l \times {{10 \times {{10}^{ - 3}}} \over {\sqrt 2 }}=19.8 \times {10^5}l$$

Equating V_d with the applied load,

$$19.8 \times {10^5}l = 75 \times {10^3} \Rightarrow 37.87{\rm{ mm}}$$

Example 2

Two plates of thickness 16mm and 12mm are to be connected by a groove weld. The joint is subjected to a tensile load of 300kN. Assume allowable tensile stress of the weld is 250MPa. Determine the length of the weld required for the following cases.

(*i*) Single V groove joint (Figure 28.5a)

(*ii*) Double V groove joint (Figure 28.5b)

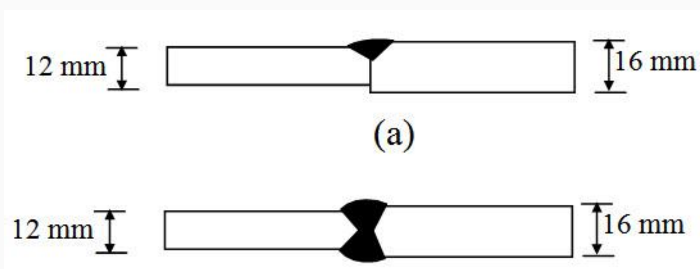


Fig. 28.5.

Solution

(*i*) case 1

Effective throat dimension $${t_e}={5 \over 8} \times {\rm{thickness of thinner plate}}={5 \over 8} \times {\rm{12}}=7.5{\rm{ mm}}$$

$${T_d}={\sigma _t}{l_e}{t_e}$$

$$\Rightarrow 300 \times {10^3} = 250 \times {10^6} \times {l_e} \times 7.5 \times {10^{ - 3}}$$

$$\Rightarrow {l_e}=160{\rm{ mm}}$$

(*ii*) case 2

Effective throat dimension $${t_e}={\rm{thickness of thinner plate}}=12{\rm{ mm}}$$

$${T_d}={\sigma _t}{l_e}{t_e}$$

$$\Rightarrow 300 \times {10^3}=250 \times {10^6} \times {l_e} \times 12 \times {10^{ - 3}}$$

$$\Rightarrow {l_e} = 100{\rm{ mm}}$$

MODULE 5. Stability Analysis of Gravity Dams

LESSON 29. Stability Analysis of Gravity Dams: Forces and General Requirements

A gravity dam is a solid structure, generally made of concrete or masonry, constructed across a river to create a reservoir on its upstream. These dams resist the various forces acting on it by its self weight and hence coined as the gravity dam. A typical section of a solid gravity dam is shown in Figure 29.1.

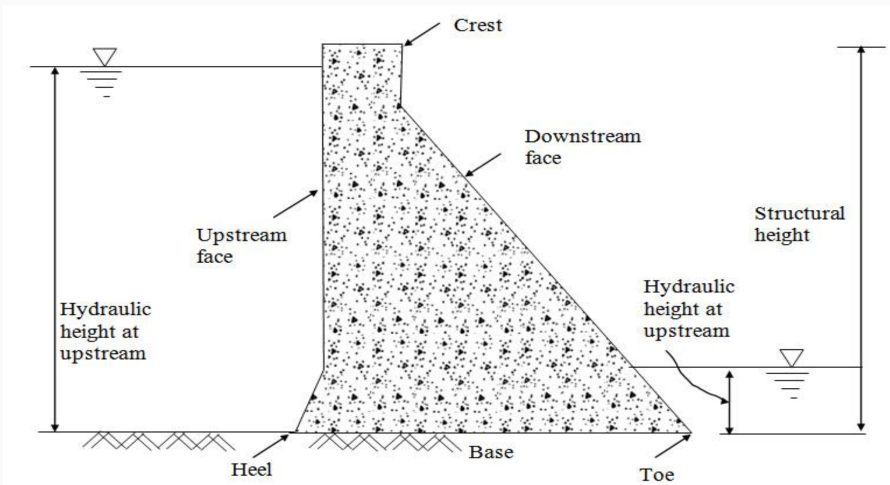


Fig. 29.2.

In this module we will learn different aspects of stability and design of concrete gravity dam.

29.1 Different Forces on Gravity Dam

A gravity dam is subjected to the following forces.

29.1.1 Self weight of the Dam

Self weight of a gravity dam is main stabilizing force.

29.1.2 Water pressure

Water pressure on the upstream side (Figure 29.2) is the main destabilizing force in gravity dam. Downstream side may also have water pressure. Though downstream water pressure produces counter overturning moment, its magnitude is much smaller as compared to the upstream water pressure and therefore generally not considered in stability analysis.

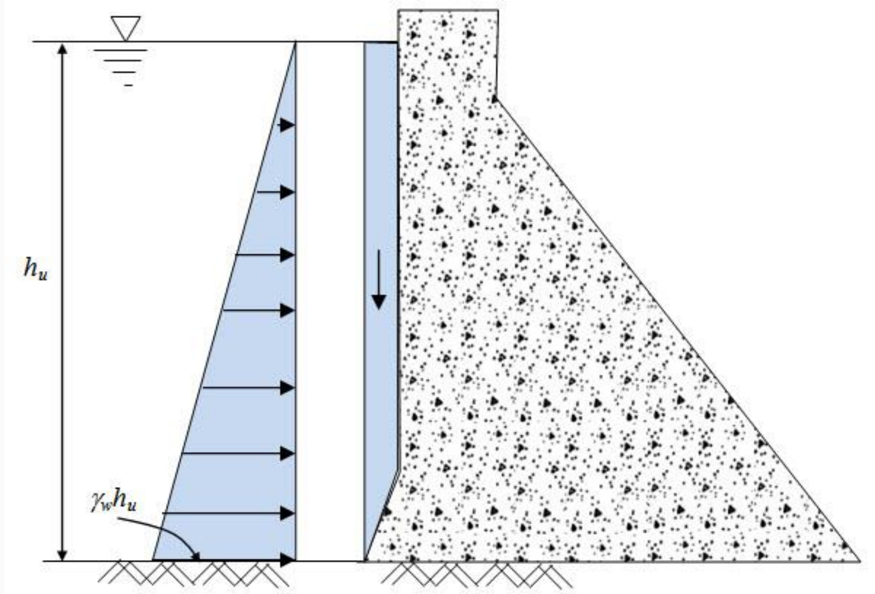


Fig. 29.2.

29.1.3 Uplift water pressure

The uplift pressure is the upward pressure of water at the base of the dam as shown in Figure 29.3. It also exists within any cracks in the dam.

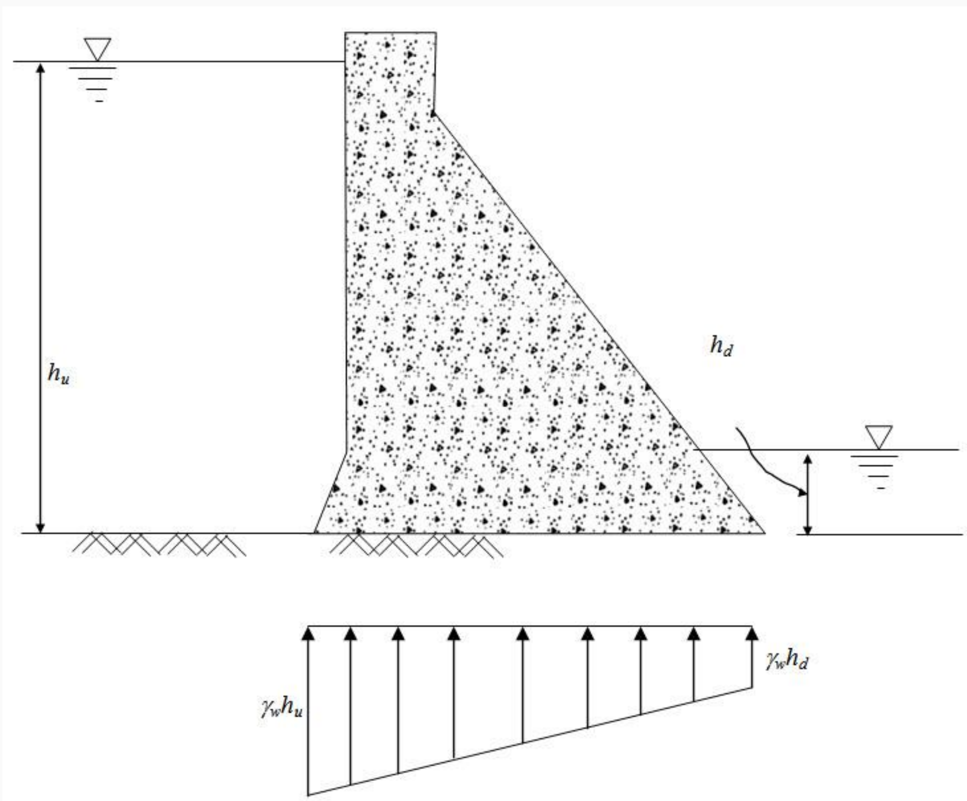


Fig. 29.3.

In addition to the above mentioned forces, a gravity dam may also subject to the following forces.

- Earth pressure
- Wave pressure
- Earthquake
- Force due to Wind
- Ice pressure

These forces have very little effect on the stability and therefore generally be neglected in stability analysis.

29.2 General Requirement for Stability

A gravity dam may fail in the following modes,

- Overturning
- Sliding
- Compression
- Tension

Therefore, the requirements for stability are,

- The dam should be safe against overturning.
- The dam should be safe against sliding.
- The induced stresses (either tension or compression) in the dam or in the foundation should not exceed the permissible value.

LESSON 30. Stability Analysis of Gravity Dams: Stability

30.1 Introduction: In this lesion we will learn two important safety requirements viz (*i*) stability against overturning and (*ii*) stability against sliding. Safety against induced stresses will be discussed in the next lesson.

Cross-section of a typical gravity dam with all relevant forces is shown in Figure 30.1.

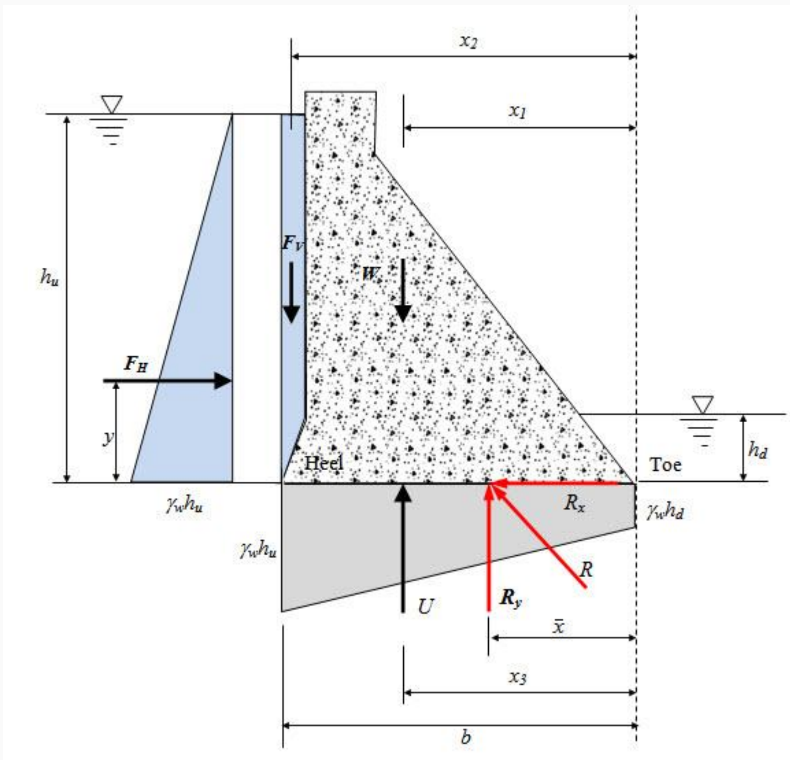


Fig. 30.1.

where,

W = Self weight of the gravity dam. It acts at a distance x_1 from the vertical line passing through the toe of the dam.

F_V = Weight of water on the inclined part of the upstream face. It acts at a distance x_2 from the vertical line passing through the toe of the dam.

F_H = ${1 \over 2}{\gamma _w}h_u^2$ = horizontal water pressure on upstream face. It acts at a distance y from the base of the dam.

U = ${1 \over 2}{\gamma _w}\left({{h_d} + {h_u}} \right)b$ Uplift force. It acts at a distance x_3 from the vertical line passing through the toe of the dam.

R = Resultant of W, F_V, F_H and U. R_x and R_y are the components of R.

Force due to horizontal water pressure (F_H) and uplift pressure (U) will cause overturning moment about the toe of the dam. This overturning moment will be stabilized mostly by the self weight of the dam W. Weight of water on the inclined part of the upstream face F_V will also produce some stabilizing moment. A gravity dam is considered to be safe against overturning if the stabilizing moment is higher than the overturning moment. The factor of safety against overturning is defined as,

$$FOS{\rm{ against overturning }}={\rm{ }}{{{\rm{Total stabilizing moment}}} \over {{\rm{Total overturning moment}}}}$$

The horizontal force acting on the dam is balanced either by friction alone or by friction and shear strength of the joint. A dam will fail in sliding at its base, or at any other level, if the net horizontal force causing sliding is more than the resistance available at that level. The factor of safety against sliding is defined as,

$$FOS \text{ against sliding} = \frac{\mu \mathrm{R_y}/F_f + \tau_c A/F_c}{\mathrm{R_x}}$$

Where,

μ = Coefficient of friction

t_c = Permissible cohesion/ shear stress

A = cross-sectional area

F_f = partial safety factor in friction (Table 1 in IS:6512 – 1984)

F_c = partial safety factor in cohesion/shear (Table 1 in IS:6512 – 1984)

LESSON 31. Stability Analysis of Gravity Dams: Stresses

31.1 Introduction: In this lesson we will derive expressions for the base pressure and stresses developed in a gravity dam.

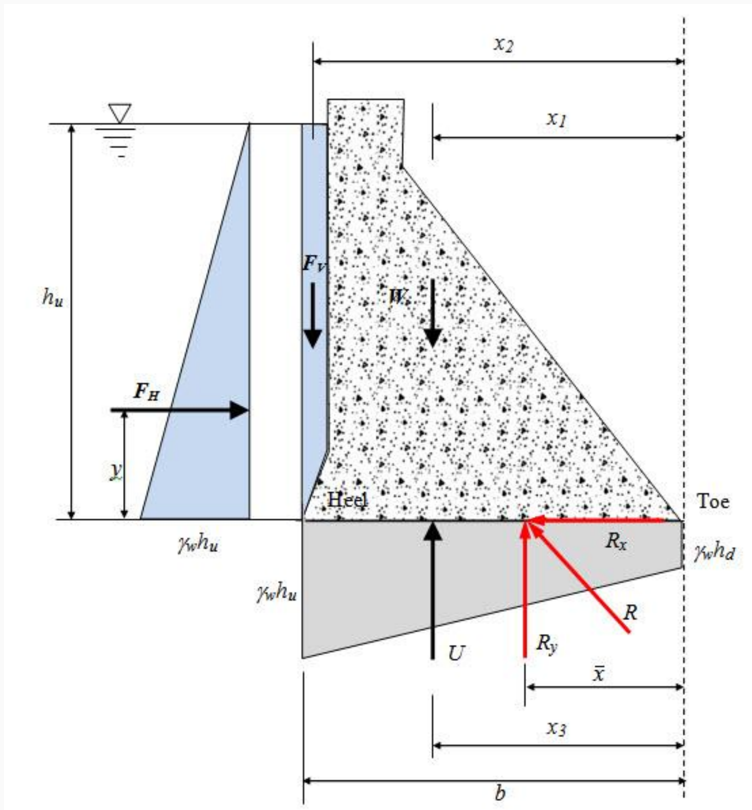


Fig. 31.1.

In the above figure let *R* be the resultant force cutting the base at a distance $\bar x$ from the toe of the dam. The components of *R* in *x* and *y* direction are obtained as,

$$R_x = F_H \qquad (31.1)$$

$$R_y = W + F_V - U \qquad (31.2)$$

$\bar x$ is obtained as,

$$\bar x = \frac{M_{toe}}{R_y} \qquad (31.3)$$

The eccentricity of from the centre of the base is given by, $e = b/2 - \bar x$ .The nominal stress at any point on the base is the sum of direct stress and bending stress.

The direct stress is always compressive and given by,

$$\sigma_{cc} = \frac{R_y}{b} \quad \text{[per unit length of the dam]} \qquad (31.4)$$

Bending moment about the centre of the base is, $M={R_y} \times e$. Corresponding bending stress at a distance x from the centre of the base is given by,

$$\sigma_{bc}=\pm {{Mx} \over I} \qquad (31.5)$$

Where, I is the second moment of area of the base per unit length of the dam. I is given by,

$$I={{1 \times {b^3}} \over {12}}={{{b^3}} \over {12}} \qquad (31.6)$$

Therefore total normal stress at a distance x from the centre of the base is,

$$p_n=\sigma_{cc} + \sigma_{bc}={{R_y} \over b} \pm {{Mx} \over I} = {{R_y} \over b} \pm {{12Mx} \over {b^3}} \qquad (31.7)$$

The resulting moment produces tension at heel and compression at toe.

Therefore,

$$p_{nheel}={{R_y} \over b} - {{12\left({R_y} \times e \right)\left(b/2 \right)} \over {b^3}}={{R_y} \over b}\left(1-{{6e} \over b} \right) \qquad (31.8)$$

$$p_{toe}={{R_y} \over b} + {{12\left({R_y} \times e \right)\left(b/2 \right)} \over {b^3}}={{R_y} \over b}\left(1 + {{6e} \over b} \right) \qquad (31.9)$$

The distributions of normal stress at the base of the dam for three different situations are shown in Figure 31.2.

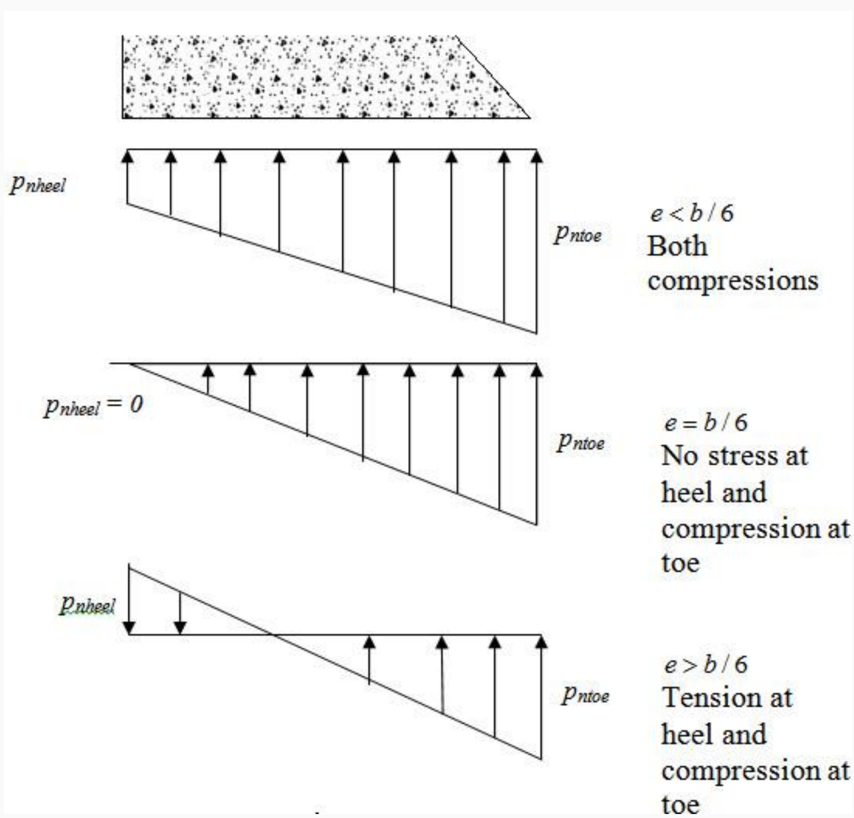


Fig. 31.2.

LESSON 32. Stability Analysis of Gravity Dams: Profile

The economy and safety of a gravity dam depend on many geometric parameters such as height of the dam, crest height, base width, upstream and downstream slopes etc. Therefore, an optimum shape design is an important problem in dam engineering. Determination of the optimum shape of a gravity dam involves several iterations and modifications. It is always better to start with an elementary profile and then do the necessary modification depending on the loading conditions, geometric feasibility etc. In this lesson we will learn how to fix an elementary profile of a gravity dam.

32.1 Elementary Profile

While determining the elementary profile of a gravity dam only pressure due to water is considered. Therefore the dam is subjected to horizontal water pressure at the upstream face and uplift pressure at the base. In such case a right angled triangular profile as shown in Figure 32.1, provides the maximum possible stabilizing force against overturning, without causing tension in the base. This profile is defined by two parameters viz, dam height (H) and base width (b). The procedure to determine the dam height and base width is given bellow.

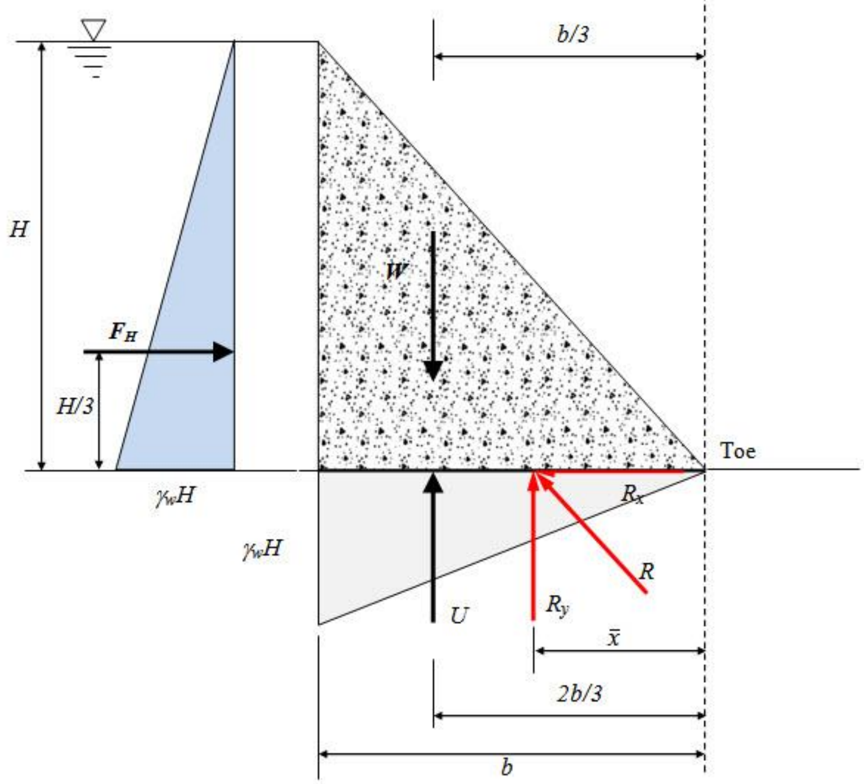


Fig. 32.1.

32.1.1 Base width of elementary profile

First the required base width is determined based on two criteria; (*i*) no sliding stress criteria and (*ii*) no tension criteria. The greater of the width given by the both criteria is taken as the width of the elementary profile.

(a) No sliding criteria

Horizontal force due to water pressure should be balanced by the frictional resistance. Therefore condition for no sliding is,

$${F_H}=\mu \left({W - U} \right) \qquad (32.1)$$

$$\Rightarrow {1 \over 2}{\gamma _w}{H^2}=\mu \left({{1 \over 2}{\gamma _C}bH - {1 \over 2}{\gamma _w}bH} \right) \qquad (32.2)$$

$$\Rightarrow b=\frac{{{\gamma _w}H}}{{\mu\left({{\gamma _C}-{\gamma _w}}\right)}}=\frac{H}{{\mu\left({\frac{{{\gamma _C}}}{{{\gamma _w}}}-1}\right)}}\Rightarrow b=\frac{H}{{\mu \left({{S_c}-1}\right)}} \qquad \text{....(32.3)}$$

$${S_c}={\gamma _c}/{\gamma _w}$$ specific gravity of dam material]

If uplift is neglected, $b=\frac{H}{\mu {S_c}}$

In this case the normal stress developed at the base of the gravity dam varies as linearly with maximum tensile stress at the heel and maximum compressive stress at the toe as shown in Figure 32.3.

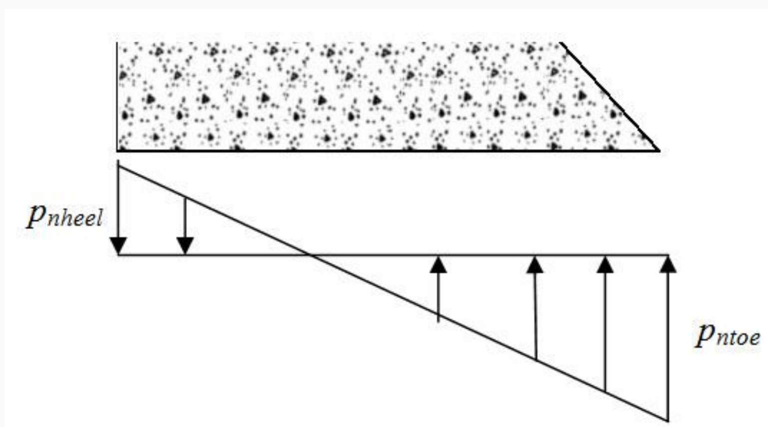


Fig. 32.2.

${p_{ntoe}}$ and ${p_{nheel}}$ are given by,

$${p_{ntoe}}={{{R_y}} \over b}\left({1 + {{6e} \over b}} \right) \qquad (32.4)$$

$${p_{nheel}}={{{R_y}} \over b}\left({1 - {{6e} \over b}} \right) \qquad (32.5)$$

where, $$e={b \over 2} - {{{M_{toe}}} \over {{R_y}}} \qquad (32.6)$$

Now,

$${R_y}=W - U = {1 \over 2}{\gamma _w}bH\left({{S_c} - 1} \right)$$ (32.7)

$${M_{toe}}=W{{2b} \over 3} - {F_H}{H \over 3} - U{{2b} \over 3}={{{b^2}} \over 3}{\gamma _w}H\left({{S_c} - 1} \right) - {{{H^3}} \over 6}{\gamma _w}$$ (32.8)

From Equations (3) and (6) – (8), we have,

$${p_{ntoe}}={\gamma _w}H{\mu ^2}{\left({{S_c} - 1} \right)^2}$$ (32.9)

$${p_{ntoe}}={\gamma _w}H\left({{S_c} - 1} \right)\left[{1 - {\mu ^2}\left({{S_c} - 1} \right)} \right]$$ (32.10)

Corresponding principal stress at toe,

$${\sigma _1}={p_{ntoe}}{\sec ^2}\varphi$$ (32.11)

$$\Rightarrow {\sigma _1}={\gamma _w}H{\mu ^2}{\left({{S_c} - 1} \right)^2}\left[{{{\left({b/H} \right)}^2} + 1} \right] \quad [\tan \varphi=b/H]$$ (32.12)

$$\Rightarrow {\sigma _1}={\gamma _w}H\left[{{\mu ^2}{{\left({{S_c} - 1} \right)}^2} + 1} \right] \quad \left[b={H \over {\mu \left({{S_c} - 1} \right)}}\right]$$ (32.13)

Similarly, shear stress is,

$$\tau={\gamma _w}H{\mu ^2}{\left({{S_c}-1} \right)^2}\tan \varphi\Rightarrow\tau={\gamma _w}H\mu \left({{S_c}-1} \right)$$ (32.14)

Following the similar approach stresses at the heel can be computed.

No tension criteria

Tension generally occurs at the heel. Condition for no tension is,

Moment of F_H about heel = moment of R_y about heel.

$$\Rightarrow {F_H}{H \over 3}=\left({W - U} \right){b \over 3}$$ (32.15)

$$\Rightarrow {1 \over 6}{\gamma _w}{H^3}=\left({{1 \over 2}{\gamma _C}bH - {1 \over 2}{\gamma _w}bH} \right){b \over 3}$$ (32.16)

$$\Rightarrow {H^2}=\left({{S_C} - 1} \right){b^2}$$ (32.17)

$$\Rightarrow b={H{\left/ {\sqrt {\left({{S_C} - 1} \right)} }\right.}}$$ (32.18)

If uplift is neglected, $b = {H{\left/ {\sqrt {{S_c}} }\right.}}$.

In this case the normal stress developed at the base of the gravity dam varies linearly with zero value at the heel and maximum at the toe as shown in Figure 32.3.

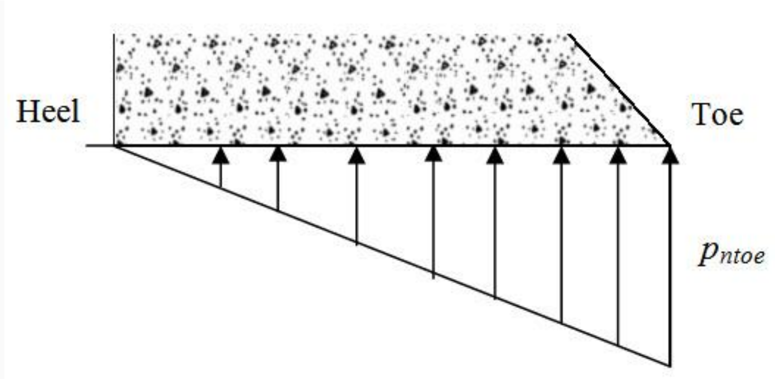


Fig. 32.3.

$${1 \over 2}b{p_{ntoe}}=W - U \Rightarrow {1 \over 2}b{p_{ntoe}}={1 \over 2}{\gamma _C}bH - {1 \over 2}{\gamma _w}bH \qquad (32.19)$$

$$\Rightarrow {p_{ntoe}}={\gamma _w}H\left({{{{\gamma _C}} \over {{\gamma _w}}} - 1} \right) \Rightarrow {p_{ntoe}}={\gamma _w}H\left({{S_c} - 1} \right) \qquad (32.20)$$

Corresponding principal stress at toe,

$${\sigma _1}={p_{ntoe}}{\sec ^2}\varphi \qquad (32.21)$$

$$\Rightarrow {\sigma _1}={\gamma _w}H\left({{S_c} - 1} \right)\left[{{{\left({b/H} \right)}^2} + 1} \right] \quad [\tan \varphi=b/H] \qquad (32.22)$$

$$\Rightarrow {\sigma _1}={\gamma _w}H\left({{S_c} - 1} \right)\left[{{1 \over {{S_C} - 1}} + 1} \right] \qquad (32.23)$$

$$\Rightarrow {\sigma _1}={\gamma _w}H{S_c} \qquad (32.24)$$

Similarly, shear stress is,

$$\tau={p_{ntoe}}\tan\varphi\Rightarrow\tau={\gamma _w}H\left({{S_c} - 1} \right){b \over H} \Rightarrow \tau={\gamma _w}H{{\left({{S_c} - 1} \right)} \over {\sqrt {\left({{S_c} - 1} \right)} }} \qquad (32.25)$$

$$\Rightarrow \tau={\gamma _w}H\sqrt {\left({{S_c} - 1} \right)} \qquad (32.26)$$

32.1.2 Limiting height of a gravity dam

The limiting height of a gravity dam is determined based on no tension criteria. The maximum value of principal stress should not exceed the permissible value.

Therefore,

$${\sigma _1}\le{\sigma _a} \qquad [{\sigma _a} \text{ is the allowable normal stress}]$$

$$\Rightarrow {\gamma _w}{H_{\lim }}{S_c} \le {\sigma _a}$$

$$\Rightarrow {H_{\lim }} \le {{{\sigma _a}} \over {{\gamma _w}{S_c}}} \qquad (32.27)$$

Generally, uplift pressure is not considered while determining the limiting height of a gravity dam. Following the approach given in section 32.1.1, it can be shown that for no uplift pressure Equation (27) is reduced to,

$$H_{\lim} \le \frac{\sigma_a}{\gamma_w \left(S_c + 1 \right)} \qquad (32.28)$$